POSITIVE COLUMN IN A LONGITUDINAL MAGNETIC FIELD

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1. Introduction

We have dealt with the positive column of a gasous discharge in an applied magnetic field. Without the applied magnetic field, the experimental results of the positive column in the low gas pressure are in good agreement with theories.[1] With the weak magnetic field, the experimental results are also well explained,[2][3] using the coefficient of the ambipolar diffusion across the line of magnetic field.[4] When the magnetic field is strong, however, the behaviors of positive column are unusual, viz. the longitudinal electric field decreases with the increasing magnetic field usually. When the magnetic field is over a critical value, however, the longitudinal electric field increases with the increasing magnetic field.[3)-5] Two plausible theories were proposed to explain the above effect. One of them is based on the instability of a sheath surrounding the positive column[6] and the other is rested on the instability of the positive column itself.[7] In this paper, we show that the contribution of circumferential electric current to the joule heat plays an important role in the effect and that the ionic part of a ambipolar heat plays also an important role.

2. Basic Equations

The hydrodynamic behaviors of a partially ionized plasma is described by the equations of ρ, ρ_p, ρ^*, $\vec{v}$, $\vec{v}_p$ and $\vec{j}$, where ρ is the mass density of plasma, ρ_p the mass density of charged particles, ρ^* the acuumulation of electric charge per unit volume, $\vec{v}$ the flow speed of the center of mass, $\vec{v}_p$ the flow speed of charged particles and $\vec{j}$ the density of electric current. They are defined by

$$\rho = \sum \rho_s, \qquad (s = e,\ i \text{ and } a), \tag{1}$$

$$\rho_p = \rho_e + \rho_i, \tag{2}$$

$$\rho^* = en_i - en_e, \tag{3}$$

$$\vec{v} = \sum \rho_s \vec{v}_s / \rho, \tag{4}$$

$$\vec{v}_p = (\rho_i \vec{v}_i + \rho_e \vec{v}_e) / \rho_p, \tag{5}$$

$$\vec{j} = en_i \vec{v}_i - en_e \vec{v}_e. \tag{6}$$

where the subscripts e, i and a mean the quantities in regard with the electrons, ions and atoms respectively, ρ_s is the mass density of the component s, e the unit of electric charge, n_s the number density of the component s, and $\vec{v}_s$ the flow speed of the component s.

The basic equations are as follows;[8)-10)]

$$\frac{\partial \rho}{\partial t} = -\operatorname{div} \cdot \rho \vec{v}, \tag{7}$$

$$\frac{\partial \rho_p}{\partial t} = -\operatorname{div} \cdot \rho_p \vec{v}_p + Q, \tag{8}$$

$$\frac{\partial \rho^*}{\partial t} = -\operatorname{div} \cdot \vec{j}, \tag{9}$$

$$\rho \frac{d\vec{v}}{dt} = \rho^* \vec{E} + [\vec{j} \times \vec{B}] - \operatorname{grad} \cdot p, \tag{10}$$

$$\rho_p \frac{d\vec{v}_p}{dt} = \rho^* \vec{E} + [\vec{j} \times \vec{B}] - \operatorname{grad} \cdot p_p - a\{\vec{v}_A + b(\vec{j} - \rho^* \vec{v}_p)\}, \tag{11}$$

$$\frac{m_e \partial}{e^2 \partial t}\left(\frac{\vec{j} - \rho^* \vec{v}_p}{n_e}\right) + \eta(\vec{j} - \rho^* \vec{v}_p) = \vec{E} + \vec{v}_p \times \vec{B} + \frac{1}{e n_e} \cdot (\operatorname{grad} \cdot p_e - [\vec{j} \times \vec{B}]) - ab\vec{v}_A, \tag{12}$$

where Q is the souce of ionization per unit time and volume, $\vec{E}$ the intensity of electric field, $\vec{B}$ the intensity of magnetic field, p the pressure, p_p the pressure of charged particles, $\vec{v}_A$ the ambipolar velocity defined by $v_A = \vec{v}_p - \vec{v}_a$, a and b are the coefficients, m_e is the mass of electron and η the electrical resistivity. Let ν_{ea} and ν_{ia} be the frequencies of the electron-atom and ion-atom interactions respectively. Then a and b are given by

$$a = n_e m_e \nu_{ea} + n_i m_i \nu_{ia}/2, \tag{13}$$

$$b = \frac{m_e}{ea}\left(\frac{\nu_{ia}}{2} - \nu_{ea}\right), \tag{14}$$

respectively, where m_i is the mass of ion.

