RESEARCH REPORTS

ON THE EQUATION OF LONGITUDINAL VIBRATION OF A CIRCULAR CYLINDER WITH MODERATE THICKNESS UNDER THERMAL STRESS

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§ 1. Preliminaries and Notations

The equation of longitudinal vibration of a thin rod under thermal stress, the cross-sectional form of which is freely chosen, is expressed by

$$\rho\frac{\partial^2}{\partial t^2}g=E\frac{\partial^2}{\partial z^2}g-E\alpha\frac{\partial T}{\partial z},$$

where g is longitudinal displacement and T deviation of temperature. Other notations used here are given at the end of this paragraph. Taking into account the lateral motion of the material points of the rod under no influence of thermal stress, the equation is modified into:[1)]

$$\rho\frac{\partial^2}{\partial t^2}g-E\frac{\partial^2}{\partial z^2}g-K^2\sigma^2\rho\frac{\partial^4}{\partial t^2\partial z^2}g+K^4\sigma^2\frac{(5,\ 20,\ 124,\ 104,\ 28)}{3\cdot 2^2(1,1)^2(1,2)(3,2)}\rho\frac{\partial^6}{\partial t^2\partial^4 z}g=0,$$

where K is the radius of gyration of a cross-section about the central line.

In the present paper, starting from the equations of motion of elastic body, the authors deduced the equation of longitudinal vibration of a circular cylinder of radius a, in which case K^2 corresponds to $a^2/2$, taking into account the higher order deformations of the cross-sectional plane under thermal stress. The radius of the cylinder being small, approximate calculation can be carried out successively. The result is given to the terms of order K^4, and agrees with the above equation in case of no thermal stress.

Notations

r, ϑ, z : cylindrical coordinates,
u_r, u_ϑ, u_z : components of displacement,
ρ : density,
t : time,
λ, μ : Lamé's constants,
T : deviation of temperature,
$\alpha/(3\lambda+2\mu)$: coefficient of linear thermal expansion,
κ : coefficient of thermal conductivity.

For the sake of simplicity, we shall write

$$(m,\ n) = m\lambda + n\mu,$$
$$(l,\ m,\ n) = l\lambda^2 + m\lambda\mu + n\mu^2, \text{ etc.}$$
$$E = \mu(3,2)/(1,1): \text{ Young's modulus,}$$
$$\sigma = \lambda/\{2(1,1)\}: \text{ Poisson's ratio.}$$

We shall take the following abbreviations:

$$[\alpha,\ \beta] = \alpha\rho\frac{\partial^2}{\partial t^2} + \beta\frac{\partial^2}{\partial z^2},$$
$$[\alpha,\ \beta,\ \gamma] = \alpha\rho^2\frac{\partial^4}{\partial t^4} + \beta\rho\frac{\partial^2}{\partial t^2}\frac{\partial^2}{\partial z^2} + \gamma\frac{\partial^4}{\partial z^4}, \quad \text{etc.}$$

§ 2. Fundamental Equations and Longitudinal Vibration of a Circular Cylinder under Thermal Stress

The equations of motion under thermal stress, when no body force exists, are written as:

$$\left.\begin{aligned}
\rho\frac{\partial^2 u_r}{\partial t^2} &= (1,\ 2)\frac{\partial\Theta}{\partial r} - \frac{2\mu}{r}\frac{\partial\tilde{\omega}_z}{\partial\vartheta} + 2\mu\frac{\partial\tilde{\omega}_\vartheta}{\partial z} - (3,\ 2)\alpha\frac{\partial T}{\partial r},\\
\rho\frac{\partial^2 u_\vartheta}{\partial t^2} &= (1,\ 2)\frac{1}{r}\frac{\partial\Theta}{\partial\vartheta} - 2\mu\frac{\partial\tilde{\omega}_r}{\partial z} + 2\mu\frac{\partial\tilde{\omega}_z}{\partial r} - (3,\ 2)\frac{\alpha}{r}\frac{\partial T}{\partial\vartheta},\\
\rho\frac{\partial^2 u_z}{\partial t^2} &= (1,\ 2)\frac{\partial\Theta}{\partial z} - \frac{2\mu}{r}\frac{\partial}{\partial r}(r\tilde{\omega}_\vartheta) + \frac{2\mu}{r}\frac{\partial\tilde{\omega}_r}{\partial\vartheta} - (3,\ 2)\alpha\frac{\partial T}{\partial z},
\end{aligned}\right\} \tag{1}$$

