ON THE VIBRATION OF AEROFOIL

I. LOCALIZED VIBRATION OF LINEAR ELASTIC CHAIN

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§ 0. Introduction

In the first paper on the vibration of aerofoil, we shall start to discuss a simple model of linear elastic chain, which consists of infinite number of particles and elastic springs connecting the adjascent particles. When the length of the linear elastic chain is finite, the model shows the dynamical behavior of the simplest aerofoil either with simple elongation in the span-wise direction or with simple transverse vibration perpendicular to the span-wise direction.

The statistical behavior of such a linear chain is of quite interest, when we consider the dynamical response of the aerofoil under the influence of external force. The solutions of the chain of infinite length with no external force and of the chain of finite length with both ends free, have been already obtained.[1) 2)] Recently, Prigogine[2)] has also discussed the statistical behavior of the chain of infinite length without external applied force.

In § 1 of the present paper, we shall at first consider the chain of finite length with one end fixed and the other end free. In § 2 we shall treat the chain of finite length with both ends fixed. In § 3 we shall discuss the chain of infinite length with external force, and in § 4 the statistical behavior of the chain under the influence of external random force. In order to take into account the initial conditions explicitly, we shall use Bessel function expansion rather than Fourier series. The emphasis is laid especially to show the existence of the localized vibration in the linear elastic chain.

§ 1. Chain with One End Fixed and the Other End Free

We shall take a linear elastic chain of infinite length as shown in Fig. 1, which consists of infinite number of particles of the same mass M and of springs of the same spring constant K.

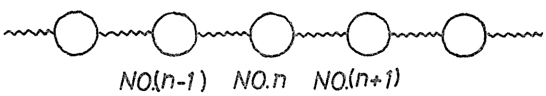

FIG. 1. Linear elastic chain of infinite lengh.

The equations of motion of the system are

$$\frac{d\xi_n}{dt} = \frac{K}{M}\{(r_{n,n+1} - \bar{r}) - (r_{n-1,n} - \bar{r})\}, \quad \text{for} \quad -\infty < n < +\infty \qquad (1.1)$$

where $r_{i,i+1}$ is the distance between the i-th and the $(i+1)$-th particles, $\bar{r}$ represents the distance at equilibrium, ξ_n the velocity of the n-th particle.

Introducing the normalized co-ordinates $q_{n,n+1}$, ξ_n^* and τ, which are defined by

$$\left.\begin{aligned} q_{n,n+1} &= \frac{r_{n,n+1}-\bar{r}}{\sigma}, \\ \xi_n^* &= \xi_n \bigg/ \left(\sigma\sqrt{\frac{K}{M}}\right), \\ \tau &= 2t\sqrt{\frac{K}{M}}, \end{aligned}\right\} \tag{1.2}$$

with σ the root mean square of $q_{n,n+1}$, and time t, and putting

$$\left.\begin{aligned} y_{2n+1} &= q_{n,n+1}, \\ y_{2n} &= \xi_n^*, \end{aligned}\right\} \tag{1.3}$$

we obtain from (1.1) the normalized equations of motion:

$$2\frac{d}{d\tau}y_n(\tau) = y_{n+1}(\tau) - y_{n-1}(\tau), \qquad \text{for} \quad -\infty < n < +\infty \tag{1.4}$$

which have a system of solutions[1)2)]:

$$y_n(\tau) = \sum_{\nu=-\infty}^{+\infty} a_\nu J_{\nu-n}(\tau), \qquad \text{for} \quad -\infty < n < +\infty \tag{1.5}$$

with constants a_ν and Bessel function $J_m(\tau)$ of order m.

Now, let us consider a system of finite length with N particles, and call such a system the chain of length N. We shall take the chain of length N with one end fixed and the other end free, *i.e.*, we demand that the first particle is fixed and that the N-th particle is set free. The equations of motion are

$$y_2(\tau) = 0, \tag{1.6}$$

$$2\frac{d}{d\tau}y_n(\tau) = y_{n+1}(\tau) - y_{n-1}(\tau), \qquad \text{for} \quad n = 3, 4, \ldots, 2N-1 \tag{1.7}$$

and

$$2\frac{d}{d\tau}y_{2N}(\tau) = -y_{2N-1}(\tau). \tag{1.8}$$

The solutions of (1.6) ~ (1.8) can be taken as

$$y_n(\tau) = \sum_{\nu=-\infty}^{+\infty} a_\nu J_{\nu-n}(\tau), \tag{1.9}$$

which satisfy (1.7), with constants a_ν. Inserting (1.9) into (1.6) and (1.8), we obtain

$$\left.\begin{aligned} a_2 &= 0, \\ a_{\nu+2} &= (-1)^{\nu+1} a_{2-\nu}, \end{aligned}\right\} \tag{1.10}$$

$$\text{for} \quad \nu = 1, 2, 3, \ldots$$

and

$$\left.\begin{array}{l} a_{2N+1}=0, \\ a_{\nu+2N+1}=(-1)^{\nu+1}a_{2N+1-\nu}, \end{array}\right\} \quad \text{for} \quad \nu=1, 2, 3, \ldots \tag{1.11}$$

