NOTE ON THE PARAMETER M IN LINEAR PROGRAMMING

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I. Introduction

A problem of linear programming is concerned with the problem of finding a set of non-negative values x which satisfy the inequalities,

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots\cdots\cdots + a_{1n}x_n &\leqq b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots\cdots + a_{2n}x_n &\leqq b_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots\cdots + a_{mn}x_n &\leqq b_m, \end{aligned}\right\} \tag{1}$$

and maximize (or minimize) a linear functional,

$$y = c_1x_1 + c_2x_2 + \cdots\cdots\cdots + c_nx_n. \tag{2}$$

It is well known that in the general simplex method, auxiliary variables, called slack variables, are introduced to obtain the optimum solution.

Especially, when constriants consist of only equalities such that,

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots\cdots\cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots\cdots + a_{2n}x_n &= b_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots\cdots + a_{mn}x_n &= b_m, \end{aligned}\right\} \tag{3}$$

the procedure generally adopted, with not so much mathematical rigour, to proceed the simplex calculation is following. The linear objective function (2) is changed by introducing artificial variables to constriants (1) and giving them unit profits of large negative value $-M$, giving the expression

$$y = c_1x_1 + c_2x_2 + \cdots\cdots\cdots + c_nx_n - M(x_{n+1} + \cdots + x_{n+m}). \tag{4}$$

The method of succeeding calculation is quite the same as before. This paper aims to establish a mathematical basis for this calculation procedure.

II. Brief Explanation of Linear Programming and Simplex Method

Now, let us explain the conventional linear programming problem. The general description of the problem is: find a set of non-negative variables x which satisfy a set of inequalities,

$$\left.\begin{array}{l} a_{11}x_1+a_{12}x_2+\cdots\cdots\cdots+a_{1n}x_n\leqq b_1, \\ a_{21}x_1+a_{22}x_2+\cdots\cdots\cdots+a_{2n}x_n\leqq b_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1+a_{m2}x_2+\cdots\cdots\cdots+a_{mn}x_n\leqq b_m, \end{array}\right\} \tag{5}$$

and maximize a linear functional,

$$y=c_1x_1+c_2x_2+\cdots\cdots\cdots+c_nx_n. \tag{6}$$

If the inequality signs of (5) are replaced with the equality signs by introducing slack variables, x_{n+1}, x_{n+2}, $\cdots$, x_{n+m}, we may rewrite (5) as:

$$\left.\begin{array}{l} a_{11}x_1+a_{12}x_2+\cdots\cdots+a_{1n}x_n+x_{n+1}=b_1, \\ a_{21}x_1+a_{22}x_2+\cdots\cdots+a_{2n}x_n+x_{n+2}=b_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1+a_{m2}x_2+\cdots\cdots+a_{mn}x_n+x_{n+m}=b_m. \end{array}\right\} \tag{7}$$

For the sake of abbreviation of the expressions, we put the column vectors as

$$\mathbf{P}_1=\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix},\ \mathbf{P}_2=\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix},\cdots,\ \mathbf{P}_n=\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix},\ \mathbf{P}_{n+1}=\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},\cdots,\ \mathbf{P}_{n+m}=\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

the row vector as $\mathbf{c}=(c_1, c_2, \ldots, c_n, 0, \ldots, 0)$ and the solution vector as $\mathbf{x}=(x_1, x_2, \ldots, x_m, x_{n+1}, \ldots, x_{n+m})$. So, we can rewrite our linear programming problem in the following compact form: "find $\mathbf{x}=(x_1, x_2, \ldots, x_{n+m})$, $x_i\geqq 0$, such that

$$L(\mathbf{x})=\sum_{j=1}^{n+m} x_j\mathbf{P}_j=\mathbf{P}_0 \tag{8}$$

and

$$y=f(\mathbf{x})=\mathbf{c}\cdot\mathbf{x} \tag{9}$$

is maximum."

A linear functional $y=f(\mathbf{x})$ defined on a convex polyhedron K takes its maximum value at an extreme point of K. Now, suppose that in any way we have found a basic solution, yielding an extreme point. Further we assume that, without loss of generality, the first m components of the extreme point $\mathbf{x}$ are positive.

