STATISTICAL CONSIDERATION ABOUT THE CRITICAL RATE OF STRENGTH OF MATERIALS

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Résumé

The "critical rate" is defined as the probability of occurrence of fracture of materials under applied stress. The "safty rate" is also defined as the probability which is equal to the unity minus the critical rate. Thus, the terminology "safty factor" is criticized. The critical rate for the body having many components of stress and for the anisotropic body, is also discussed. The probability distribution of the ratio of the stress of fracture to the available stress is briefly mentioned.

§ 1. Tensor of Safty Coefficient

The maximum strength of materials, *i.e.* the stress of materials when fracture occurs, shall be denoted by $\mathbf{M}$. The stress which is allowed to appear in materials shall be denoted by $\mathbf{F}^\circ$. We call $\mathbf{F}^\circ$ as *available stress* but not as *allowable stress, i.e.* $\mathbf{F}^\circ$ is not the *greatest value* of stresses which appears in materials, but other values of stresses which do appear in materials. We define the *safty coefficient* by means of the following equation:

$$\mathbf{M} = \mathbf{S} \cdot \mathbf{F}^\circ. \tag{1}$$

Let the rectangular components of $\mathbf{M}$ and $\mathbf{F}^\circ$ be M_{ij} $(i, j=1, 2, 3)$ and F_{ij}° $(i, j=1, 2, 3)$, respectively. Then, the components of $\mathbf{S}$ are to be denoted by S_{ijkl} $(i, j, k, l=1, 2, 3)$, and (1) is written as follows:

$$M_{ij} = S_{ijkl} F_{kl}^\circ, \tag{2}$$

with

$$S_{ijkl} = S_{klij} = S_{ikjl}, \tag{3}$$

by physical consideration. As usual in tensorial calculation, the summation convention over dummy indices is used. $\mathbf{M}$ and $\mathbf{F}^\circ$ are symmetric tensors of second rank. Accordingly $\mathbf{S}$ is expressed as a symmetric tensor of fourth rank. Engineers commonly name $\operatorname{Max}_{i,j,k,l} S_{ijkl}$ or $\operatorname{Max}_{i,j} S_{ijkk}$ as a *safty factor*.

In an isotropic material, components F_{ij} $(i, j=1, 2, 3)$ of stress tensor $\mathbf{F}$ is written as:

$$F_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}, \qquad (i, j=1, 2, 3) \tag{4}$$

where ε_{ij} are components of strain tensor $\boldsymbol{\varepsilon}$, with Lamé's constans λ and μ.

In anisotropic materials, however, we have

$$F_{ij}=a_{ijkl}\varepsilon_{kl}, \qquad (i,\ j,\ k,\ l=1,\ 2,\ 3) \tag{5}$$

with elastic constants:

$$a_{ijkl}=a_{klij}=a_{ikjl}. \tag{6}$$

Similar consideration leads to write the components of M_{ij} in an isotropic material

$$M_{ij}=S_\alpha F^\circ_{kk}\delta_{ij}+2\,S_\beta F^\circ_{ij}, \qquad (i,\ j=1,\ 2,\ 3) \tag{7}$$

where S_α and S_β may be called *safty coefficient for expansion* and *safty coefficient for shearing*.

By means of (7), we can write the force of the fracture acting on unit volume of the material:

$$\frac{\partial M_{ij}}{\partial x_j}=\{\lambda(3\,S_\alpha+2\,S_\beta)+2\,\mu S_\alpha\}\frac{\partial}{\partial x_i}\varepsilon^\circ_{kk}+4\,\mu S_\beta\frac{\partial\varepsilon^\circ_{ij}}{\partial x_j}, \qquad (i,\ j=1,\ 2,\ 3) \tag{8}$$

with ε°_{ij} $(i,j=1,2,3)$: components of strain in the case of fracture of the material.