Thus the solutions of (1.6)~(1.8) are given by (1.9) with (1.10) and (1.11). Let the initial values of y_i ($i=2, 3, 4, \ldots, 2N$) be y_i° ($i=2, 3, 4, \ldots, 2N$), respectively. Then we have

$$a_i = y_i^\circ, \qquad (i=2, 3, 4, \ldots, 2N) \tag{1.12}$$

and we can express all the a_ν's in terms of y_i° ($i=2, 3, 4, \ldots, 2N$). The schematic diagram for a_ν's is given in Fig. 2. As is easily seen, the repetition of a_ν has a period of four times larger than the length of the chain.

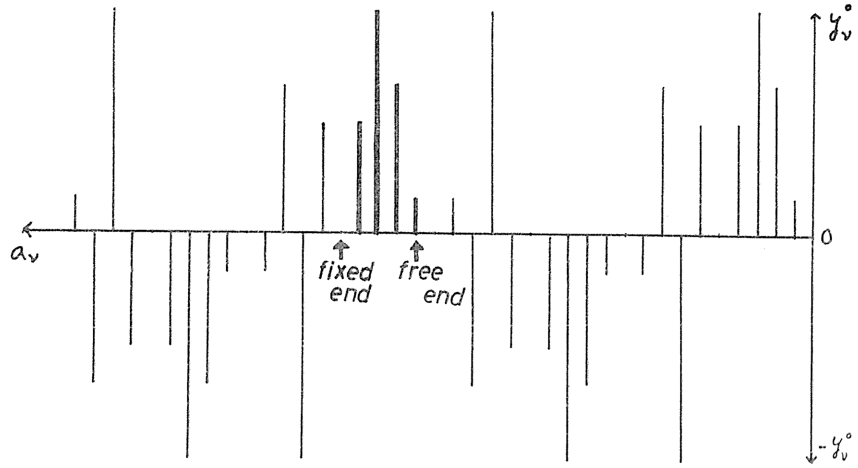

FIG. 2. Example of distribution of y_ν° versus a_ν in a chain with one end fixed and the other end free.

§2. Chain with Both Ends Fixed

Let us consider the chain of length N with the first and the N-th particles both fixed.

The equations of motion are

$$2\frac{d}{d\tau}y_n(\tau)=y_{n+1}(\tau)-y_{n-1}(\tau), \quad \text{for} \quad n=3, 4, \ldots, 2N-1 \tag{2.1}$$

$$y_2(\tau)=0, \tag{2.2}$$

$$y_{2N}(\tau)=0. \tag{2.3}$$

We assume the solutions of the form:

$$y_n(\tau)=\sum_{\nu=-\infty}^{+\infty} a_\nu J_{\nu-n}(\tau), \tag{2.4}$$

which satisfy (2.1), with constants a_ν. From (2.2) and (2.3), we obtain

$$\left.\begin{array}{ll} a_2=0, & \\ a_{\nu+2}=(-1)^{\nu+1}a_{2-\nu}, & \text{for } \nu=1,2,\ldots \\ a_{2N}=0, & \\ a_{\nu+2N}=(-1)^{\nu+1}a_{2N-\nu}. & \text{for } \nu=1,2,\ldots \end{array}\right\} \tag{2.5}$$

Accordingly, the solutions of (2.1) ~ (2.3) are given by (2.4) with (2.5). If we take the initial values of y_i $(i=2, 3, \ldots, 2N)$ as y_i° $(i=2, 3, \ldots, 2N)$, we obtain

$$a_i=y_i^\circ, \qquad (i=2, 3, 4, \ldots, 2N) \tag{2.6}$$

and all the a_ν's is expressed by use of y_i°. Fig. 3. shows the schematic representation of a_ν. The period of the repetition of a_ν is twice greater than the length of the chain.

