THEORY OF SPACE CHARGE IN VACUUM DIODE TUBE

PART I. HYDRODYNAMIC DESCRIPTION

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ABSTRACT

In this paper we describe a space charge theory developed on the basis of hydrodynamics, treating many thermal electrons in vacuum diode tube as fluid. We solve the problem under the two conditions. Under condition (1), as usual in hydrodynamics, the temperature changes adiabatically. Under condition (2) the temperature is nearly constant in the system. We call these conditions "adiabatic" and "constant temperature" approximations respectively.

With constant temperature, the results under condition (2) are in good agreement with those of the space charge theories of Fry and Langmuir based on the assumption that electrons are emitted from a hot cathode with the Maxwllian velocity distribution. With the adiabatic approximation, the results are in poor agreement with experimental results for a weak electric current, because in this case the temperature may be constant due to the large thermal conductivity of the electron gas. This approximation is valid under the conditions which we discuss in more detail in part II.

I. Introduction

It is well known that the current-voltage characteristics of a hot cathode vacuum diode tube show negative anode voltage when the electric current is small compared with the saturated current. To explain this phenomenon, Fry[1] and Langmuir[2][3] proposed a space charge theory in which electrons are considered to be emitted from the cathode with the Maxwellian velocity distribution. Langmuir's theory was found to be in agreement with experimental results, provided that the well-known Boltzmann density distribution is valid,

$$n = n_0 \exp\left(\frac{eV}{kT_0}\right) \tag{1}$$

where V = electric potential, n = the number density of electrons and n_0 = the density for $V=0$, k = the Boltzmann constant and $-e$ = the electron charge. However, if the velocity of flow, or the mean velocity (hereafter referred to only as "velocity") is not equal to zero, Eq. (1) does not apply.[4] Furthermore, Langmuir's equation for the electric current, Eq. (57), which is similar in form to Eq. (1), does not satisfy the continuity of electric current.

Meanwhile Jaffe[5] proposed a theory based on the consideration that all the emitted electrons have the same velocity. According to his theory, however, the strength of electric field at the cathode is too complicated to be determined easily.

Both theories treated the phenomenon as an effect that modifiers the velocity distribution of electron at the cathode on the basis of the dynamics of one particle. From a macroscopic point of view, the motion of thermal electrons can not always be expressed by the dynamic equation of one particle. As we shall see later. The dynamic equation of one particle is valid only when the velocity is smaller than the mean thermal velocity. If this is not the case, Euler's hydrodynamic equation might be used.

We treat the motion of an electron gas on the basis of the hydrodynamics. It may be reasonable to assume that the electron temperature of the electron gas changes adiabatically an approximation usually adopted in hydrodynamics of compressible fluids. We call this assumption the "adiabatic approximation". However, if the electron temperature is kept nearly constant in the system, we can then obtain equations similar to those of Langmuir's theory. It is found desirable to adopt an approximation of constant temperature if we are to obtain good agreement with experimental results. This is called the "constant temperature approximation". We deal with these two extreme cases independently.

In Sec. 2 we describe a set of fundamental equations and in Sec. 3 they are reduced to practical ones suitable to a steady one-dimensional system. The equations are solved for two extreme cases in Secs. 4 and 5. Section 6 contains discussions on the results.

II. Fundamental Equations

1. Conservation of mass. We denote the mass density and the velocity by ρ and v respectively. The equation of continuity, expressed by the conservation of mass, is then

$$\frac{d\rho}{dt} = -\rho \operatorname{div} v, \tag{2}$$

where $d/dt = \partial/\partial t + (v \cdot \operatorname{grad})$.

2. Conservation of momentum. We denote the force per unit mass and the pressure by K and p respectively. The equation of motion is given by the Euler equation;

$$\frac{dv}{dt} = K - \operatorname{grad} p. \tag{3}$$

For charged particles in an electric field, by ignoring the force associated with the magnetic field caused by the electric current, K becomes

$$K = -\frac{eE}{m}. \tag{4}$$

Here $-e$ and m are the charge and the mass of electron respectively; E is the electric field strength. Equation (3) holds only for a non viscous fluid.