In general, M_{ij} take different values for different materials of different chemical composition, *i.e.* elastic constants (*e.g.* λ and μ etc.) are different for different kinds of materials. Moreover, even if we take a material of definite chemical composition, M_{ij} vary with pieces of the material, that is, elastic constants and other material constants take variable values timally and spacially. The stress which acts on the material, varies also timally and spacially in general case. Thus we are led to consider $\mathbf{M}$ and $\mathbf{F}^\circ$ to be random variables.

§ 2. Statistical Consideration of Safty Coefficient in the One-Dimensional Case*) 1)

In equation (3), we put

$$\left.\begin{aligned}&F^\circ_{ij}=0, \qquad (i\neq j)\\&F^\circ_{kk}=F^\circ_{11}+F^\circ_{22}+F^\circ_{33}\equiv F^\circ,\\&M_{11}=M_{22}=M_{33}\equiv M,\\&M_{ij}=0, \qquad (i\neq j)\\&S_\alpha=S,\end{aligned}\right\} \tag{7}$$

and obtain

$$M=S\cdot F^\circ. \tag{8}$$

If we put $\underset{i,j,k,l}{\mathrm{Max}}\ S_{ijkl}=S$, then we have a similar expression to (8).

Let
$$E\{F^\circ\}=F_0,$$
and
$$E\{(F^\circ-F_0)^2\}=f^2, \tag{9}$$

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* The discussion in § 2 is mainly due to the reference 1).

where E means mathematical expectation.

$$\left.\begin{array}{ll}\text{Let} & E\{M\} = M_0, \\ \text{and} & E\{(M-M_0)^2\} = m^2.\end{array}\right\} \tag{10}$$

Then we obtain from (9) and (10)

$$\left.\begin{array}{ll} & E\{M-F\} = M_0 - F_0, \\ \text{and} & E\{[M-F-(M_0-F_0)]^2\} = m^2+f^2.\end{array}\right\} \tag{11}$$

If the random variables F and M are distributed normally, *i.e.* their distributions are respectively $N(F_0, f^2)$ and $N(M_0, m^2)$, then we obtain[2]

$$P_r[M \leq F] = P_r[M-F \leq 0] = \frac{1}{\sqrt{2\pi(m^2+f^2)}}\int_{-\infty}^{0} \exp\left[-\frac{\{x-(M_0-F_0)\}^2}{2(m^2+f^2)}\right]dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{\frac{M_0-F_0}{\sqrt{m^2+f^2}}}^{\infty} \exp\left[-\frac{z^2}{2}\right]dz. \tag{12}$$

In many cases, we can take $M_0 \gg F_0$, and we obtain from (12) by means of asymptotic expansion:

$$P_r[M \leq F] \approx \frac{1}{\sqrt{2\pi}} \cdot \exp[-y^2/2] \cdot \left[\frac{1}{y} - \frac{1}{y^3} + \frac{3}{y^5} - \cdots\right], \tag{13}$$

where

$$y = \frac{M_0 - F_0}{\sqrt{m^2+f^2}}.$$

Accordingly the *critical rate* (*i.e.* the probalility of occurrence of fracture), increases rapidly with vanishing y, *i.e.* when F_0 approaches M_0. The larger $\sqrt{m^2+f^2}$ becomes, the more increases the critical rate. That means, the larger m^2 or f^2 becomes, the more increases the critical rate. In other words, the more the strength of materials deviates (or, the greater the standard deviation of stress becomes), the more increases the critical rate.

When $F_{ij} \neq 0$ $(i \neq j)$, we can say the same thing about the safty coefficient S_3, if we put $F_{kk} = 0$.