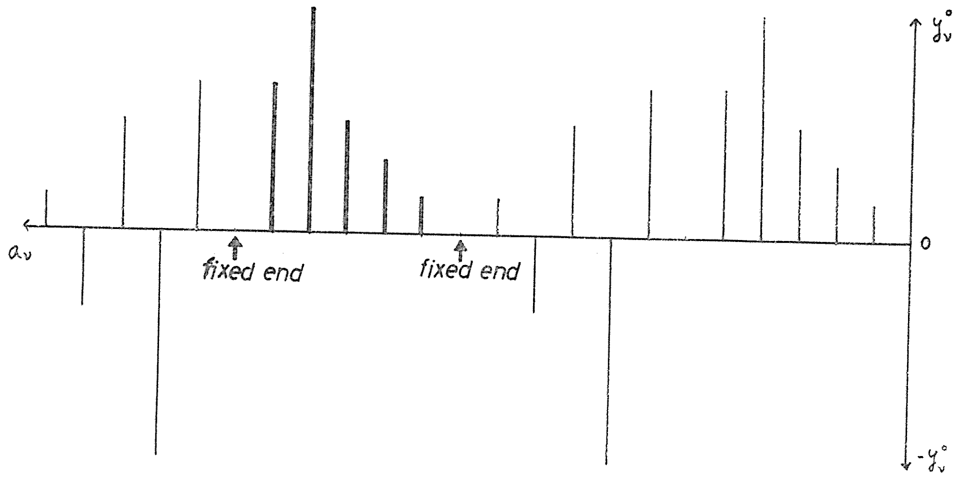

FIG. 3. Example of distribution of y_ν° versus a_ν in a chain with both ends fixed.

§ 3. Chain of Infinite Length with External Force

Let us take a chain of infinite length with an external force $2f(\tau)$ acting on the k-th particle.

The equations of motion are

$$2\frac{d}{d\tau}y_n(\tau)=y_{n+1}(\tau)-y_{n-1}(\tau)+2f(\tau)\cdot\delta_{n,2k}, \tag{3.1}$$

$$\text{for} \quad -\infty<n<+\infty$$

with Kronecker's delta $\delta_{n,m}$. We shall take the solutions of the form:

$$y_n(\tau)=\sum_{\nu=-\infty}^{+\infty}a_\nu(\tau)J_{\nu-n}(\tau). \tag{3.2}$$

Inserting (3.2) into (3.1), we obtain the equation

$$\sum_{\nu=-\infty}^{+\infty} \frac{da_\nu(\tau)}{d\tau} J_{\nu-n}(\tau) = f(\tau) \cdot \delta_{n,2k}. \tag{3.3}$$

Operating $\sum_{n=-\infty}^{+\infty} J_{l-n}(\tau)$ to the both sides of (3.3), and by means of the relation[3]:

$$\sum_{n=-\infty}^{+\infty} J_{\nu-n}(\tau) J_{l-n}(\tau) = \delta_{\nu,l},$$

we have

$$\frac{da_l(\tau)}{d\tau} = f(\tau) J_{l-2k}(\tau),$$

i.e.

$$a_\nu(\tau) = a_\nu^\circ + \int_0^\tau f(\tau) J_{\nu-2k}(\tau) d\tau, \tag{3.4}$$

with integration constants a_ν°.

Accordingly, the solutions of (3.1) are expressed by (3.2) and (3.4), *i.e.*

$$y_n(\tau) = \sum_{\nu=-\infty}^{+\infty} (y_\nu^\circ + \int_0^\tau f(\tau) J_{\nu-2k}(\tau) d\tau) J_{\nu-n}(\tau), \tag{3.5}$$

with initial values $y_\nu^\circ (= a_\nu^\circ)$ of y_ν.

§4. Statistical Behavior of the Chain with External Random Force

Starting from the initial conditions of the chain which are given statistically, we consider the timal behavior of the chain under the influence of external random force.

Let the initial values y_ν° of y_ν in (3.5) be distributed statistically, and their second moments $\langle y_\mu^\circ y_\nu^\circ \rangle_{Av}$ be given at $\tau = 0$. We do not discuss the functional form of the joint probability density function of y_ν°, but we only assume that their second moments are initially known to us. If we take the initial probability density function of y as $f^\circ(\mathrm{y}^\circ, \tau = 0)$ and the probability density function of $\mathrm{y}(\tau)$ at a time point τ as $f(\mathrm{y}, \tau)$, then we have the relation

$$f^\circ(\mathrm{y}^\circ, 0) d\mathrm{y}^\circ = f(\mathrm{y}, \tau) d\mathrm{y}, \tag{4.1}$$

with two vectors of infinite components $\mathrm{y}^\circ = (\ldots, y_{\nu-1}^\circ, y_\nu^\circ, y_{\nu+1}^\circ, \ldots)$ and $\mathrm{y}(\tau) = (\ldots, y_{\nu-1}, y_\nu, y_{\nu+1}, \ldots)$. Equation (4.1) represents Liouville's theorem.