3. Equation of state. The pressure is connected with the density n and the temperature T in the equation of state as

$$p = nkT, \tag{5}$$

where k is the Boltzmann constant. As described in the preceding section, we assume the following two extreme cases for temperature variations:

(1) In the adiabatic approximation, the temperature is

$$T = \propto n^{\gamma - 1}, \tag{6}$$

where γ, ratio of the specific heat of constant pressure to the specific heat of constant volume, is given by

$$\gamma = (f+2)/f,$$

where f is the degree of freedom.

(2) In the constant temperature approximation, the temperature is

$$T = T_0 = \text{const.} \tag{7}$$

4. Poisson's equation. We have described fundamental equations in hydrodynamics; let us now consider basic equations in electromagnetism. There is a connection between the electric field strength in Eq. (4) and the charge density q through Poisson's equation, namely

$$\operatorname{div} \boldsymbol{E} = 4\pi q, \tag{8}$$

where q is equal to $-en$.

When we neglect the displacement current, the current density is

$$i = qv. \tag{9}$$

III. Some Expression for a Steady One-dimensional System

In a steady one-dimensional system, the five equations (9), (3), (6), (7) and (8) in the preceding section become, respectively

$$env = j, \tag{10}$$

$$mnv\frac{dv}{dx} = -neE - \frac{dp}{dx}, \tag{11}$$

$$p = nkT \propto n^3, \tag{12}$$

$$p = kT_0 n, \tag{13}$$

$$\frac{dE}{dx} = -4\pi en. \tag{14}$$

Here $j = -i$ is a constant and use has been made of $f = 1$ (namely $\gamma = 3$) for one-dimensional compression. Furthermore, we introduce the electric potential V by

$$E = -\frac{dV}{dx}. \tag{15}$$

1. Conservation of energy. Combining Eqs. (11), (12) and (15) and integrating the result, we obtain

$$\frac{1}{2}mv^2 = eV - \frac{3}{2}kT + K_{1a}, \tag{16}$$

where K_{1a} is a constant of integration. For the constant temperature approximation, using Eq. (13) instead of Eq. (12), we obtain

$$\frac{1}{2}mv^2 = eV - kT_0 \ln n + K_{1b}, \tag{17}$$

where K_{1b} is another constant of integration. If v is equal to zero, Boltzmann's density distribution, Eq. (1), can be obtained from Eq. (17). If v is not equal to zero, from Eq. (17) we get

$$n = n_0 \exp\left[\left\{eV - \frac{1}{2}m(v^2 - v_1^2)\right\}\Big/kT_0\right], \tag{18}$$

where v_1 is the velocity at the point for $V=0$. When the temperature is nearly constant in the system, Eq. (18) is more general than Eq. (1).

2. *Equation of stress.* Inserting Eq. (10) into the left-hand side of Eq. (11) and eliminating n on the right-hand side of Eq. (11) by using Eq. (14) and integrating the result, we have the following equation

$$\frac{mjv}{e} = \frac{1}{8\pi}E^2 - p + K_2, \tag{19}$$

where K_2 is a constant of integration. This stress equation is valid for two approximations. When $E^2/8\pi \gg p - K_2$, the velocity is given by

$$v = \frac{eE^2}{8\pi mj}\,. \tag{20}$$

Equation (20) is the most interesting when it is compared with the velocity of charged particles in an atomic or molecular gas at high pressure, in which by neglecting diffusion and thermo-diffusion, the velocity equation is such as

$$v = b_1 E \quad \text{or} \quad v = b_2 E^{1/2},$$

where b_1 and b_2 are constants depending on the kind of gas.[6]

3. *Condition under which dynamics of one particle holds.* Let us assume that the temperature is nearly constant. When we substitute Eq. (13) for Eq. (11) and make use of Eq. (10), then Eq. (11) can be rewritten

$$mnv\left\{1 - \left(\frac{v}{v_0}\right)^2\right\}\frac{dv}{dx} = -neE$$

where v_0 equals $(kT_0/m)^{1/2}$. If the condition that

$$v^2 \gg v_0^2 \tag{21}$$

be satisfied, the pressure gradient on the right-hand side of Eq. (11) can be neglected and dynamic equation of one particle can be used. For the adiabatic approximation, we have a similar condition. Therefore, if Eq. (21) be not satisfied, Euler's hydrodynamic equation must be used.