3. The radial Distribution

For the positive column of small electric current in a steady state, the degree of ionization is much small so that $\vec{v}_A \simeq \vec{v}_p$. Under the conditions of the cylindrcal symmetry of positive column and $\rho^* = 0$, the basic equations described in the preceding section become

$$\frac{1}{r}\frac{d}{dr}(r \rho_p v_{pr}) = Q, \tag{15}$$

$$\frac{1}{r}\frac{d}{dr}(r j_r) = 0, \tag{16}$$

$$j_\theta B - \frac{dp}{dr} = a v_{pr}, \tag{17}$$

$$v_{p\theta} = -b j_\theta, \tag{18}$$

$$v_{pz} = -b j_z, \tag{19}$$

$$0 = E_r + v_{p\theta} B + \frac{1}{e n_e}\left(\frac{dp_e}{dr} - j_\theta B\right) - ab v_{pr}, \tag{20}$$

$$\eta j_\theta = -v_{pr} B - ab v_{p\theta}, \tag{21}$$

$$\eta j_z = E_z, \tag{22}$$

where $\vec{B} = (0, 0, B)$ and B is a constant spatially. Furthermore, E_z is a constant and $E_\theta = 0$ from $\mathrm{rot} \cdot \vec{E} = 0$.
Since

$$p_p \simeq n_e k T_e + n_i k T_i, \tag{23}$$

$$p_e \simeq n_e k T_e, \tag{24}$$

$$ab^2 \ll \eta, \tag{25}$$

where k is the Boltzmann constant, T_e the electron temperature and T_i the ion temperature, so that

$$v_{pr} \simeq -D_{AH} \frac{1}{n_e} \frac{dn_e}{dr}. \tag{26}$$

Here D_{AH} is the coefficient of the ambipolar difiusion across the line of magnetic field and is defined by

$$D_{AH} = \frac{n_e k (T_e + T_i)}{a(1+x^2)}. \tag{27}$$

where $x^2 = B^2/a\eta$.
From Eqs. (15) ~ (26), we can deduce the following equations:

$$n_e = n_{e0} J_0(\alpha r), \tag{28}$$

$$\vec{v}_p = \left(\alpha D_{AH} \frac{J_1(\alpha r)}{J_0(\alpha r)}, \quad -bj_\theta, \quad -bj_z\right), \tag{29}$$

$$\vec{j} = \left(0, \ \frac{\alpha B D_{AH}}{\eta} \cdot \frac{J_1(\alpha r)}{J_0(\alpha r)}, \ \frac{E_z}{\eta}\right), \tag{30}$$

$$\vec{E} = \left(\alpha A^2 \cdot \frac{kT_e}{e} \cdot \frac{J_1(\alpha r)}{J_0(\alpha r)}, \ 0, \ E_z\right), \tag{31}$$

where J_0 and J_1 are the Bessel function of the order zero and the order first, n_{e0} is the electron density at $r = 0$, $\alpha^2 = z/D_{AH}$, z the number of ionization per unit time and an electron which is defined by $z = Q/\rho_p$ and $A = \left\{1 - \frac{x^2(T_e + T_i)}{(1+x^2)T_e}\right\}$.

It is noted that from Eq. (31) $E_r = 0$ when $x^2 = x_c^2 = T_e/T_i$. Since $a\eta \simeq m_e m_i \nu_{ea} \nu_{ia} / 2e^2$, so that

$$x_c^2 = \frac{2(eB_c/m)(eB_c/m_i)}{\nu_{ea}\nu_{ia}}. \tag{32}$$

where B_c is a critical magnetic field. For a small magnetic field, the sheath of ion space charge is present. With the increasing magnetic field, the space charge of ions is canceled by the space charge of electrons gradually and when $B = B_c$ the net space charge disappears. For a strong magnetic field the sheath of electron space charge is formed. These effects were experimentally observed[5] and are caused by the fact that the gyration radius of electron is much smaller than that of ion.

The radial distribution of electric current $I(r)$ is given by

$$I(r) = \int_0^r 2\pi (E_z/\eta) r dr. \tag{33}$$

For the positive column of small electric current, η in Eq. (33) is approximately as follows, namely

$$\eta \simeq \nu_{ea} m_e / e^2 n_e,$$

so that the electric current becomes

$$I(r) = \frac{2\pi e^2 n_{e0} E_z}{\alpha^2 m_e \nu_{ea}} \alpha r \cdot J_1(\alpha r). \tag{34}$$