We shall calculate the second moments of $y_n(\tau)$ by means of (3.5) at any time point τ.

$$\langle y_n(\tau) y_m(\tau) \rangle_{Av} = \Big\langle \sum_{\nu=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} (y_\nu^\circ + \int_0^\tau f(\tau) \cdot J_{\nu-2k}(\tau) d\tau)$$
$$\times (y_{\nu+j}^\circ + \int_0^\tau f(\tau) J_{\nu+j-2k}(\tau) d\tau) \cdot J_{\nu-n}(\tau) J_{\nu+j-m}(\tau) \Big\rangle_{Av}$$

$$= \sum_{\nu=-\infty}^{+\infty} \sum_{j=-\infty}^{\infty} [\langle y_\nu^\circ y_{\nu+j}^\circ \rangle_{Av} + \langle y_\nu^\circ \int_0^\tau f(\tau) \cdot J_{\nu+j-2k}(\tau)\, d\tau \rangle_{Av}$$

$$+ \langle y_{\nu+j}^\circ \int_0^\tau f(\tau) \cdot J_{\nu-2k}(\tau) d\tau \rangle_{Av}$$

$$+ \langle \int_0^\tau f(\tau)\, J_{\nu-2k}(\tau) d\tau \cdot \int_0^\tau f(\tau) J_{\nu+j-2k}(\tau) d\tau \rangle_{Av}] \cdot J_{\nu-n}(\tau) J_{\nu+j-m}(\tau). \qquad (4.2)$$

Further we assume for the random force $f(\tau)$ to be

$$1) \quad \langle f(\tau) \rangle_{Av} = 0, \qquad (4.3)$$

and

$$2) \quad \langle f(\tau_1) \cdot f(\tau_2) \rangle_{Av} = D \cdot \delta(\tau_1 - \tau_2). \qquad (4.4)$$

The expression (4.4) means that the random function $f(\tau)$ has a white spectrum.
Then we obtain, from (4.3) and (4.4),

$$\langle y_\nu^\circ \int_0^\tau f(\tau)\, J_{\nu+j-2k}(\tau) d\tau \rangle_{Av} = \langle y_\nu^\circ \rangle_{Av} \int_0^\tau \langle f(\tau) \rangle_{Av} J_{\nu+j-2k}(\tau) d\tau = 0, \qquad (4.5)$$

and

$$\langle \int_0^\tau f(\tau) J_{\nu-2k}(\tau)\, d\tau \cdot \int_0^\tau f(\tau) J_{\nu+j-2k}(\tau) d\tau \rangle_{Av}$$

$$= \int_0^\tau \int_0^\tau \langle f(\tau_1) f(\tau_2) \rangle_{Av} J_{\nu-2k}(\tau_1) \cdot J_{\nu+j-2k}(\tau_2) d\tau_1 d\tau_2$$

$$= D \int_0^\tau J_{\nu-2k}(\tau) \cdot J_{\nu+j-2k}(\tau) d\tau. \qquad (4.6)$$

From (4.2), (4.5) and (4.6), we obtain

$$\langle y_n(\tau) y_m(\tau) \rangle_{Av}$$

$$= \sum_{\nu=-\infty}^{+\infty} \sum_{j=-\infty}^{\infty} \langle y_\nu^\circ y_{\nu+j}^\circ \rangle_{Av} J_{\nu-n}(\tau) \cdot J_{\nu+j-m}(\tau)$$

$$+ D \sum_{\nu=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} J_{\nu-n}(\tau) \cdot J_{\nu+j-m}(\tau) \int_0^\tau J_{\nu-2k}(\tau) \cdot J_{\nu+j-2k}(\tau)\, d\tau. \qquad (4.7)$$