IV. Constant Temperature Approximation

In the first place, we calculate the constant temperature approximation because the results are in good agreement with those of Fry and Langmuir theory and we have no need of numerical integration.

1. Initial velocity. We take the cathode surface as the origin of our coordinate system and, at this point, the electric potential is taken as zero. There is a contact potential difference which consists of the work functions of anode and cathode material, and therefore an external potential difference between anode and cathode contains the contact potential difference. In this paper, however, V is used to express the potential difference which contains no contact potential difference.

At the cathode surface, the density of electrons emitted from the cathode, denoted by n_0, is given by the following equation:

$$n_0 = j_0 / ev_0 \tag{22}$$

where j_0 is the saturated current density and $v_0 = (kT_0/m)^{1/2}$. Since n is a constant in so far as the cathode temperature T_0 is constant, the initial velocity is

$$v_c = v_0(j/j_0) \tag{23}$$

2. Dimensionless equations and determination of the field strength at cathode surface. To avoid a complicated calculation, we introduce dimensionless equation with the following new quantities:

$$\left.\begin{array}{ll} N = n/n_0 & U = v/v_0 \\ \xi = x/x_0 & x_0 = (kT_0/8\pi n_0 e^2)^{1/2} \\ \zeta = E/E_0 & E_0 = kT_0/ex_0 \end{array}\right\} \tag{24}$$

$$\left.\begin{array}{ll} \eta = V/V_0 & V_0 = kT_0/e \\ J = j/j_0 & j_0 = en_0 v_0 \end{array}\right\} \tag{25}$$

Quantities n_0, v_0, x_0, E_0 and V_0 can be calculated when j_0 and T_0 are given.

Since n, v and V at the cathode are equal to n_0, v_0 and zero respectively, K_{1b} in Eq. (17) can be determined by using Eq. (23). The dimensionless electric potential η can be written as follows:

$$\eta = \frac{J}{2}\left(\frac{1}{N^2} - 1\right) - \ln N, \tag{26}$$

using $J = NU$. In the same way, dimensionless electric field ζ is given by

$$\zeta^2 - \zeta_c^2 = J^2\left(\frac{1}{N} - 1\right) - (1 - N), \tag{27}$$

where ζ_c is the electric field strength at the cathode and is theoretically determined as follows:

Equations (26) and (27) are plotted in Fig. 1 and Fig. 2 respectively. The straight-line in Fig. 1 shows the Boltzmann distribution, *viz.* Eq. (1). Both η and $\zeta^2 - \zeta_c^2$ have minimum values at the same value of N. Differentiating Eq. (26) and

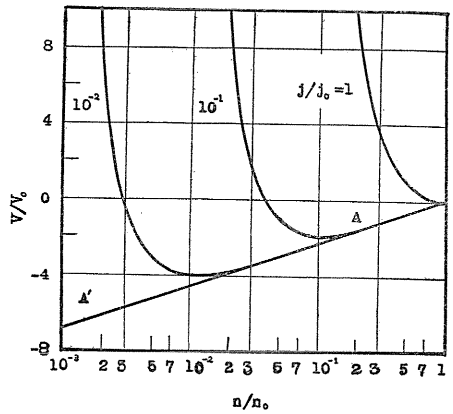

FIG. 1. V/V_0 *vs.* n/n_0. Straight-line shows Boltzmann's relation for density distribution, Eq. (1).