Then, (8) and (9) reduce to the following expressions;

$$\sum_{i=1}^{m} x_i\mathbf{P}_i=\mathbf{P}_0, \qquad (x_1, x_2, \ldots, x_m>0). \tag{10}$$

and

$$\sum_{i=1}^{m} c_ix_i=f(\mathbf{x})=y_0. \tag{11}$$

As the vectors $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_m$ form a basis of the m-dimensional space R_m, every other vector is expressed in terms of this basis; or formaly,

$$\sum_{i=1}^{m} x_{ij}\mathbf{P}_i = \mathbf{P}_j \qquad (j=1, 2, \ldots, n+m). \tag{12}$$

Using (12), we may rewrite the solution (10), giving

$$\mathbf{P}_0 = \sum_{i=1}^{m} x_i\mathbf{P}_i - \theta\mathbf{P}_k + \theta\mathbf{P}_k \tag{13}$$

$$= \sum_{i=1}^{m} x_i\mathbf{P}_i - \theta\sum_{i=1}^{m} x_{ik}\mathbf{P}_i + \theta\mathbf{P}_k$$

$$= \sum_{i=1}^{m} (x_i - \theta x_{ik})\mathbf{P}_i + \theta\mathbf{P}_k. \tag{14}$$

Choosing the value of θ such that,

$$\theta = \min_i\left(\frac{x_i}{x_{ik}}\right) \tag{15}$$

for all i's where $x_{ik}>0$, the non-negative conditions,

(a) $\theta \geqq 0$,

(b) $x_i - \theta x_{ik} \geqq 0$

are obvious.

Then, these values are really the solution in the case where one of the basic vectors, $\mathbf{P}_r$, is replaced by new vector $\mathbf{P}_k$. It must be noticed, however, that for any other vector than $\mathbf{P}_r$, the relation $x_i - \theta x_{ik} = 0$ is also possible.

Referring (11) and (13), the corresponding value of y_0' is

$$y_0' = \sum_{i=1}^{m} c_i(x_i - \theta x_{ik}) + \theta c_k$$

$$= \sum_{i=1}^{m} c_i x_i + \theta\,(c_k - \sum_{i=1}^{m} c_i x_{ik}). \tag{16}$$

Let,

$$y_j = \sum_{i=1}^{m} c_i x_{ij}, \qquad (j=1, 2, \ldots, n+m), \tag{17}$$

then, (16) becomes

$$y_0' = y_0 + \theta(c_k - y_k). \tag{18}$$

If $c_k - y_k > 0$ and $y_k > 0$, then $y_0' > y_0$.

There exist three possibilities in (12);

(a) For some j, $y_j - c_j < 0$, and $x_{ij} \leqq 0$ for every i. The case where the value of linear functional y_0 is unlimited.

(b) For some j, $y_j - c_j < 0$, and $x_{ij} > 0$ for some i. The case where a larger value of y_0 is obtained.

(c) $y_j - c_j \geqq 0$, for all $j = 1, 2, \ldots, n+m$. The case where the solution of (10) is maximum.

III. Mathematical Explanation of the Parameter M

Now, let us consider a problem of linear programming in which variables satisfy a set of the following equations,

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots\cdots\cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots\cdots + a_{2n}x_n = b_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots\cdots + a_{mn}x_n = b_m, \end{array}\right\} \tag{19}$$

and maximize the linear functional,

$$y = \sum_{j=1}^{n} c_j x_j. \tag{20}$$

As a matter of course, we set here the following two assumptions:

(1) For every column vector $\mathbf{P}_i$ $(i = 1, 2, \ldots, n)$ in the coefficient matrix of (19),

$$\mathbf{P}_i \neq 0 \qquad (i = 1, 2, \ldots, n)$$

and for a real quantity a, $\mathbf{P}_i \neq a\mathbf{P}_j$, $(i \neq j)$.

(2) There is an n-dimensional vector $\mathbf{x}(x_1, x_2, \ldots, x_m) \geqq 0$, the components of which are the coefficients in $\sum_{j=1}^{n} x_j \mathbf{P}_j = \mathbf{P}_0$.

In this case, by introducing the artificial variables and vectors in the same manner with the preceding paragraph, (19) and (20) change to

$$\mathbf{P}_0 = \sum_{j=1}^{n+m} x_j \mathbf{P}_j, \tag{19$'$}$$

$$y = \sum_{j=1}^{n} c_j x_j - M \sum_{j=n+1}^{n+m} x_j \tag{20$'$}$$

respectively.

Suppose, that a basis is formed with some structural and some artificial vectors at a certain stage in the process of simplex calculations. Denoting the basic vector by $\mathbf{P}_i$, the set of suffix of structural vector by I and that of artificial vector by I', we obtain the expressions,

$$\mathbf{P}_0 = \sum_{i \in I} x_i \mathbf{P}_i + \sum_{i \in I'} x_i \mathbf{P}_i, \tag{21}$$

where

$$x_i \geqq 0,$$

and

$$\mathbf{P}_j = \sum_{i \in I} x_{ij} \mathbf{P}_i + \sum_{i \in I'} x_{ij} \mathbf{P}_i. \tag{22}$$