The second term in the right hand side of (4.7) is easily calculated, and we obtain

$$\sum_{\nu=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} J_{\nu-n}(\tau) \cdot J_{\nu+j-m}(\tau) \int_0^\tau J_{\nu-2k}(\tau) \cdot J_{\nu+j-2k}(\tau) d\tau$$

$$= \lim_{\tau^* \to \tau} \int_0^\tau \sum_{\nu=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} J_{\nu-n}(\tau^*)\, J_{\nu+j-m}(\tau^*)\, J_{\nu-2k}(\tau)\, J_{\nu+j-2k}(\tau) d\tau$$

$$= \lim_{\tau^* \to \tau} \int_0^\tau \sum_{\nu=-\infty}^{+\infty} J_{\nu-n}(\tau^*)\, J_{\nu-2k}(\tau) \sum_{j=-\infty}^{+\infty} J_{m-\nu-j}(-\tau^*) J_{m-2k-(m-\nu-j)}(\tau) d\tau$$

$$= \lim_{\tau^* \to \tau} \int_0^\tau \sum_{\nu=-\infty}^{+\infty} J_{\nu-n}(\tau^*)\, J_{\nu-2k}(\tau) J_{m-2k}(\tau - \tau^*)\, d\tau$$

$$= \lim_{\tau^* \to \tau} \int_0^\tau J_{m-2k}(\tau - \tau^*) \sum_{\nu=-\infty}^{+\infty} J_{n-\nu}(-\tau^*) J_{n-2k-(n-\nu)}(\tau) d\tau$$

$$= \lim_{\tau^*\to\tau}\int_0^{\tau} J_{m-2k}(\tau-\tau^*)J_{n-2k}(\tau-\tau^*)d\tau, \tag{4.8}$$

where the relation[3]:

$$J_\nu(z+t)=\sum_{m=-\infty}^{+\infty} J_{\nu-m}(t)\cdot J_m(z)$$

is used.

In evaluating the integral (4.8), we take two cases for τ, namely, (a) sufficiently large τ and (b) sufficiently small τ.

Case (a): for $\tau\to+\infty$:

$$\begin{aligned}
&\lim_{\tau^*\to\tau}\int_0^{\infty} J_{m-2k}(\tau-\tau^*)J_{n-2k}(\tau-\tau^*)d\tau\\
&=\lim_{\tau^*\to\tau}\int_{-\tau^*}^{\infty} J_{m-2k}(x)\cdot J_{n-2k}(x)dx\\
&=\int_{-\infty}^{\infty} J_{m-2k}(x)J_{n-2k}(x)dx\\
&=(-1)^{m+n}\int_0^{\infty} J_{m-2k}(x)J_{n-2k}(x)dx+\int_0^{\infty} J_{m-2k}(x)J_{n-2k}(x)dx\\
&=\begin{cases} 0, \quad \text{for} \quad m+n=\text{odd integer}\\ 2\int_0^{\infty} J_{m-2k}(x)\cdot J_{n-2k}(x)dx. \quad \text{for} \quad m+n=\text{even integer}\end{cases}
\end{aligned} \tag{4.9}$$

When $m+n$ is an even integer, we put $m-n=2s$ and obtain

$$\begin{aligned}
&\int_0^{\infty} J_{m-2k}(x)\cdot J_{n-2k}(x)dx\\
&=\frac{1}{\pi}\left\{2(-1)^{s-1}\sum_{n=0}^{s-1}\frac{1}{2s-(2n+1)}+(-1)^s\left[\log\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\right]_{\phi=0}^{\frac{\pi}{2}}\right\},
\end{aligned} \tag{4.10}^{*)}$$

for $s\geq 1$,
and

$$=\frac{1}{\pi}\left[\log\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\right]_{\phi=0}^{\frac{\pi}{2}}, \tag{4.11}^{*)}$$

for $s=0$.

Accordingly, we have the following results:

(i) when $m=n$,

$$\int_0^{\infty}\{J_{n-2k}(x)\}^2dx=\frac{1}{\pi}\left(\log\tan\frac{\pi}{2}-\log\tan\frac{\pi}{4}\right),$$

and

$$\lim_{\tau\to+\infty}\sum_{\nu=-\infty}^{+\infty}\sum_{j=-\infty}^{+\infty}\langle y_\nu^\circ y_{\nu+j}^\circ\rangle_{Av}J_{\nu-n}(\tau)J_{\nu+j-n}(\tau)=\frac{1}{2}(\mathring{\mu}_{v,0,v}+\mathring{\mu}_{d,0,d}). \tag{4.12}^{**)}$$

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*) cf. Appendix.

**) cf. Reference 2) p. 1057.

If we assume that the initial distribution of y_ν is homogeneous, *i.e.* $\langle y_\nu^\circ y_{\nu+j}^\circ\rangle_{Av}$ depend merely upon j. The first suffix of μ° expresses the parity of ν, *i.e.* if ν is odd we write d, and if ν even, we use v. The second suffix expresses the value of j, and the third expresses the parity of $\nu+j$. For example,

$$\overset{\circ}{\mu}_{v,0,v}=\langle \overset{\circ}{y}_{\text{even}}\cdot \overset{\circ}{y}_{\text{even}}\rangle_{Av},$$
$$\overset{\circ}{\mu}_{d,0,d}=\langle \overset{\circ}{y}_{\text{odd}}\cdot \overset{\circ}{y}_{\text{odd}}\rangle_{Av},$$

etc.