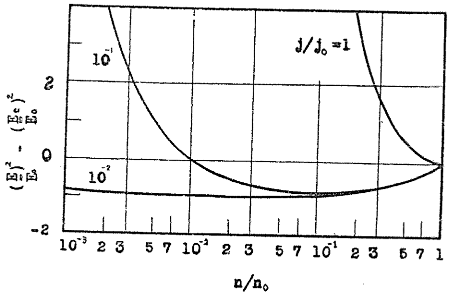

FIG. 2. $(E^2-E_0^2)/E_0^2$ *vs.* n/n_0, for j/j_0 $=1$, 10^{-1} and 10^{-2}.

Eq. (27) with respect to N and setting them equal to zero, we get the value of N corresponding to the minimum as given by the following equation:

$$(N)_{\min}=J. \tag{28}$$

Because ζ is a function of N alone, we can rewrite Eq. (15) as follows,

$$\zeta=-\frac{d\eta}{d\xi}=-\frac{d\eta}{dN}\cdot\frac{dN}{d\xi}. \tag{29}$$

Since $d\eta/d\xi=0$ for $N=J$, ζ must be zero for $N=J$. Thus, ζ_c is given by

$$\zeta_c=\pm(1-J) \tag{30}$$

and then Eq. (27) becomes

$$\zeta=\pm(N-J)/N^{1/2}. \tag{31}$$

Plus and minus signs in Eqs. (30) and (31) are to be physically determined, *i.e.* the dimensionless velocity U is monotonously increased as ξ and N are monotonously decreased. Therefore, $dN/d\xi$ is always negative and ζ is positive when η is decreased with decreasing N. Thus from Fig. 1 we can determine that we should take the plus sign.

3. Solution and results. Since η and ζ can be expressed as functions of N alone, from Eq. (29) ξ can be related to N by the following integration,

$$\xi=-\int_1^N\frac{d\eta/dN}{\zeta}dN. \tag{32}$$

Using Eq. (26) and Eq. (31), Eq. (32) becomes

$$\xi=2\left(\frac{1}{N^{1/2}}-1\right)+\frac{2}{3}J\left(\frac{1}{N^{3/2}}-1\right). \tag{33}$$

From the above, we can deduce the following:

(1) From Eq. (33), which is an algebraical equation of the third order with respect to $N^{-1/2}$, the distance dependency of the density is

$$\frac{1}{N} = \frac{n_0}{n} = 4\frac{j}{j_0}\sinh^2\theta, \tag{34}$$

where

$$\sinh 3\theta = \frac{3}{2}\left(\frac{j}{j_0}\right)^{1/2}\left(1+\frac{j}{3j_0}+\frac{x}{2x_0}\right). \tag{35}$$

(2) The density and the velocity at a point where the electric potential i is minimum can be given by

$$\left(\frac{n}{n_0}\right)_{\min} = \frac{j}{j_0} \quad \text{and} \quad \left(\frac{v}{v_0}\right)_{\min} = 1, \tag{36}$$

respectively.

(3) The distance dependency of electric potential is obtained by eliminating N from Eqs. (26) and (34), as

$$\frac{V}{V_0} = 8\sinh^4\theta - \frac{1}{2}\left(\frac{j}{j_0}\right)^2 - \ln 4\frac{j_0}{j}\sinh^2\theta. \tag{37}$$

(4) The current dependency of the minimum electric potential V_m, substituting Eq. (37) for Eq. (26), is given by

$$\frac{V_m}{V_0} = \ln\frac{j}{j_0} + \frac{1}{2}\left\{1-\left(\frac{j}{j_0}\right)^2\right\}. \tag{38}$$

(5) The distance corresponding to the minimum electric potential x_m by inserting Eq. (37) into Eq. (33), is

$$\frac{x_m}{x_0} = \frac{8}{3}\left(\frac{j_0}{j}\right)^{1/2}\left\{1-\frac{3}{4}\left(\frac{j}{j_0}\right)^{1/2}-\frac{1}{4}\left(\frac{j}{j_0}\right)^{3/2}\right\}. \tag{39}$$

If the bracket on the right-hand side of Eq. (39) is nearly equal to 1, then we have

$$x_m = 3.24\times 10^{-6}\frac{T_0^{3/4}}{j^{1/2}},$$

where j and T_0 are measured in units of A/cm^2 and °K, respectively.