4. The Radius of a Plasma Column

Since $J_0=0$ at $\alpha r=2.405$, the quantities proportional to J_1/J_0 are all infinite at the point. From Poisson's equation, ρ^* is also infinite at the point so that our assumption of $\rho^*=0$ is not valid. We have, therefore, to take into account of the electro-static force $\rho^* \vec{E}$ in the equation of motion of the charged particles. Now we divide the positive column into a plasma column and a space charge layer. In the former $\rho^* \vec{E} \ll -\operatorname{grad} p_p$, whereas in the latter $\rho^* \vec{E} \gg -\operatorname{grad} p_p$. If the radius of the plasma colnmn is denoted by R_1, R_1 is temporarily determined as follows; using the calculated results in the preceding section, $\rho^* E_r$ and $-dp_p/dr$ bemome

$$\rho^* E_r = \varepsilon_0 \left(\frac{kT_e}{e}\right)^2 A^2 \alpha^3 \cdot \left[\left\{\frac{J_1(\alpha r)}{J_0(\alpha r)}\right\}^2 + 1\right] \cdot \frac{J_1(\alpha r)}{J_0(\alpha r)} \tag{35}$$

$$-\frac{dp_p}{dr} = n_0 k (T_e + T_i) \alpha \cdot J_1(\alpha r), \tag{36}$$

respectively, where ε_0 is the dielectric constant in a vacum space. Then we give R_1 by the following equation, namely

$$\frac{n_{e0} e^2}{\alpha^2 A^2 \varepsilon_0 k} \cdot \left(\frac{T_e + T_i}{T_e^2}\right) = \left[\left\{\frac{J_1(\alpha R_1)}{J_0(\alpha R_1)}\right\}^2 + 1\right] \cdot \frac{1}{J_0(\alpha R_1)} \tag{7}$$

equating Eq. (35) to Eq. (36). When n_0/α^2 in Eq. (37) is eleiminated by the use of Eq. (34), Eq. (37) becomes

$$\frac{m_e}{2\pi \varepsilon_0 k} \cdot \frac{\nu_{ea} I(R_1)}{E_z} \cdot \frac{(1+x^2)}{\{1-x^2(T_i/T_e)\}} \cdot \frac{(T_e+T_i)}{T_e^2} = \left[\left\{\frac{J_1(\alpha R_1)}{J_0(\alpha R_1)}\right\}^2 + 1\right] \cdot \frac{J_1(\alpha R_1)}{J_0(\alpha R_1)} \alpha R_1 \tag{38}$$

From Eq. (38) we can determine the value of αR_1 and hence the value of the upper limit of $J_1(\alpha R_1)/J_0(\alpha R_1)$. In earlier works, two methods was proposed to determine the radius of positive column although the basic ideas in these two methods are identical. It is the balance of charged particles between the charged particles created in the column and the charged particles escaped from the column. In the equation of the balance, the ion temperature is used either an atom temperature[11)12)] or a half of electron temperature.[13)] The latter is based on the stability of ion sheath discussed by D. Bohm.[14)] For the ion temperature we shall discuss in the next section.

From Eq. (38), we can find that when the electric current is very small and when the gas pressure is very low, the positive column can not be considered as the plasma column. For the very small electric current the positive column is called, "The subnormal discharge column".[15] For the very low pressure just as $\lambda_i \gg R_w$ (where λ_i is the mean free path of ion and R_w the radius of discharge tube), however, the ambipolar diffusion theory is not applicable.[16]

5. The Electron Temperature and the Ion Temperature

Neglecting both terms of adiabatic change and heat conduction, the electron- and ion-temperatures are respectively given[17] by

$$\eta_{ea}\mathbf{j}^2 + \eta_{ea}(en_e v_{pr})^2 = n_e \nu_{ea}\kappa_e \cdot \frac{3}{2}k(T_e - T_a), \tag{39}$$

$$\frac{\eta_{ea}^2}{\eta_{ia}}\mathbf{j}^2 + \eta_{ia}(en_e v_{pr})^2 = n_e \nu_{ia}\kappa_i \cdot \frac{3}{2}k(T_i - T_a), \tag{40}$$

where,

$$\mathbf{j}^2 = j_\theta^2 + j_z^2, \tag{41}$$

$$\eta_{ea} = \frac{m_e \nu_{ea}}{e^2 n_e}, \quad \eta_{ia} = \frac{m_i \nu_{ia}}{2\,e^2 n_e}. \tag{42}$$

Here κ_e and κ_i are respectively the fractions of the energy loses of electron and ion due to the collisions.