From (4.7), (4.9) and (4.12), we obtain

$$\lim_{\tau\to\infty}\langle y_n^2(\tau)\rangle_{Av}=\frac{1}{2}(\overset{\circ}{\mu}_{v,0,v}+\overset{\circ}{\mu}_{d,0,d})+\frac{2D}{\pi}\left[\log\tan\frac{\pi}{2}-\log\tan\frac{\pi}{4}\right]. \tag{4.13}$$

(ii) when $m=n+1$,

$$\lim_{\tau\to\infty}\sum_{\nu=-\infty}^{+\infty}\sum_{j=-\infty}^{+\infty}\langle y_\nu^\circ y_{\nu+j}^\circ\rangle_{Av}J_{\nu-n}(\tau)J_{\nu+j-n-1}(\tau)=\frac{1}{2}(\overset{\circ}{\mu}_{v,1,d}+\overset{\circ}{\mu}_{d,1,v}). \tag{4.14}$$

From (4.7), (4.9) and (4.14), we have

$$\lim_{\tau\to\infty}\langle y_n(\tau)\cdot y_{n+1}(\tau)\rangle_{Av}=\frac{1}{2}(\overset{\circ}{\mu}_{v,1,d}+\overset{\circ}{\mu}_{d,1,v}). \tag{4.15}$$

Case (b): for τ sufficiently small, *i.e.*, $0<\tau\ll 1$.

Expanding Bessel function $J_\mu(\tau-\tau^*)$ in Taylor series of τ, and neglecting terms of $O(\tau^3)$, we obtain for (4.8)

$$\begin{aligned}&\lim_{\tau^*\to\tau}\int_0^\tau J_{m-2k}(\tau-\tau^*)\cdot J_{n-2k}(\tau-\tau^*)d\tau\\&\fallingdotseq(-1)^{m+n}\Big[J_{m-2k}(\tau)\cdot J_{n-2k}(\tau)\tau\\&+\frac{1}{4}J_{m-2k}(\tau)\{J_{n-2k+1}(\tau)-J_{n-2k-1}(\tau)\}\tau^2\\&+\frac{1}{4}J_{n-2k}(\tau)\{J_{m-2k+1}(\tau)-J_{m-2k-1}(\tau)\}\tau^2\Big].\end{aligned} \tag{4.16}$$

Assuming that the initial distribution of y_ν is homogeneous, *i.e.*, $\langle y_\nu^\circ y_{\nu+j}^\circ\rangle_{Av}$ depend merely on j, and taking $\langle y_\nu^\circ y_{\nu+j}^\circ\rangle_{Av}=0$ for $|j|\geq 2$, we see that

(i) when $m=n$,

$$\begin{aligned}\langle y_n^2\rangle_{Av}&=\frac{1}{2}\{\overset{\circ}{\mu}_{v,0,v}+\overset{\circ}{\mu}_{d,0,d}\pm J_0(2\tau)(\overset{\circ}{\mu}_{v,0,v}-\overset{\circ}{\mu}_{d,0,d})\\&\pm J_1(2\tau)(\overset{\circ}{\mu}_{v,1,d}-\overset{\circ}{\mu}_{d,1,v}+\overset{\circ}{\mu}_{d,-1,v}-\overset{\circ}{\mu}_{v,-1,d})\}\\&+D\Big[\{J_{n-2k}(\tau)\}^2\tau+\frac{1}{2}J_{n-2k}(\tau)\{J_{n-2k+1}(\tau)-J_{n-2k-1}(\tau)\}\tau^2\Big].\end{aligned} \tag{4.17}$$

and

(ii) when $m=n+1$,

$$\langle y_n(\tau)y_{n+1}(\tau)\rangle_{Av}=\frac{1}{2}\{\mathring{\mu}_{v,1,d}+\mathring{\mu}_{d,1,v}\pm J_0(2\tau)(\mathring{\mu}_{v,1,d}-\mathring{\mu}_{d,1,v})$$
$$\pm J_1(2\tau)(\mathring{\mu}_{d,0,d}-\mathring{\mu}_{v,0,v})$$
$$\pm J_2(2\tau)(\mathring{\mu}_{d,-1,v}-\mathring{\mu}_{v,-1,d})\}$$
$$-D[J_{n-2k+1}(\tau)\cdot J_{n-2k}(\tau)\tau$$
$$+\frac{1}{4}J_{n-2k+1}(\tau)\{J_{n-2k+1}(\tau)-J_{n-2k-1}(\tau)\}\tau^2$$
$$+\frac{1}{4}J_{n-2k}(\tau)\{J_{n-2k+2}(\tau)-J_{n-2k}(\tau)\}\tau^2]. \tag{4.18}$$