(6) When j/j_0 is limited to zero, the electric potential becomes

$$V_l = -2V_0\ln\left(1+\frac{x}{2x_0}\right). \tag{40}$$

(7) The field strength and the pressure gradient at the cathode surface are

$$\frac{E_c}{E_0} = \left(1-\frac{j}{j_0}\right) \tag{41}$$

and

$$\left(-\frac{dp}{dx}\right)_{x=0} = \frac{p_0}{x_0}\left(1+\frac{j}{j_0}\right)^{-1} \tag{42}$$

respectively, where $p_0 = n_0 kT_0$. For $j/j_0 = 1$, E_c is zero but the pressure gradient is finite. Therefore, the electrons emitted from the cathode are accelerated only by the pressure gradient.

V. The Adiabatic Approximation

In the preceding section, we concerned with the constant temperature approximation. Electrons accelerated by a high voltage lose the character of the gas, *i.e.* lose the thermal random motion when separated from the hot cathode, and are oriented in the direction of the electric field. To better describe such behavior, it is reasonable to assume that the electron temperature decreases adiabatically. The procedure of calculation here is the same as in Sec. 4, except that we use $p \propto n^3$ instead of $p \propto n$.

The density of electrons at the cathode is

$$n_{0a} = \frac{j_0}{ev_{0a}}, \tag{43}$$

where

$$v_{0a} = \left(\frac{3kT_0}{m}\right)^{1/2}. \tag{44}$$

Here the quantities j_0 and T_0 are respectively the saturated current density and the cathode temperature. The velocity at the cathode surface, i.e. the initial velocity, is

$$v_{ca} = v_{0a}\frac{j}{j_0}. \tag{45}$$

Determining the K_{1a} in Eq. (16) and K_2 in Eq. (19), the dimensionless electric potential and the dimensionless electric field are respectively given by

$$\eta = \frac{3}{2}J\left(\frac{1}{N_a^2} - 1\right) - \frac{3}{2}(1 - N_a^2) \tag{46}$$

and

$$\zeta_a^2 - \zeta_{ca}^2 = 3J^2\left(\frac{1}{N_a} - 1\right) - (1 - N_a^3), \tag{47}$$

where $\zeta_a = E/E_{0a}$, $E_{0a} = kT_0/ex_{0a}$, $x_{0a} = (kT_0/8\pi n_{0a}e^2)^{1/2}$, $N_a = n/n_{0a}$ and where ζ_{ca} is the dimensionless electric field at the cathode.

Both η and $\zeta_a^2 - \zeta_{ca}^2$ have minimum values at

$$(N_a)_{\min} = J^{1/2} \tag{48}$$

and ζ_{ca} can be determined in the same way as in Sec. 4. Equation (47) becomes

$$\zeta_a = (N_a^4 - 3J^2 - 4J^{3/2}N_a)^{1/2}N_a^{-1/2}. \tag{49}$$

In the same way as in the preceding section, $\xi_a(=x/x_{0a})$ is connected with N_a by the following integration:

$$\xi_a = -\int \frac{d\eta/dN_a}{\zeta_a}dN_a. \tag{50}$$

Let us now introduce the following new quantities, *viz.*,

$$y = N_a/J^{1/2}, \quad y_0 = 1/J^{1/2} \quad \text{and} \quad Z = \xi_a/3J^{1/4}. \tag{51}$$

Using these quantities, Eq. (50) becomes

$$Z = Z_\alpha + Z_\beta, \tag{52}$$

where

$$Z_\alpha = \int_1^{y_0} \frac{y^3+y^2+y+1}{(3+y^2+2y)^{1/2}} \frac{dy}{y^{5/2}}$$

$$Z_\beta = -\int_1^{y} \frac{y^3+y^2+y+1}{(3+y^2+2y)^{1/2}} \frac{dy}{y^{5/2}} .$$

The result of the numerical integration of Z_α is plotted in Fig. 3. Since $N_a = 1$ expresses the density at which the electric potential equals zero, Z_α is a measure of the distance between the cathode and the point of the minimum electric potential. The distance corresponding to Z_α, denoted by x_m, is given by

$$x_m = 3J^{1/4} x_{0a} Z_\alpha. \tag{53}$$

If $y_0 \gg 1$, then Z_α becomes

$$Z_\alpha \approx y_0^{1/2}/2. \tag{54}$$

The straight-line $A - A'$ in Fig. 3 shows Eq. (54).