where

$$\left.\begin{aligned}
2\tilde{\omega}_r &= \frac{1}{r}\frac{\partial u_z}{\partial\vartheta} - \frac{\partial u_\vartheta}{\partial z},\\
2\tilde{\omega}_\vartheta &= \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r},\\
2\tilde{\omega}_z &= \frac{1}{r}\left(\frac{\partial(ru_\vartheta)}{\partial r} - \frac{\partial u_r}{\partial\vartheta}\right),
\end{aligned}\right\} \tag{2}$$

and

$$\Theta = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\vartheta}{\partial\vartheta} + \frac{\partial u_z}{\partial z}. \tag{3}$$

The equation of conservation of energy is expressed by:[2)]

$$\rho\frac{\partial U}{\partial t} = A_{ij}\sigma_{ij} + \kappa\Delta T, \tag{4}$$

where U is the internal energy per unit mass, A_{ij} components of stress, and σ_{ij} components of strain in rectangular coordinates.

The expression of internal energy can be written as:[3)4)5)]

$$U = CT + b_{ij}\sigma_{ij}, \qquad (i,\ j = 1,\ 2,\ 3) \tag{5}$$

where C represents specific heat at constant volume. b_{ij} measure the heat effect of deformation of the material at isothermal state, and are seen to be positive in an elastic solid body. If an elastic body undergoes infinitesimal deformation adiabatically, U represents the work done by the surface force and is an infinitesimal small quantity of second order. The change in temperature T is determined by $U_{\text{adiabatic}} = 0$. For example, when we take volume dilatation Θ, and take $b = b_{11} = b_{22} = b_{33}$, we have

$$T = -\frac{b}{C}\Theta.$$

Taking $\Theta > 0$, we find by experiments that $T < 0$. This results in the positive value of b.

If we deal with the problem of the so-called "thermal shock", two terms should be taken into consideration. The first is the term containing b, which makes the propagation of the change in temperature finite. The second is the temperature rate with time $\partial T/\partial t$. Then the equation (4), neglecting terms of second order, becomes

$$\rho\frac{\partial}{\partial t}(CT + b_{ij}\sigma_{ij}) = \kappa\Delta T. \tag{6}$$

In the present treatise, we put $b = 0$ in (6), *i.e.* in cylindrical coordinates the energy equation takes the form:

$$\frac{\partial T}{\partial t} = \nu^2\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \vartheta^2} + \frac{\partial^2 T}{\partial z^2}\right\}, \tag{7}$$

with $\nu^2 = \kappa/(\rho C)$.
The equations (1) and (7) are the fundamental equations.

Let the cylinder occupy the region $r \leq a$, $-\infty < z < +\infty$, and let us take no surface traction acting on the peripheral surface. Considering the symmetric properties of the displacement of the longitudinal vibration and the distribution of temperature, which are finite at the central line $r = 0$, we put

$$\left.\begin{aligned} u_r &= \sum_{n=0}^{\infty} f_n(z, t)\cdot r^{2n+1}, \\ u_z &= \sum_{n=0}^{\infty} g_n(z,\ t)\cdot r^{2n}, \\ T &= \sum_{n=0}^{\infty} T_n(z,\ t)\cdot r^{2n}, \\ u_\vartheta &= 0, \qquad \frac{\partial}{\partial \vartheta} = 0. \end{aligned}\right\} \tag{8}$$

Then, from (2) and (3), we obtain

$$\tilde{\omega}_r = \tilde{\omega}_z = 0,$$

$$\tilde{\omega}_\theta = \sum_{n=0}^{\infty}\left[\frac{1}{2}\frac{\partial f_n}{\partial z} - (n+1)g_{n+1}\right]\cdot r^{2n+1},$$

$$\Theta = \sum_{n=0}^{\infty}\left[2(n+1)f_n + \frac{\partial g_n}{\partial z}\right]\cdot r^{2n}.$$