In Eq. (39), the first term of the left-hand side is the electronic part of joule heat and the second term the electronic part of ambipolar heat. On the other hand, the terms in the left-hand side of Eq. (40) are the ionic part of the joule and ambipolar heats. In order to guess the order of magnitude in regard with the terms in the left-hand side of Eqs. (39) and (40), we utilize the calculated results in the section 3. By the monotonous calculations, we get the following equntions, namely

$$\eta_{ea} j_z^2 = \frac{e^2 E_z^2}{m_e \nu_{ea}} \cdot n_{e0} J_0(\alpha r) \tag{43}$$

$$\frac{j_\theta^2}{j_z^2} = \frac{m_e \nu_{ea} \cdot zk(T_e + T_i)}{e^2 E_z^2} \cdot \frac{x^2}{(1+x^2)} \cdot \left\{\frac{J_1(\alpha r)}{J_0(\alpha r)}\right\}^2, \tag{44}$$

$$\frac{(en_e v_{pr})^2}{j_z^2} = \frac{m_e \nu_{ea} zk(T_e + T_i)}{e^2 E_z^2} \cdot \left(\frac{2\,m_e \nu_{ea}}{m_i \nu_{ia}}\right) \cdot \frac{1}{(1+x^2)} \cdot \left\{\frac{J_1(\alpha r)}{J_0(\alpha r)}\right\}^2. \tag{45}$$

Using Eqs. (43)$\sim$(45), Eq. (39) becomes approximately as follows,

$$E_z^2 + \frac{m_e \nu_{ea} zk(T_e + T_i)}{e^2} \cdot \frac{(x^2 + 2\,m_e \nu_{ea}/m_i \nu_{ia})}{1+x^2} \cdot \left\{\frac{J_1(\alpha r)}{J_0(\alpha r)}\right\}^2 = \frac{m_e \kappa_e \nu_{ea}^2}{e^2} \cdot \frac{3}{2}k(T_e - T_a). \tag{46}$$

Since E_z is independent of r because of rot $\vec{\mathrm{E}} = 0$, hence the electron temperature has a tendency to rise with the increasing radius.

For the ion temperature, the term of ambipolar heat plays an important role. At $r=R_1$, the ion temperature is determined by

$$\left(\frac{m_i v_{pr}^2}{2}\right)_{r=R_1} = \left\{\frac{3}{2}\,\kappa_i k(T_i - T_a)\right\}_{r=R_1} \tag{47}$$

If we define v_{ia} by

$$v_{ia} = \{3\,\kappa_i k(T_i - T_a)/m_i\}_{r=R_1}^{1/2} \tag{48}$$

then,

$$v_{ia} = (v_{pr})_{r=R_1}. \tag{49}$$

Since $v_{pr} \neq 0$ except the point of $r=0$, $(T_i - T_a)_{r=R_1} \neq 0$. When $(T_i)_{r=R_1} \gg (T_a)_{r=R_1}$, v_{ia} is compared with the mean thermal velocity of ions at $r=R$. Using Eqs. (43) and (45), Eq. (49) is rewritten as follows, namely

$$v_{ia}^2 = \frac{2\,k}{m_i} \cdot \left\{\frac{(T_e+T_i)z}{(1+x^2)\nu_{ia}}\right\}_{r=R_1} \cdot \left\{\frac{J_1(\alpha R_1)}{J_0(\alpha R_1)}\right\}. \tag{50}$$

The ion temperature at the surface of plasma column, therefore, strongly depends on the upper limit of $J_1(\alpha R_1)/J_0(\alpha R_1)$.

6. The Behaviors of Positive Column

In this section, we restrict our discussions to the situation that the radius of positive column R is nearly equal to the radius of plasma column R_1. Then $I(R_1)$ is nearly equal to the discharge current I and from Eq. (38) the upper limit of $J_1(\alpha R)/J_0(\alpha R)$ is approximately given by

$$\frac{J_1(\alpha R)}{J_0(\alpha R)} = 7.85 \cdot \left\{\frac{(T_e+T_i)}{T_e^2} \cdot \frac{\nu_{ea} I}{E_z} \cdot \frac{(1+x^2)^2}{(1-x^2 T_i/T_e)^2}\right\}^{1/3}. \tag{51}$$

Here αR included explicitly in Eq. (38) was set to 2.405, namely

$$\alpha^2 R^2 = \frac{m_i \nu_{ia} z(1+x^2)}{2\,k(T_e+T_i)} \cdot R^2 = (2.405)^2. \tag{52}$$

For the example of $J_1(\alpha R)/J_0(\alpha R)$, we use a set of conditions that $T_e = T_i = 10^4$ °K, $\nu_{ea} I/E_z = 5\times 10^5$ in MKS unit and $x^2=0$, then $J_1/J_0 = 36.4$.