§ 3. Safty Coefficient with Many Components

The discussion in § 1 can be easily extended to the many dimensional case, where the safty coefficient has many components. Let M_{ij} and F_{ij}° in (4) be rearranged as M_r and F_s° in a column vector, respectively. We rewrite (4) as follows:

$$\begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_9 \end{pmatrix} = \mathrm{A} \cdot \begin{pmatrix} F_1^{\circ} \\ F_2^{\circ} \\ \vdots \\ F_9^{\circ} \end{pmatrix}, \tag{14}$$

where $\mathbf{A}$ is a matrix of nineth order, whose components involve S_α and S_β. For example, if we put $F_1^\circ = F_{11}^\circ$, $F_5^\circ = F_{22}^\circ$, $F_9^\circ = F_{33}^\circ$ etc., the elements in the first low of $\mathbf{A}$ are to be

$$(S_\alpha + 2S_\beta,\ 0,\ 0,\ 0,\ S_\alpha,\ 0,\ 0,\ 0,\ S_\alpha).$$

Let the components of stress be F_s $(s=1, 2, 3, \ldots, 9)$ which correspond to the components F_s° $(s=1, 2, 3, \ldots, 9)$ of allowable stress. Let F_s $(s=1, 2, 3, \ldots, 9)$ be distributed normally $N(F_s^\circ, f_s^2)$ respectively, and M_r $(r=1, 2, 3, \ldots, 9)$ be distributed normally $N(M_r^\circ, m_r^2)$ respectively.

Then $M_r - \sum_{s=1}^{9} B_{rs}F_s$ obeys[2)] normal distribution $N(M_r^\circ - \sum_s B_{rs}F_s^\circ,\ m_r^2 + \sum_s B_{rs}^2 f_s^2)$, and the probability for the sum of the stress $\sum_s B_{rs}F_s$ to be greater than the components M_r of the stress of fracture, is given by

$$P_r[M_r \leq \sum_{s=1}^{9} B_{rs}F_s] = \frac{1}{\sqrt{2\pi}\sqrt{m_r^2 + \sum_s B_{rs}^2 f_s^2}} \int_{-\infty}^{0} \exp\left[-\frac{\{x - (M_r^\circ - \sum_s B_{rs}F_s^\circ)\}^2}{2(m_r^2 + \sum_s B_{rs}^2 f_s^2)} \right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\frac{M_r^0 - \sum B_{rs}F_s^\circ}{\sqrt{m_r^2 + \sum_s B_{rs}^2 f_s^2}}}^{\infty} \exp\left[-\frac{z^2}{2} \right] dz, \qquad (r=1, 2, 3, \ldots, 9) \tag{15}$$

where B_{rs} $(r, s=1, 2, 3, \ldots, 9)$ are constants previously given.

In the analogous consideration, we can also calculate the critical rate of fracture, *i.e.* the probability for the fracture to occur, for the case expressed by equation (2). We shall consider the case, in which the components of stress F_{kl} $(k, l=1, 2, 3)$ obey normal distributions $N(F_{kl}^\circ, f_{kl}^2)$ respectively and independently, and in which the components of stress M_{ij} $(i, j=1, 2, 3)$ of fracture also obey normal distributions $N(M_{ij}^\circ, m_{ij}^2)$ respectively and independently.

Then, the variable

$$M_{ij} - \sum_{k,l} C_{ijkl}F_{kl}$$

obeys normal distribution $N(M_{ij}^\circ - \sum_{k,l} C_{ijkl}F_{kl}^\circ,\ m_{ij}^2 + \sum_{k,l} C_{ijkl}^2 f_{kl}^2)$.

Accordingly, we obtain the critical rate, *i.e.* the probability for the sum of stress $\sum_{k,l} C_{ijkl}F_{kl}$ to exceed the stress of fracture,

$$P_r[M_{ij} \leq \sum_{k,l} C_{ijkl}F_{kl}] = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}\sqrt{m_{ij}^2 + \sum_{k,l} C_{ijkl}^2 f_{kl}^2}} \exp\left[-\frac{\{x - (M_{ij}^\circ - \sum_{k,l} C_{ijkl}F_{kl}^\circ)\}^2}{2(m_{ij}^2 + \sum_{k,l} C_{ijkl}^2 f_{kl}^2)} \right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\beta}^{\infty} \exp\left[-\frac{z^2}{2} \right] dz, \tag{16}$$

with

$$\beta = \frac{M_{ij}^\circ - \sum_{k,l} C_{ijkl}F_{kl}^\circ}{\sqrt{m_{ij}^2 + \sum_{k,l} C_{ijkl}^2 f_{kl}^2}},$$

where C_{ijkl} $(i, j, k, l=1, 2, 3)$ are constants previously given.