The result of the numerical integration of Z_β is shown in Fig. 4. Here Z_β is a measure of the distance between the point of the minimum electric potential and an arbitrary position towards the anode. If $y \ll 1$, then Z_β becomes

$$Z_\beta \approx 2/(3y)^{3/2}. \tag{55}$$

The straight-line $A - A'$ in Fig. 4 shows Eq. (55).

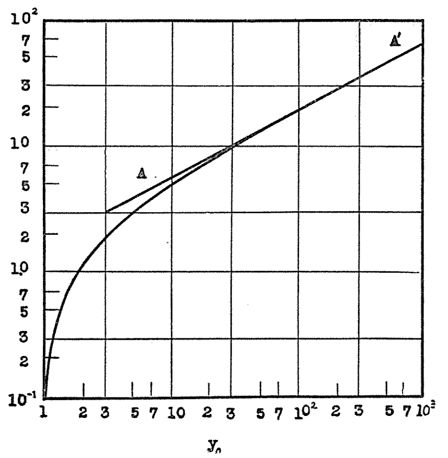

FIG. 3. Z_α *vs.* y_0. Straight-line $A-A'$ shows Eq. (54).

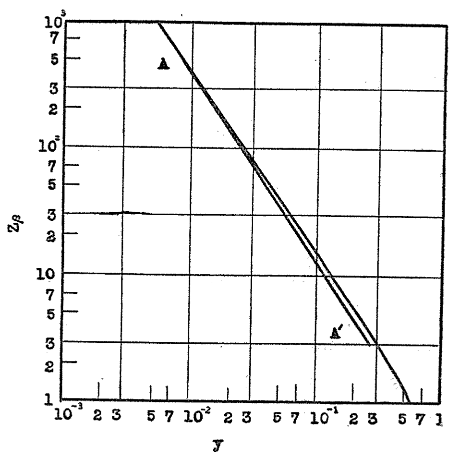

FIG. 4. Z_β *vs.* y. Straight-line shows Eq. (55).

From the above, we can deduce the following and describe a method to find the anode potential V for $J=J_1$ and $\xi_a=\xi_{a1}$.

(1) As J_1 is given, we can find Z_{a1} which corresponds to $y_{01}=J_1^{-1/2}$ in Fig. 3. The distance between cathode and minimum electric potential can be calculated by Eq. (53).

(2) Substituting Eq. (48) for Eq. (45), the minimum electric potential is

$$\frac{V_m}{V_0} = -\frac{3}{2}\left\{1-\left(\frac{j}{j_0}\right)^2\right\}.$$

If $j/j_0 \ll 1$, then $V_m = -3V_0/2$.

(3) Denoting Z by Z_1 where ξ_a equals ξ_{a1}, we can find y_1 which corresponds to $Z_1 - Z_{a1}$ in Fig. 4. Because η is rewritten from Eq. (46) as follows,

$$\eta = -\frac{3}{2}(1+J^2) + \frac{3}{2}J\left(y^2+\frac{1}{y^2}\right),$$

we can calculate the electric potential V for $J=J_1$ and $\xi_a=\xi_{a1}$.

VI. Discussions

We have dealt with the adiabatic approximation in the preceding section. Let us now discuss our theory qualitatively and quantitatively.

1. In Jaffe's theory, where he considered all emitted electrons to have the same velocity, the field strength at the cathode can not be easily determined. The difficulty lies in his assumption that the equation of the motion of a particle is valid. From a macroscopic point of view, however, this assumption is not true unless the condition (21) can be satisfied. Unfortunately, the initial velocity Eq. (23) or Eq. (45) does not satisfy this condition. In our theory, when we use Euler's hydrodynamic equation, the field strength at the cathode can be easily determined.