The equations of motion and of conservation of energy become:

$$\left.\begin{aligned}
&\sum_{n=0}^{\infty}\rho\frac{\partial^2 f_n}{\partial t^2}r^{2n+1} = \sum_{n=0}^{\infty}\Big[2(n+1)(1,2)\Big\{2(n+2)f_{n+1} + \frac{\partial g_{n+1}}{\partial z}\Big\} \\
&\qquad + \mu\Big\{\frac{\partial^2 f_n}{\partial z^2} - 2(n+1)\frac{\partial g_{n+1}}{\partial z}\Big\} - 2(3,2)\alpha(n+1)T_{n+1}\Big]r^{2n+1}, \\
&\sum_{n=0}^{\infty}\rho\frac{\partial^2 g_n}{\partial t^2}r^{2n} = \sum_{n=0}^{\infty}\Big[(1,2)\Big\{2(n+1)\frac{\partial f_n}{\partial z} + \frac{\partial^2 g_n}{\partial z^2}\Big\} \\
&\qquad - 2(n+1)\mu\Big\{\frac{\partial f_n}{\partial z} - 2(n+1)g_{n+1}\Big\} - (3,2)\alpha\frac{\partial T_n}{\partial z}\Big]r^{2n}, \\
&\text{and} \\
&\sum_{n=0}^{\infty}\frac{\partial T_n}{\partial t}r^{2n} = \sum_{n=0}^{\infty}\Big[4\nu^2 T_{n+1}(n+1)^2 + \nu^2\frac{\partial^2 T_n}{\partial z^2}\Big]r^{2n}.
\end{aligned}\right\} \quad (9)$$

Taking the coefficients of the same power of r in (9), we obtain a system of equations as follows:

$$\left.\begin{aligned}
&\rho\frac{\partial^2 f_n}{\partial t^2} = \mu\frac{\partial^2 f_n}{\partial z^2} + 4(1,2)(n+1)(n+2)f_{n+1} + 2(1,1)(n+1)\frac{\partial g_{n+1}}{\partial z} - 2(3,2)(n+1)\alpha T_{n+1}, \\
&\rho\frac{\partial^2 g_n}{\partial t^2} = (1,2)\frac{\partial^2 g_n}{\partial z^2} + 4\mu(n+1)^2 g_{n+1} + 2(1,1)(n+1)\frac{\partial f_n}{\partial z} - (3,2)\alpha\frac{\partial T_n}{\partial z}, \\
&\text{and} \\
&\frac{\partial T_n}{\partial t} = 4\nu^2(n+1)^2 T_{n+1} + \nu^2\frac{\partial^2 T_n}{\partial z^2}. \qquad (n = 0, 1, 2, 3, \ldots)
\end{aligned}\right\} \quad (10)$$

The boundary conditions, that the cylinder is free from surface traction, are: at $r = a$;

$$\lambda\Theta + 2\mu\frac{\partial u_r}{\partial r} - (3,2)\alpha T = 0,$$

$$\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} = 0,$$

i.e.

$$\left.\begin{aligned}
&0 = \sum_{n=0}^{\infty}\Big[2(n+1,\ 2n+1)f_n + \lambda\frac{\partial g_n}{\partial z} - (3,2)\alpha T_n\Big]a^{2n}, \\
&0 = \sum_{n=0}^{\infty}\Big[\frac{\partial f_n}{\partial z} + 2(n+1)g_{n+1}\Big]a^{2n}.
\end{aligned}\right\} \quad (11)$$

§ 3. Approximate Procedures for Moderately Thick Bar

From (10) we obtain:

$$\left.\begin{aligned}
f_{n+1} &= \frac{\left[1,\ \dfrac{\lambda(1,2)}{\mu}\right]}{2^2(1,2)(n+1)(n+2)} f_n - \frac{(1,1)[1,\ -(1,2)]}{2^3\mu(1,2)(n+1)^2(n+2)} \frac{\partial g_n}{\partial z} \\
&\quad - \frac{(3,2)\alpha}{2^3\mu(n+1)^2(n+2)} \frac{\partial^2 T_n}{\partial z^2} + \frac{(3,2)\alpha}{2^3\nu^2(1,2)(n+1)^2(n+2)} \frac{\partial T_n}{\partial t}, \\
g_{n+1} &= \frac{[1,\ -(1,2)]}{2^2\mu(n+1)^2} g_n - \frac{(1,1)}{2\mu(n+1)} \frac{\partial f_n}{\partial z} + \frac{(3,2)\alpha}{2^2\mu(n+1)^2} \frac{\partial T_n}{\partial z}, \\
T_{n+1} &= \frac{1}{2^2\nu^2(n+1)^2} \frac{\partial T_n}{\partial t} - \frac{1}{2^2(n+1)^2} \frac{\partial^2 T_n}{\partial z^2}.
\end{aligned}\right\} \quad (12)$$