Now we consider the ion temperature at $r=R$ and eliminate z from Eqs. (50) and (52) and put into the results the following relation

$$\nu_{ia} = v_{it} p_a/\lambda_{i0}, \tag{53}$$

where v_{it} is the mean thermal velocity of ions, p_a the pressure of atoms and λ_{i0} the mean free path of ions for $p_a = 1$. We, then, obtain the following equation, namely

$$v_{ia}^2 v_{it}^2 = (4.81)^2 \cdot \left\{\frac{k(T_i+T_e)}{m_i}\right\}^2 \cdot \left(\frac{1}{1+x^2}\right)^2 \left(\frac{\lambda_{i0}}{p_a R}\right)^2 \cdot \left\{\frac{J_1(\alpha R)}{J_0(\alpha R)}\right\}^2. \tag{54}$$

When $v_{ia} \simeq v_{it}$, Eq. (54) becomes

$$v_{it}^2 = \frac{g}{1-g} \cdot \frac{3\,kT_e}{2\,m_i}. \tag{55}$$

where,

$$v_{it}^2 \simeq \frac{3\,kT_i}{2\,m_i},$$

$$g = 3.21 \frac{1}{1+x^2} \cdot \frac{\lambda_{i0}}{p_a R} \cdot \frac{J_1(\alpha R)}{J_0(\alpha R)}. \tag{56}$$

For example, when $\lambda_{i0}/p_a R = 5 \times 10^{-3}$, $J_1/J_0 = 40$ and $x^2 = 0$, then $T_i \simeq 1.5\ T_e$. From Eq. (55), we can see that the ion temperature increases with the increasing magnetic field for a strong magnetic field. If $g = 1$ the ion temperature is infinite but it is apparent as we shall see later. Using Eq. (51), Eq. (56) becomes

$$g^{2/3}\{g^{1/2} + A_0(1-g)^{1/2}\}^{1/3} \cdot \{(1-g)^{1/2} - A_0 g^{1/2}\}^{2/3} = A_1(p_a R)^{-1} \cdot (E_z/p_a)^{-1/3}, \tag{57}$$

where,

$$A_1 = 27 \cdot \left(\frac{\lambda_{i0}}{\lambda_{e0}^{1/3}}\right) \cdot \left(\frac{k}{m_e}\right)^{1/6} \cdot T^{-1/2} I^{1/3},$$

and we used the following relations, namely

$$x^2 = A_0 \left\{\frac{1-g}{g}\right\}^{1/2},$$

$$x^2 \frac{T_i}{T_e} = A_0 \left\{\frac{g}{1-g}\right\}^{1/2},$$

in which,

$$A_0 = \frac{e^2 \lambda_{e0} \lambda_{i0}}{3(m_i m_e)^{1/2}} \cdot T_e^{-1} \cdot \left(\frac{B}{p_a}\right)^2.$$

The left-hand side of Eq. (57) is equal to infinity when $g = 0$ and $g = g_c = 1/(1 + A_0^2)$ (where g_c is the value of g for $x = x_c$. For $g = 0$, no such positive column is present and for $g = 1$ it also does not occur, as we shall see later.

Using the theoretical expression of z,[11] namely

$$z = z_0 p_a T_e^{1/2} \exp(-eV_i/kT_e), \tag{59}$$

where z_0 is the constant depending on the nature of atom and V_i the ionizatoin voltage, then Eq. (52) becomes

$$(1-g)^{1/2} \cdot \{g^{1/2} + A_0(1-g)^{1/2}\} = A_2 \cdot (p_a R)^{-2} \exp(eV_i/kT_e), \tag{60}$$

where,

$$A_2 = 9.45 \cdot \frac{\lambda_{i0}}{z_0} \cdot \left(\frac{k}{m_i}\right)^{1/2}.$$

From Eq. (60), we can see that only when T_e is infinite $g=1$ and that such positive column as $g=1$ is no present.

Let us now calculate the relation between T_e and E_z. We assume that the form of κ_e is given by

$$\kappa_e = \kappa_{e0}\exp(-eV_e/kT_e), \tag{61}$$

where κ_{e0} is the constant depend on the nature of atom and V_e the effective excitation voltage. Then Eq. (46) can be rewritten as follows;

$$\left(\frac{E_z}{p_a}\right)^{8/3} + A_3(p_aR)^{-2}\cdot\frac{(A_0+A_5)}{g^{2/3}(1-g)\{g^{1/2}+A_0(1-g)^{1/2}\}^{2/3}\{(1-g)^{1/2}-A_0g^{1/2}\}^{4/3}}$$
$$= A_4\left(\frac{E_z}{p_a}\right)^{2/3}\cdot\exp(-eV_e/kT_e), \tag{62}$$

where λ_{e0} is the mean free path of electrons defined by

$$\lambda_{e0} = v_{et}p_a/\nu_{ea}, \qquad v_{et} = 3kT/2m_e, \tag{63}$$

and A_3, A_4 and A_5 are

$$A_3 = 9.02\times10^2\cdot\left(\frac{m_e}{m_i}\right)^{1/2}\cdot\frac{k^{7/3}}{m_e^{1/3}}\cdot\frac{\lambda_{i0}}{\lambda_{e0}^{5/3}}\cdot I^{2/3}T_e^{5/3},$$

$$A_4 = \kappa_{e0}\left(\frac{3kT_e}{2e}\right)^2,$$

$$A_5 = \frac{\lambda_{i0}}{2\lambda_{e0}}\cdot\left(\frac{m_e}{m_i}\right)^{1/2}.$$