While, from the assumption (2), we have:

$$\mathbf{P}_0 = z_1 \mathbf{P}_1 + z_2 \mathbf{P}_2 + \cdots + z_n \mathbf{P}_n \tag{23}$$

where

$$z_i \geqq 0 \qquad (i = 1, 2, \ldots, n).$$

Substitution of (22) to $\mathbf{P}_j$ in (23) leads to

$$\mathbf{P}_0 = z_1(\sum_{i\in I} x_{i1}\mathbf{P}_i + \sum_{i\in I'} x_{i1}\mathbf{P}_i) + z_2(\sum_{i\in I} x_{i2}\mathbf{P}_i + \sum_{i\in I'} x_{i2}\mathbf{P}_i) + \cdots$$
$$\cdots + z_n(\sum_{i\in I} x_{in}\mathbf{P}_i + \sum_{i\in I'} x_{in}\mathbf{P}_i)$$
$$= \sum_{i\in I} (z_1x_{i1} + z_2x_{i2} + \cdots + z_nx_{in})\mathbf{P}_i + \sum_{i\in I'} (z_1x_{i1} + z_2x_{i2} + \cdots + z_nx_{in})\mathbf{P}_i. \qquad (24)$$

Now using the fact that $\mathbf{P}_0$ is uniquely expressible by $\mathbf{P}_i$, then by comparing (21) and (24), the following identity is obtained;

$$x_i = z_1x_{i1} + z_2x_{i2} + \cdots + z_nx_{in}. \qquad (25)$$

It can be supposed without loss of generality that the m basic vectors $\mathbf{P}_i$ ($i = 1, 2, \ldots, m$) consist of q structural vectors, $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_q$ and p artificial vectors, $\mathbf{P}_{n+1}, \ldots, \mathbf{P}_{n+p}$. Restricting the attention to the second term on the right side of (24), and further considering the fact that vectors $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_q$ are included in the basis, it is obvious that

$$x_{i1} = x_{i2} = \cdots = x_{iq} = 0 \qquad (i \in I'),$$

then

$$\left.\begin{aligned} x_{n+1} &= z_1x_{n+1,\,q+1} + z_2x_{n+1,\,q+2} + \cdots\cdots + z_nx_{n+1,\,n} \geqq 0,\\ x_{n+2} &= z_1x_{n+2,\,q+1} + z_2x_{n+2,\,q+2} + \cdots\cdots + z_nx_{n+2,\,n} \geqq 0,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ x_{n+p} &= z_1x_{n+p,\,q+1} + z_2x_{n+p,\,q+2} + \cdots\cdots + z_nx_{n+p,\,n} \geqq 0, \end{aligned}\right\} \qquad (26)$$

where

$$z_i \geqq 0 \qquad (i = 1, 2, \ldots, n).$$

Thus, we have:

$$\begin{aligned} &x_{n+1} + x_{n+2} + \cdots + x_{n+p}\\ &\quad = z_1(x_{n+1,\,q+1} + x_{n+2,\,q+1} + \cdots + x_{n+p,\,q+1})\\ &\quad + z_2(x_{n+1,\,q+2} + x_{n+2,\,q+2} + \cdots + x_{n+p,\,q+2})\\ &\quad + z_n(x_{n+1,\,n} + x_{n+2,\,n} + \cdots + x_{n+p,\,n})\\ &\qquad z_i \geqq 0 \qquad (i = 1, 2, \ldots, n). \end{aligned} \qquad (27)$$

Then, it follows that for at least one suffix, say k, of column suffixes, $q+1, q+2, \ldots, n$, non-negativeness,

$$x_{n+1,\,k} + x_{n+2,\,k} + \cdots + x_{n+p,\,k} > 0 \qquad (28)$$

holds.

Providing M be a sufficiently large possitive value, the shadow cost for vector $\mathbf{P}_k$ is

$$y_k - c_k = c_1x_{1k} + c_2x_{2k} + \cdots + c_qx_{qk} + (-M)(x_{n+1,\,k} + x_{n+2,\,k} + \cdots + x_{n+p,\,k}) - c_k. \qquad (29)$$

Then referring (28), the inequalities,

$$y_k - c_k < 0 \tag{30}$$

and, for at least one suffix, say r, which belongs to I',

$$x_{rk} > 0 \tag{31}$$

are ascertained.

IV. Conclusion

When the artificial vectors are in a basis with possitive coefficients at a certain stage of simplex table, we obtain (30) and (31) as a result of the study of mathematical property of the table. These findings are conceived as an assertion of methodological reasonability of introducing the parameter M into the linear programming problem, where the constriants are given by a set of equalities. As the results given above just belong to the case (b) of three possibilities (a), (b), (c) in II, then we can introduce the new vector $\mathbf{P}_k$ of the structural one to the basis in place of one of the basic vector, $\mathbf{P}_r$, thus obtaining a new solution and a larger y_0. Therefore, as far as some of the artificial vectors with positive coefficient appear in the basis, the simplex table is not optimum and further improvement is possible. Thus the final conclusion is: even if the linear programming problem is given under the constriants of a set of equalities, it is mathematically plausible to carry out the general simplex calculation by introducing the parameter M.

References

1) Charnes, W. W., Cooper, A. A. Henderson: *An Introduction to Linear Programming*, John Wiley and Sons, 1953.