We introduce a new variable

$$\varphi = f_0 + \frac{\lambda}{2(1,1)} \frac{\partial g_0}{\partial z} - \frac{(3,2)\alpha}{2(1,1)} T_0,$$

and take operators τ_n, G_n, $\overline{G}_n$, $\overline{\overline{G}}_n$, F_n, $\overline{F}_n$, and $\overline{\overline{F}}_n$, which are defined successively as follows:

$$\left.\begin{aligned}
T_n &= \tau_n T_0, \\
g_n &= G_n g_0 + \overline{G}_n \varphi + \overline{\overline{G}}_n T_0, \\
f_n &= F_n g_0 + \overline{F}_n \varphi + \overline{\overline{F}}_n T_0.
\end{aligned}\right\} \quad (13)$$

Inserting (13) into (11), we obtain

$$\left.\begin{aligned}
0 &= \varphi + \sum_{n=1}^{\infty} [C_n g_0 + \overline{C}_n \varphi + \overline{\overline{C}}_n T_0] a^{2n}, \\
0 &= [1,\ -E] g_0 + H\varphi + LT_0 + \sum_{n=1}^{\infty} [D_n g_0 + \overline{D}_n \varphi + \overline{\overline{D}}_n T_0] a^{2n},
\end{aligned}\right\} \quad (14)$$

with

$$\left.\begin{aligned}
C_n &= \frac{(n+1,\ 2n+1)}{(1,1)} F_n + \frac{\lambda}{2(1,1)} \frac{\partial}{\partial z} G_n, \\
\overline{C}_n &= \frac{(n+1,\ 2n+1)}{(1,1)} \overline{F}_n + \frac{\lambda}{2(1,1)} \frac{\partial}{\partial z} \overline{G}_n, \\
\overline{\overline{C}}_n &= \frac{(n+1,\ 2n+1)}{(1,1)} \overline{\overline{F}}_n + \frac{\lambda}{2(1,1)} \frac{\partial}{\partial z} \overline{\overline{G}}_n - \frac{(3,2)\alpha}{2(1,1)} \tau_n, \\
D_n &= -2\lambda \frac{\partial}{\partial z} F_n + \frac{[1,\ -(1,2)]}{n+1} G_n, \\
\overline{D}_n &= -2\lambda \frac{\partial}{\partial z} \overline{F}_n + \frac{[1,\ -(1,2)]}{n+1} \overline{G}_n,
\end{aligned}\right\} \quad (15)$$

and

$$\overline{\overline{D}}_n = -2\lambda \frac{\partial}{\partial z} \overline{\overline{F}}_n + \frac{[1,\ -(1,2)]}{n+1} \overline{\overline{G}}_n + \frac{(3,2)\alpha}{n+1} \frac{\partial}{\partial z} \tau_n,$$

where

$$H = -2\lambda\frac{\partial}{\partial z} \quad \text{and} \quad L = E\alpha\frac{\partial}{\partial z}.$$

Some of the explicit expressions of the operators in (13) and (14), are as follows:

$$G_1 = \frac{[1,\ -2\mu]}{4\mu}, \quad \overline{G}_1 = -\frac{(1,1)}{2\mu}\frac{\partial}{\partial z}, \quad \overline{\overline{G}}_1 = 0,$$

$$G_2 = \frac{1}{2^6\mu^2}\left[1,\ -\mu\frac{(3,7)}{(1,2)},\ 3\mu^2\right], \quad \overline{G}_2 = -\frac{(1,1)}{2^5\mu^2}\left[\frac{(1,3)}{(1,2)},\ -2\mu\right]\frac{\partial}{\partial z},$$