From Eqs. (57), (60) and (62), we can solve either E_z/p_a, T_e and g or p_aR, T_e and g, although the solutions have the parameters of p_aR, B and I or E_z/p_a, B and $\bar{I}$. For simplification of the calculations, we assume that the changes of the electron temperature have a infiuence on Eqs. (60) and (62) through only the exponential terms. Then we can write as follows, namely

$$1 = G_1(1-G_2), \tag{64}$$

$$E_z/p_a = C_3(1-G_2)^{1/2}[(1-g)^{1/2}\{g^{1/2}+A_0(1-g)^{1/2}\}]^{-\gamma/2}, \tag{65}$$

where,

$$G_1 = C_1g^5(1-g)^{-1}[(1-g)^{1/2}\{g^{1/2}+A_0(1-g)^{1/2}\}]^{2-\gamma}\{(1-g)^{1/2}-A_0g^{1/2}\}^4, \tag{66}$$

$$G_2 = C_2(A_0+A_5)(1-g)^{-1}g[\{g^{1/2}+A_0(1-g)^{1/2}\}(1-g)^{1/2}]^{\gamma}, \tag{67}$$

$$C_1 = A_1^{-6}A_2^{\gamma}A_4(p_aR)^{-2\gamma+6},$$

$$C_2 = A_3A_2^{-\gamma}A_4^{-1}(p_aR)^{2\gamma},$$

$$C_3 = A_2^{\gamma/2}A_4^{1/2}(p_aR)^{-\gamma},$$

$$\gamma = V_e/V_i.$$

The ratio of V_e to V_i, γ, is 3/4 for Hg. For He it is nearly equal to 0.7 from the result calculated by Lehnert and for Ar, $\gamma=0.764$ from the study of a gaseous breakdown.

Without the magnetic field, if $g \ll 1$, then Eqs. (64) and (65) are respectively reduced to the following equations, namely

$$G_1 = C_1 g^{6-\gamma/2},$$
$$E_z/p_a = C_3 g^{-\gamma/4},$$

from which

$$E_z/p_a = C_3 C_1^{\gamma/2(12-\gamma)}.$$

Therefore, $E_z/p_a \propto I^{1/15}$ for $\gamma = 3/4$.

Neglecting G_2 in Eqs. (64) and (65), G_1/C_1 and E_z/p_a are respectively plotted in Figs. 1 and 2 for $A_0 = 0$, 0.5, 1, 2 and 5 and for $\gamma = 3/4$. Each of the curves in Fig. 1, except the curve A, is composed with two branches, viz. one of them is in the range $g < g_c = 1/(1+A_0^2)$ and the other is in the range $g > g_c$. The curves $A \sim F$ belong to the former and the curves $B' \sim F'$ to the latter. Each of the curves $A \sim F$ has a maximum respectively and the maximum value increases as A_0 is decreased. Two values of g, therefore, are found to satisfy the equation $G_2 = 1$. This, however, depends on both C_1 and A_0, because if A_0 is over a critical value differentiated from $A_0 = (1-g_c)^{1/2}/g_c^{1/2}$ then the value of g satisfied $G_1 = 1$ is not present in the range $g < g_c$. In the above two valucs of g, it is plausible to take the value of g which is corresponding to the value of E_z/p_a lower than the other. When A_0 is over a critical value, however, we have to turn our attention in the range $g > g_c$. Then we find the value of g satisfying $G_1 = 1$ for the all values of A_0.

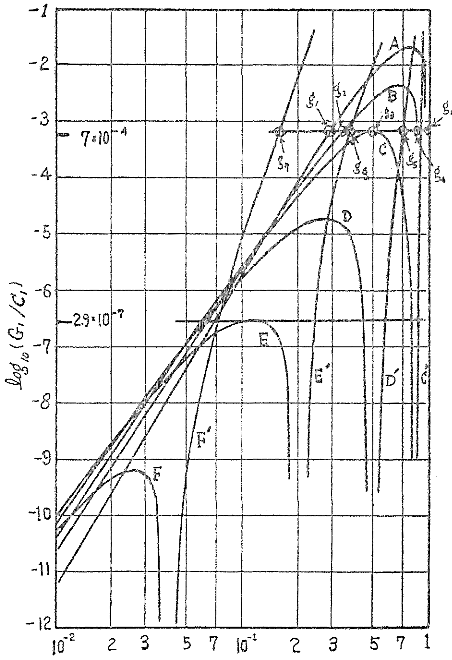

FIG. 1. The curves of G_1/C_1 as a function of g. The curve A is $A_0=0$, the curves B and B' are $A_0=0.2$, C and C' $A_0=0.5$, D and D' $A_0=1$, E and E' $A_0=2$ and F and F' $A_0=5$. The curves A, B, C, D, E and F are in the range $g<g_c$, whereas the curves B', C', D', E' and F' are in the range $g>g_c$, where $g_c=1/(1+A_0^2)$ and A_0 is proportinal to B^2/p_a^2.