§ 4. The Ratio of the Stress of Fracture to the Available Stress

For the sake of simplicity, we take a case

$$M=S\cdot F^{\circ}. \tag{8}$$

Just as in § 1, we assume that F is normally distributed $N(F_0, f^2)$ and M is also normal $N(M_0, m^2)$.

If we take n random samples F_i $(i=1, 2, 3, \ldots, n)$ from $N(F_0, f^2)$, then a new variable

$$\chi_1^2=\sum_{i=1}^{n}(F_i-F_0)^2/f^2 \tag{17}$$

obeys χ^2-distribution of degrees of freedom n. Accordingly a variable

$$t=\frac{M-M_0}{m}\bigg/\frac{\chi_1}{\sqrt{n}} \tag{18}$$

obeys t-distribution of degrees of freedom n. And another variable

$$\frac{M-M_0}{n}\bigg/\frac{F-F_0}{f} \tag{19}$$

obeys t-distribution of degree of freedom unity. Accordingly, if we give previously the numerical value of α or β, then we can obtain probability such that

$$\frac{M-M_0}{F-F_0}\leq\alpha \quad \text{or} \quad \frac{M-M_0}{F-F_0}>\beta.$$

As another case, if we take r random samples M_j $(j=1, 2, 3, \ldots, r)$ from $N(M_0, m^2)$, then a variable

$$\chi_2^2=\sum_{j=1}^{r}(M_j-M_0)^2/m^2 \tag{20}$$

obeys χ^2-distribution of degrees of freedom r. Accordingly a new variable

$$y=\frac{\chi_1^2/n}{\chi_2^2/r} \tag{21}$$

obeys F-distribution of degrees of freedom (n, r).

By means of (17), (18), (20) and (21) etc., we can calculate the probabilities for F, M, and their ratio to be greater than some refference values from the realized values of the samples.

§ 5. Critical Rate for Anisotropic Body

Let the components M_{ij} of the stress of fracture be represented by a ν-

dimensional column vector $\mathbf{M}$. In the same manner, we shall write the components F_{ij} of the available stress by a ν-dimensional column vector $\mathbf{F}$. These two vectors obey the ν-dimensional normal distributions $N(\mathbf{M}_0, \rho)$ and $N(\mathbf{F}_0, \sigma)$ respectively, with the mean vectors $\mathbf{M}_0$ and $\mathbf{F}_0$ and with matrices ρ and σ. *i.e.* the probability density functions for $\mathbf{M}$ and $\mathbf{F}$ in the ν-dimensional vector space $\mathbf{R}^\nu$ shall be denoted by

$$f_1(\mathrm{x}) = \frac{\sqrt{\det \rho}}{(2\pi)^{\nu/2}} \exp\left[-\frac{1}{2} \tilde{\mathrm{x}} \rho \mathrm{x} \right], \tag{22}$$

and

$$f_2(\mathrm{y}) = \frac{\sqrt{\det \sigma}}{(2\pi)^{\nu/2}} \exp\left[-\frac{1}{2} \tilde{\mathrm{y}} \sigma \mathrm{y} \right], \tag{23}$$

where $\mathrm{x} = \mathbf{M} - \mathbf{M}_0$, $\mathrm{y} = \mathbf{F} - \mathbf{F}_0$, and $\tilde{\mathrm{x}}$ is the transposed matrix of x.