$$\overline{\overline{G}}_2 = -\frac{(3,2)\alpha\rho}{2^6\mu(1,2)}\frac{\partial^3}{\partial t^2\partial z} + \frac{(3,2)\alpha}{2^6\nu^2(1,2)}\frac{\partial^2}{\partial z\partial t},$$

$$F_1 = -\frac{[(1,3,1),\ -(1,2)\mu(2,1)]}{2^4\mu(1,1)(1,2)}\frac{\partial}{\partial z}, \quad \overline{F}_1 = \frac{\left[1,\ \frac{\lambda(1,2)}{\mu}\right]}{2^3(1,2)},$$

$$\overline{\overline{F}}_1 = \left[\frac{(3,2)\alpha}{2^4(1,1)(1,2)},\ -\frac{(3,2)\alpha}{2^4(1,1)}\right] + \frac{(3,2)\alpha}{2^4\nu^2(1,2)}\frac{\partial}{\partial t}.$$

$$F_2 = -\frac{1}{3\cdot 2^7(1,1)(1,2)\mu^2}\left[\frac{(1,5,8,3)}{(1,2)},\ -\mu(3,10,5),\ \mu^2(1,2)(3,2)\right]\frac{\partial}{\partial z}.$$

$$\overline{F}_2 = \frac{1}{3\cdot 2^6(1,2)}\left[\frac{1}{(1,2)},\ \frac{(1,4,1)}{\mu^2},\ -\frac{(1,2)(2,1)}{\mu}\right].$$

$$\overline{\overline{F}}_2 = \left[\frac{(3,2)\alpha}{3\cdot 2^7(1,1)(1,2)^2},\ \frac{(3,2)\alpha(1,-1)}{3\cdot 2^7(1,1)(1,2)\mu},\ \frac{(3,2)\alpha}{3\cdot 2^7(1,1)}\right]$$
$$+\left[\frac{(3,2)\alpha}{3\cdot 2^7(1,2)^2\nu^2},\ -\frac{(3,2)\alpha}{2^7\nu^2(1,2)}\right]\frac{\partial}{\partial t} + \frac{(3,2)\alpha}{3\cdot 2^7\nu^4(1,2)}\frac{\partial^2}{\partial t^2}.$$

$$\tau_1 = \frac{1}{2^2\nu^2}\frac{\partial}{\partial t} - \frac{1}{2^2}\frac{\partial^2}{\partial z^2}, \quad \tau_2 = \frac{1}{2^6\nu^4}\frac{\partial^2}{\partial t^2} - \frac{1}{2^5\nu^2}\frac{\partial^3}{\partial t\partial z^2} + \frac{1}{2^6}\frac{\partial^4}{\partial z^4},$$

$$C_1 = -\frac{1}{2^4(1,1)^2}\left[\frac{(3,7,3)}{(1,2)},\ -\mu(4,3)\right]\frac{\partial}{\partial z}, \quad \overline{C}_1 = \frac{1}{2^3(1,1)}\left[\frac{(2,3)}{(1,2)},\ \lambda\right],$$

$$\overline{\overline{C}}_1 = \frac{(3,2)\alpha}{2^4(1,1)^2(1,2)}[(2,3),\ -\mu(1,2)] - \frac{(3,2)\mu\alpha}{2^4\nu^2(1,1)(1,2)}\frac{\partial}{\partial t},$$

$$C_2 = \frac{-1}{3\cdot 2^7(1,1)^2(1,2)\mu}\left[\frac{(5,25,37,15)}{(1,2)},\ -\mu(15,44,25),\ 2\mu^2(1,2)(6.5)\right]\frac{\partial}{\partial z},$$

$$\overline{C}_2 = \frac{1}{3\cdot 2^6(1,1)(1,2)}\left[\frac{(3,5)}{(1,2)},\ \frac{(5,14,5)}{\mu},\ -(1,2)(7,5)\right].$$