Now, we consider what will happen if the magnetic field increases from zero. To see this we write the straight lines of $C_1^{-1} = 7\times10^{-4}$ and 2.9×10^{-7} in Fig. 1. For $C_1^{-1} = 7\times10^{-4}$, the values of g corresponding to the intersecting points with the curve A are $g_1 = 0.28$ and $g_0 > 0.9$. From Fig. 2 we make choice of $g_1 = 0.28$ as the real, since it takes the value of E_z/p_a lower than that of g_0. For $A_0 = 0.2$ and 0.5 $g_2 = 0.34$ and $g_3 = 0.5$ respectively. For $A_0 = 0.5$, however, the intersection is identical to the maximum of the curve C, so that when A_0 becomes larger than 0.5 no intersection is present in

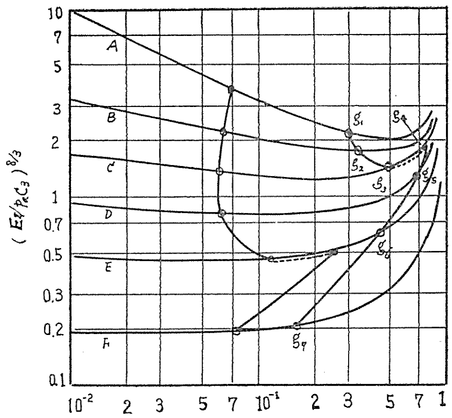

FIG. 2. The curves $A \sim F$ are the curves of $(E_z/p_aC_3)^{8/3}$ as a function of g, where the curves A, B, C, D, E and F are $A_0=0$, 0.2, 0.5, 1, 2 and 5 respectively.

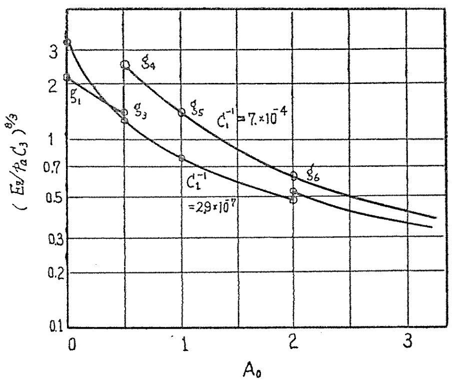

FIG. 3. The curves of $(E_z/p_aC_3)^{8/3}$ as a function of A_0, (where A_0 is proportional to B^2/p_a^2) for two values of C. From the curves we can see the abrupt rise of E_z/p_a.

the range $g<g_c$. We, now, turn to the range $g>g_c$. For $A_0=1$, 2 and 5 $g_5=0.7$, $g_6=0.39$ and $g_7=0.19$. Using these values of g, viz. g_1, g_2 etc., we can write the curve of E_z/p_a as a function of A_0, as is shown in Fig. 3. In the figure, we see the abrupt rises of E_z/p_a for two values of C_1, although for $C_1^{-1}=2.9\times10^{-7}$ it is very little.

The abrupt rise of E_z/p_a predicted here may be connected with the experimental results by Lehnert and other workers.[3)-5)] Their results, however, show that the increase of E_z/p_a is not abrupt but gradual. The difference may be caused by various simplifications of calculations.

The ion temperature can be calculated from g_1, g_2 etc. using Eq. (55). For $C_1^{-1}=7\times10^{-4}$, T_i is equal to 0.39 for $A_0=0$. For $A_0=0.2$, 0.5, 1, 2 and 5, $T_i=0.52$ T_e, $1.0\times T_e$, $0.3\times T_e$, $0.64\times T_e$ and $0.19\times T_e$ respectively. Except the abrupt rises of both E_z/p_a and T_i, E_z/p_a has a tendency of decrease as A_0 increses, whereas T_i has not always a tendency of decrease depending on C_1 strongly. For example, when $C_1^{-1}=7\times10^{-4}$, T_i increases with the magnetic field and after the abrupt rise T_i decreases as A_0 increases.