The characteristic functions of (22) and (23) are respectively

$$\varphi_1(\mathrm{t}) = \int e^{i\tilde{\mathrm{t}}\mathrm{x}} f_1(\mathrm{x}) d\mathrm{x} = \exp\left[-\frac{1}{2} \tilde{\mathrm{t}} \mu \mathrm{t} \right], \tag{24}$$

and

$$\varphi_2(\mathrm{t}) = \int e^{i\tilde{\mathrm{t}}\mathrm{y}} f_2(\mathrm{y}) d\mathrm{y} = \exp\left[-\frac{1}{2} \tilde{\mathrm{t}} \phi \mathrm{t} \right], \tag{25}$$

where $\mu = \rho^{-1}$ and $\phi = \sigma^{-1}$ are covariance matrices.[3] Accordingly, a new column vector

$$\mathrm{z} = \mathbf{M} - \mathbf{F} - (\mathbf{M}_0 - \mathbf{F}_0) = \mathrm{x} - \mathrm{y},$$

also obeys the ν-dimensional normal distribution $N(0, \lambda)$, where λ is the inverse matrix of $\mu + \phi$.

Accordingly, the probability for $\mathbf{M}$ to be smaller than $\mathbf{F}$, *i.e.* the probability of occurence of fracture (critical rate), is

$$P_r[\mathbf{M} \leq \mathbf{F}] = \frac{\sqrt{\det \lambda}}{(2\pi)^{\nu/2}} \int\int \cdot \overset{\alpha}{\underset{-\infty}{\vdots}} \cdot \int \exp\left[-\frac{1}{2} \tilde{\mathrm{z}} \lambda \mathrm{z} \right] d\mathrm{z}, \tag{26}$$

with the integration limit α.

As the matrices ρ and σ are symmetric, the matrix λ is also symmetric. One can diagonalize the matrix λ by means of a suitable orthogonal transformation

$$\eta = \mathbf{C}\mathrm{z}.$$

Taking any integer p in $1 \leq p \leq \nu$, we obtain an expression

$$P_r[M_p \leq F_p,\ M_q > F_q(q \neq p)] = \frac{\sqrt{\det \lambda}}{(2\pi)^{\nu/2}} \int_{z_p=-\infty}^{z_p=\alpha_p} \int_{z_q=\alpha_q}^{z_q=+\infty} \cdots \int \exp\left[-\frac{1}{2} \tilde{\mathrm{z}} \lambda \mathrm{z} \right] d\mathrm{z}, \tag{27}$$

where M_i $(i = 1, 2, \ldots, \nu)$ is a component of vector $\mathbf{M}$.

It may be perhaps useful to take

$$\max_{\nu \in (1, 2, \dots, \nu)} P_r[M_p \leq F_p], \tag{28}$$

as a practical critical rate (probability of occurrence of fracture), and to define $\{1-(\text{craitical rate})\}$ as the *safty rate* (probability of no occurrence of fracture).

§ 6. Applications and Remarks

1) The idea given above refers to the two kinds of quantities, between which the linear relation such as (1), (2), or (8) holds. In the same manner, our idea can be conveniently applied to the quantities linearly dependent each other. Especially, the distribution of the difference of two random variables or the distribution of their ratio comes into question. Accordingly, if the random variables are commutative with space-coordinates (or time), we can apply our idea to many physical phenomena just as in our consideration. Moreover, it is also very useful to take the normal distribution, because many random variables appeared in natural phenomena are likely to obey normal distribution.

2) We shall call S_0 which is defined by (8):

$$M_0 = S_0 \cdot F_0, \tag{8'}$$

as the "mean safty coefficient" (the mean value of the safty coefficient). The reason why it is called by such a name, is as follows:

$$M_0 = E[M] = E[S \cdot F_0] = E[S] \cdot F_0 = S_0 \cdot F_0,$$

where E represents the mathematical expectation. By means of the "mean safty coefficient" S_0, we can write (12) as follows:

$$P_r[M \leq F] = \frac{1}{\sqrt{2\pi}} \int_{\frac{(S_0-1)F_0}{\sqrt{m^2+f^2}}}^{\infty} \exp\left[-\frac{z^2}{2}\right] dz. \tag{12'}$$

It is sometimes perhaps useful to define $\{1-(12')\}$ as the *mean safty rate*, which is the probability for the fracture not to occur under the mean available stress F_0.

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