$$\overline{\overline{C}}_2 = \left[\frac{(3,2)(3,5)\alpha}{3\cdot 2^7(1,1)^2(1,2)^2},\ -\frac{(3,2)(1,5)\alpha}{3\cdot 2^7(1,1)^2(1,2)},\ \frac{(3,2)\mu\alpha}{3\cdot 2^6(1,1)^2}\right]$$
$$+\left[\frac{(3,2)(3,5)\alpha}{3\cdot 2^7(1,1)(1,2)^2\nu^2},\ -\frac{(3,2)\mu\alpha}{2^7\nu^2(1,1)(1,2)}\right]\frac{\partial}{\partial t} - \frac{(3,2)\mu\alpha}{3\cdot 2^7\nu^4(1,2)(1,1)}\frac{\partial^2}{\partial t^2}.$$

$$D_1 = \frac{1}{2^3\mu(1,1)}\left[(1,1),\ -\mu\frac{(4,13,8)}{(1,2)},\ \mu^2(5,4)\right].$$

$$\overline{D}_1 = -\frac{1}{4\mu(1,2)}[(1,4,2),\ -\mu(3,2)(1,2)]\frac{\partial}{\partial z},$$

$$\overline{\overline{D}}_1 = -\frac{\lambda(3,2)\alpha}{2^3(1,1)(1,2)}\left[1, \frac{\mu(1,2)}{\lambda}\right]\frac{\partial}{\partial z} + \frac{(3,2)\mu\alpha}{2^2\nu^2(1,2)}\frac{\partial^2}{\partial z\partial t},$$

$$D_2 = \frac{1}{3\cdot 2^6(1,1)(1,2)\mu^2}\left[(1,1)(1,2),\ -\mu\frac{(5,26,44,22)}{(1,2)},\ \mu^2(9,31,20),\ -\mu^3(1,2)(7,6)\right],$$

$$\overline{D}_2 = -\frac{1}{3\cdot 2^5\mu^2(1,2)}\left[\frac{(1,6,12,6)}{(1,2)},\ -2\mu(2,8,5),\ \mu^2(1,2)(5,4)\right]\frac{\partial}{\partial z},$$

$$\overline{\overline{D}}_2 = \frac{(3,2)\alpha}{3\cdot 2^6\mu(1,1)(1,2)^2}[-(1,4,2),\ 2\mu(1,2)(2,1),\ \mu^2(1,2)^2]\frac{\partial}{\partial z}$$
$$+\frac{(3,2)\mu\alpha}{3\cdot 2^5\nu^2(1,2)^2}[1,\ -3(1,2)]\frac{\partial^2}{\partial t\partial z} + \frac{(3,2)\mu\alpha}{3\cdot 2^5\nu^4(1,2)}\frac{\partial^3}{\partial t^2\partial z}.$$

§ 4. Approximate Calculations

In the following, $\varphi_r (r=0,1,2)$ means the value of φ in the r-th approxination.

I. Zero-th order approximation

The radius of the bar being small compared with the wave-length of the longitudinal wave in the bar, we find, neglecting the terms of $0(a^2)$ in (14):

$$\left.\begin{aligned} &\varphi_0 = 0, \\ \text{and}\quad &0 = [1,\ -E]g_0 + E\alpha\frac{\partial T_0}{\partial z}, \end{aligned}\right\} \tag{16}$$

which is nothing but the usual equation for a thin rod or bar.

II. First order approximation

Inserting (16) into the terms of $0(a^2)$ of (14), and neglecting the terms of $0(a^4)$, we obtain

$$\left.\begin{aligned} \varphi_1 &= -(C_1 g_0 + \overline{\overline{C}}_1 T_0)a^2, \\ 0 &= [1,\ -E]g_0 + LT_0 + (D_1 - HC_1)g_0 a^2 + (\overline{\overline{D}}_1 - H\overline{\overline{C}}_1)T_0\cdot a^2. \end{aligned}\right\} \tag{17}$$

Replacing $\rho\frac{\partial^2 g_0}{\partial t^2}$ by the expression obtained from (16) into the terms of $0(a^2)$ of (17), we find

$$0 = [1,\ -E]g_0 + E\alpha\frac{\partial T_0}{\partial z} - \frac{a^2}{2}\sigma^2 E\frac{\partial^4 g_0}{\partial z^4} + \frac{E\alpha(2,2,1)}{2^3(1,1)^2}\frac{\partial^3 T_0}{\partial z^3}a^2$$
$$+\frac{E\alpha}{2^3\nu^2}\frac{\partial^2 T_0}{\partial t\partial z}a^2 - \frac{E\alpha}{2^3(1,1)}\rho\frac{\partial^3 T_0}{\partial t^2\partial z}\cdot a^2. \tag{18}$$