From (4.13) we can see that the mean square of velocities of the particles and the mean square of the relative displacements of adjacent particles do diverge when τ goes to infinity. The expression (4.15) shows that the expected value of the product of the velocity of a particle and the relative displacement of its adjacent particle (relative displacement of particles at the nearest neighbour) remains constant even if τ goes to infinity. In other words, the velocity-displacement correlation of the adjacent particles still exists even after sufficiently long time.

In a chain of finite length, if sufficiently long, the similar relation can still hold. In a chain of finite length, however, the reflection of disturbances at the free end and at the fixed end causes somewhat complicated phenomena, but the localized vibration may still remain also in such a system under a certain condition.[6] (cf. §5)

§5. Localized Vibrations

We take

$$x=n\bar{r}, \tag{5.1}$$

and

$$\vartheta=\frac{d}{dx}, \qquad \bar{r}\vartheta=\frac{d}{dn}, \tag{5.2}$$

in equations (1.4), and obtain

$$2\frac{d}{d\tau}y_n=[e^{\bar{r}\vartheta}-e^{-\bar{r}\vartheta}]\cdot y_n$$
$$=2\sinh(\bar{r}\vartheta)\cdot y_n, \tag{5.3}$$

where the shifting operators $e^{\bar{r}\vartheta}$ and $e^{-\bar{r}\vartheta}$ are used, which are defined by

$$e^{\bar{r}\vartheta}\cdot y_n=y_{n+1}, \tag{5.4}$$

and

$$e^{-\bar{r}\vartheta}\cdot y_n=y_{n-1}. \tag{5.5}$$

Then the equations (1.4) are written as follows:

$$\frac{d^2}{d\tau^2} y_n = \sinh^2(\bar{r}\vartheta) \cdot y_n$$
$$= [\cosh^2(\bar{r}\vartheta) - 1] \cdot y_n$$
$$= \frac{1}{2}[\cosh(2\bar{r}\vartheta) - 1] \cdot y_n. \qquad (5.6)$$

Considering the limitting case: $\bar{r}\vartheta \to 0$, which corresponds to the case of short spring and of diminishing mass of each mass-point, we obtain the corresponding classical wave equation:

$$\frac{\partial^2 y}{\partial \tau^2} = \bar{r}^2 \frac{\partial^2 y}{\partial x^2}\left(+ \frac{1}{3}\bar{r}^4 \frac{\partial^4 y}{\partial x^4} + \cdots \right). \qquad (5.7)$$

The solntions of (5.6) and those of (5.7) give almost similar waves for long wave-length. For waves of short wave-length, however, it is well known that the expressions of solutions of (5.6) and (5.7) are quite different. Moreover, the solution of (5.6) shows dispersion, while the wave equation (5.7) for continuum does not show any dispersion phenomena.

In the system of a chain of finite length, the number of eigen-frequencies of (5.6) is equal to the number of mass-points of the chain. That is, there do not exist in the chain of finite length the waves of higher frequency than the maximum frequency ω_{max} of the chain. Accordingly the external applied force whose frequency is higher than ω_{max}, can not propagate into the system of the chain, and the vibration due to the external force does localize[6] in the neighbourhood of the mass-point on which the external force is applied.

The same effect is observed by changing the mass of a mass-point in the chain, instead of applying the external force. That is, if the mass of a mass-point in the one-dimensional chain is reduced, the localized vibration exists in the vicinity of this mass-point. Because the frequency of a mass-point is represented by the square root of the ratio of spring constant to its mass, and the reduced mass results in the comparatively high frequency locally in the chain. Thus the localized vibration takes place.

By the same reason, we can see easily that the existence of the localized vibration do occur in the vicinity of a mass-point, when the mass-point has harder springs (*i.e.* greater spring constants) than the other mass-points in the one-dimensional elastic chain.