We consider the influence of G_2 in Eqs. (64) and (65) on the above discussions. The quantity G_2 means the contributions of two heats to the energy balance of electrons: One of them is the electronic part of the joule heat due to the circumferential current and the other is the electronic part of ambipolar heat. The contribution of the latter to G_2 is expressed by A_5 in Eq. (67). We write the curves of G_2/C_2 in Fig. 4 for $A_0=0.2$, 0.5, 1, 2 and 5 neglecting A_5 because A_5 is nearly equal to 2×10^{-3} at the largest. Since G_2 becomes larger than one depending on g and A_0, therefore, $(1-G_2)$ becomes negative. In the other hand, however, $(1-G_2)$ has to be positive from Eq. (65), so it seems to be natural that to hold $(1-G_2)>0$ the value of C_2 tends to the decrease with increasing magnetic field, viz. p_aR tends to the decrease. If the gas pressure is relatively high, then the positive column may be contracted. If, however, the gas pressure is relatively low, then the column may be not so. On the effect of G_2, we remember that Eq. (64) is the balance equation of electron energy at $r = R$. It is, therefore, not the total energy balance of electrons per unit length of column, viz. the contribution of $\int \eta_{ea} j_\theta^2 2\pi r \cdot dr$ to the total energy balance is less than the contribution of $(\eta_{ea} j_\theta^2)_{r=R}$ to Eq. (64). In the vicinity of the plasma surface, however, the effect of $\eta_{ea} j_\theta^2$ brings the value of E_z/p_a to be smaller than that of $j_\theta = 0$. Since the small value of E_z/p_a turns into the small value of longitudinal current, the radius of current channel, therefore, becomes small for the strong magnetic field.

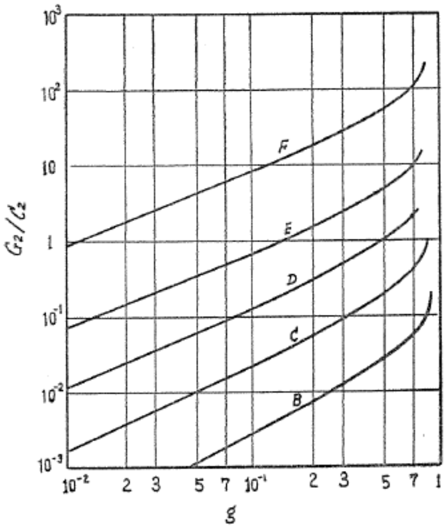

FIG. 4. The curves of G_2/C_2 as a function of g, where the curves B, C, D, E and F are $A_0 = 0.2$, 0.5, 1, 2 and 5 respectively.

7. Conclusion

We have dealt with the positive column of a gaseous discharge with an applied magnetic field on the basis of magnetohydrodynamics and thermdynamics.

The radial distribution of electron density is given by the Bessel function of the zero order using the coefficient of the ambipolar diffusion perpendicular to the magnetic field. Although the quantities proportinal to J_0^{-1}, therefore, is infinite for $J_0 = 0$, the lower limit of J_0 is determined by the condition that the electro-static force due to the charge accumulation is nearly equal to the pressure gradient of charged particles, see Eq. (38). From this condition we also determine the radius of the plasma column in which the electric charge of electrons is nearly equal to that of ions.

The radial component of electric filed is zero for a critical magnetic field, B_c, given by Eq. (32). For the strong magnetic field, the radial component of electric field is so negative as the electrons are accelarated towards a wall. The reason is that the gyration radius of electron is much smaller than that of ion.

We have given the equations of electron and ion energies by Eqs. (39) and (40) respectively. In the equation of electron energy, the joule heat of circumferential electric current plays an important role, whereas in the equation of ion energy the ionic part of ambipolar heat plays an important role. The ion temperature at the surface of plasma column is determined by Eq. (55).

In the section 6, we have discussed the bahaviors of plasma column and have solved E_z/p_a, T_i and T_e as the functions of p_aR, B and I. When the joule heat of circumferential current is neglected in the equation of electron energy, E_z/p_a decreases with the increasing weak magnetic field, whereas the ion temperature not always decreases depending on the conditions. For a certain value of magnetic field which differs from B_c, the abrupt rises of both E_z/p_a and T_i are found. For the further increase of magnetic field E_z/p_a decreases again and T_i also decreases.

This effect predicted here may be connected with the experimental results, although the experiments show that E_z/p_a increases gradually as the magnetic field increases. The difference between theory and experiments may be caused by simplification of calculations.

The influence of the joule heat of circumferential current on the lecal balance of electron energy becomes important with increasing magnetic field. The influence becomes also important with the increasing radius in the radial distribution. When the influence is taken into account, we can find that the radius of current channel tends to become small as the magnetic field increase. In the vicinity of the plasma surface, the joule heat of circumferential current is so large that it is sufficient to compensate the loss of electron energy. This, therefore, means that the joule heat of longitudinal current is not needed to heat the plasma near the surface. In the other hand, however, discharge current has to flow, so that it is unlikely to be $E_z/p_a=0$. The channel of electric current, thus, may be constricted.

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