In case of vanishing T_0, the equation (18) coincides with the result taking into account the lateral deformation in the cross-sectional plane of the bar.[6]

III. Second order approximation

Inserting (16) and (17) respectively into the terms of $0(a^4)$ and $0(a^2)$ of (14), we have

$$\left.\begin{aligned}\varphi_2 &= -(C_1g_0+\overline{\overline{C}}_1T_0)a^2-\{(C_2-\overline{C}_1C_1)g_0+(\overline{\overline{C}}_2-\overline{C}_1\overline{\overline{C}}_1)T_0\}\cdot a^4,\\ 0 &= [1,\ -E]g_0+LT_0+(D_1-HC_1)g_0\cdot a^2+(\overline{\overline{D}}_1-H\overline{\overline{C}}_1)T_0a^2\\ &\quad -\{C_1\overline{D}_1+H(C_2-\overline{C}_1C_1)-D_2\}g_0\cdot a^4-\{\overline{D}_1\overline{\overline{C}}_1+H(\overline{\overline{C}}_2-\overline{C}_1\overline{\overline{C}})-\overline{\overline{D}}_2\}T_0a^4.\end{aligned}\right\}\quad(19)$$

Putting $\rho\dfrac{\partial^2 g_0}{\partial t^2}$ and $\rho^2\dfrac{\partial^4 g_0}{\partial t^4}$ obtained from (18) into the terms of $0(a^2)$ of (19) and putting $\rho^3\dfrac{\partial^6 g_0}{\partial z^6}$, $\rho^2\dfrac{\partial^4 g_0}{\partial z^4}$ and $\rho\dfrac{\partial^2 g_0}{\partial z^2}$ obtained from (16), into the terms of $0(a^4)$, we find, after some complicated calculations:

$$\begin{aligned}0 &= [1,\ -E]g_0+E\alpha\frac{\partial T_0}{\partial z}\\ &\quad +K^2\left\{-E\sigma^2\frac{\partial^4 g_0}{\partial z^4}+\frac{E\alpha(2,2,1)}{2^2(1,1)^2}\frac{\partial^3 T_0}{\partial z^3}-\frac{E\alpha}{2^2(1,1)}\rho\frac{\partial^3 T_0}{\partial t^2\partial z}+\frac{E\alpha}{2^2\nu^2}\frac{\partial^2 T_0}{\partial t\partial z}\right\}\\ &\quad +K^4\left\{\frac{\mu\sigma^2(-2,13,56,52,14)}{3\cdot 2(1,1)^3(1,2)}\frac{\partial^6 g_0}{\partial z^6}+\frac{E\alpha(6,12,5,14,11,2)}{3\cdot 2^4(1,1)^4(1,2)}\frac{\partial^5 T_0}{\partial z^5}\right.\\ &\quad +\frac{E\alpha}{3\cdot 2^4\nu^4}\frac{\partial^3 T_0}{\partial t^2\partial z}+\frac{E\alpha(5,14,12,4)}{2^4\nu^2(1,1)^2(1,2)}\frac{\partial^4 T_0}{\partial t\partial z^3}-\frac{E\alpha(3,8,8)}{3\cdot 2^4\nu^2\mu(1,1)(1,2)}\rho\frac{\partial^4 T_0}{\partial t^3\partial z}\\ &\quad \left.-\frac{\alpha(3,2)(14,69,119,82,24)}{3\cdot 2^4(1,1)^4(1,2)}\rho\frac{\partial^5 T_0}{\partial t^2\partial z^3}+\frac{\alpha(3,2)(4,13,18,12)}{3\cdot 2^4\mu(1,1)^3(1,2)}\rho^2\frac{\partial^5 T_0}{\partial t^4\partial z}\right\},\end{aligned}\quad(20)$$

where $K=a/\sqrt{2}$ is the radius of gyration of the cross-section of the bar around the central line.

In case of no thermal stress, the equation (20) agrees with the result for a thick bar[1] taking into account the terms of K^4.

Referencies

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