§ 6. Remarks

1) In a continuous medium, there occurs no localized vibration as sharply contrasted to the vibration in a discrete elastic chain of different masses under a certain condition. In a continuous system, however, such as an aerofoil mounted by propelling devices and fuel tanks, we can take a dynamical model of discrete elastic chain with particles of different masses and elastic springs of different elastic constants. It is, therefore, very important to take a model of discrete elastic chain discussed above and the localization of the vibration in such a system should be more emphasized, not in the statical sense (such as in the discussion of spacial change of rigidity of aerofoil) but in the dynamical sense such as in

the case of some kind of flutter.[5] Futhermore, the reference should be made to consider the statistical response of such a system especially under the existence of external random inputs.

2) The dynamical behavior of an elastic chain serves as a model of irreversible phenomena (such as heat propagation) in thermodynamics and statistical mechanics.[6] The statistical dynamical problems of such a system under the influence of external random force is also needed in the theory of Brownian motion.[4][6]

3) The chain with both ends free can be also served to calculate the relaxation spectrum[7] in the rheology of high-polymeric substances. The model may be also useful to calculate the light-absorption spectrum of some organic dyes.[8]

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Appendix

$$\int_0^{\infty} J_{m-2k}(x)\cdot J_{n-2k}(x)dx=\frac{2}{\pi}\int_0^{\infty}\int_0^{\frac{\pi}{2}} J_{m+n-4k}(2x\cos\phi)\cdot\cos(m-n)\phi d\phi dx$$

$$=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\cos(m-n)\phi\cdot\left\{\int_0^{\infty}J_{m+n-4k}(2x\cos\phi)\,dx\right\}d\phi$$

$$=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\cos(m-n)\phi\cdot\left\{\int_0^{\infty}\frac{J_{m+n-4k}(\alpha)}{2\cos\phi}d\alpha\right\}d\phi$$

$$=\begin{cases}\dfrac{1}{\pi}\displaystyle\int_0^{\frac{\pi}{2}}\frac{\cos(m-n)\phi}{\cos\phi}d\phi, & \text{for}\quad m+n-4k>-1\\[2ex](-1)^{m+n}\dfrac{1}{\pi}\displaystyle\int_0^{\frac{\pi}{2}}\frac{\cos(m-n)\phi}{\cos\phi}d\phi. & \text{for}\quad m+n-4k\leq -1\end{cases}$$

When $m-n=2s$,

$$\int_0^{\frac{\pi}{2}}\frac{\cos 2s\phi}{\cos\phi}d\phi=2\int_0^{\frac{\pi}{2}}\cos(2s-1)\phi\,d\phi-\int_0^{\frac{\pi}{2}}\frac{\cos(2s-2)\phi}{\cos\phi}d\phi,$$

$$-\int_0^{\frac{\pi}{2}}\frac{\cos(2s-2)\phi}{\cos\phi}d\phi=-2\int_0^{\frac{\pi}{2}}\cos(2s-3)\phi\,d\phi+\int_0^{\frac{\pi}{2}}\frac{\cos(2s-4)\phi}{\cos\phi}d\phi,$$

$$\vdots\qquad\qquad \cdot\quad\cdot\quad\cdot\quad\cdot$$

$$+)\qquad(-1)^{s-1}\int_0^{\frac{\pi}{2}}\frac{\cos 2\phi}{\cos\phi}d\phi=(-1)^{s-1}\int_0^{\frac{\pi}{2}}\cos\phi d\phi+(-1)^s\int_0^{\frac{\pi}{2}}\frac{1}{\cos\phi}d\phi,$$

$$\int_0^{\frac{\pi}{2}}\frac{\cos 2s\phi}{\cos\phi}d\phi=2\Big[(-1)^0\int_0^{\frac{\pi}{2}}\cos(2s-1)\phi\,d\phi+(-1)^1\int_0^{\frac{\pi}{2}}\cos(2s-3)\phi d\phi$$

$$+\cdots+(-1)^{s-1}\int_0^{\frac{\pi}{2}}\cos\phi d\phi\Big]+(-1)^s\int_0^{\frac{\pi}{2}}\frac{d\phi}{\cos\phi}$$

$$=\begin{cases}2(-1)^{s-1}\displaystyle\sum_{n=0}^{s-1}\frac{1}{2s-(2n+1)}+(-1)^s\Big[\log\tan\Big(\frac{\pi}{4}+\frac{\phi}{2}\Big)\Big]_{\phi=0}^{\frac{\pi}{2}}, & \text{for}\quad s\geq 1\\[2ex]\Big[\log\tan\Big(\frac{\pi}{4}+\frac{\phi}{2}\Big)\Big]_{\phi=0}^{\frac{\pi}{2}}. & \text{for}\quad s=0\end{cases}$$

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