2. In Langmuir's theory, the determination of minimum electric potential is based on the following equation:

$$j/j_0 = \exp(eV/kT_0) \tag{57}$$

When Eq. (57) is differentiated with respect to x, we find that the current density does not satisfy the equation of continuity unless the field strength equals zero. In our theory, multiplying v by Eq. (18) and setting $j=env$ and $j_0=en_0v_0$, we have

$$\frac{j}{j_0} = \frac{v}{v_0}\cdot\exp\left\{\frac{eV}{kT_0} - \frac{m(v^2-v_c^2)}{2kT_0}\right\}, \tag{58}$$

where v_1 in Eq. (18) equals v_c for the present case. If $v=v_0$, the difference between Eq. (57) and Eq. (58) is only kinetic energy term $m(v^2-v_c^2)/2$ in the bracket. The velocity where the electric potential is minimum is equal to v_0 so that using Eq. (23) for v Eq. (58) becomes

$$\frac{j}{j_0} = \exp\left\{\frac{eV}{kT_0} - \frac{1}{2}\left(1-\frac{j^2}{j_0^2}\right)\right\}.$$

If v is not equal to v_0, *i.e.* if the position of the anode differs from the position of the minimum electric potential, Eq. (58) must be used. In this paragraph we are concerned with the constant temperature approximation.

3. In Fig. 5 the current dependency of the anode voltages is shown under the following set of conditions: $j_0 = 0.16\,A/cm^2$, $T_0 = 2{,}400°K$ and the distance between the anode and the cathode equals 0.5 cm. In addition, we tabulate the current dependency of the minimum electric potential and the distance from the cathode to the position of the minimum electric potentail in Table 1. We can conclude that with the adiabatic approximation the results are in poor agreement with those of Langmuir's theory for $j/j_0 \ll 1$ because the heat conduction which is neglected in the adiabatic approximation is large, as discussed in part II, and thus the temperature is nearly constant for $j/j_0 \ll 1$. For $j/j_0 \simeq 1$, we can find no appreciable difference between the two approximations except for temperature variation, because under the above set of conditions, d/x_0 is very large ($d/x_0 \approx 10^3$) and the pressure gradient does not play an important role.

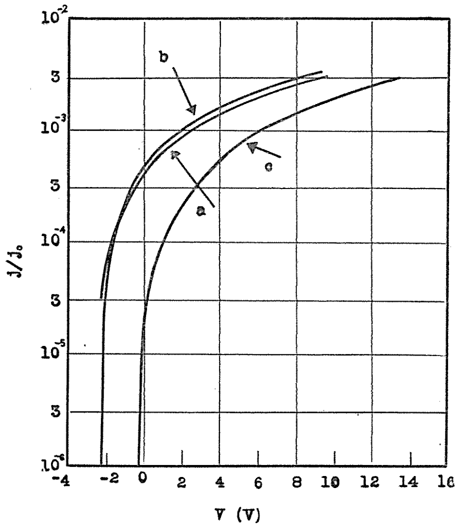

FIG. 5

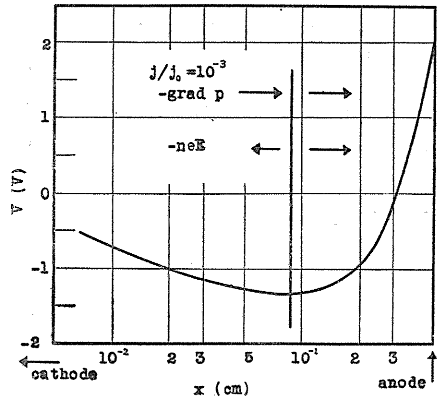

FIG. 6

FIG. 5. The anode voltage, V, versus j/j_0, for $j_0 = 0.16\,A/cm^2$, $T_0 = 2{,}400°K$ and $d = 0.5$ cm. Curves a, b and c show calculated results by Langmuir's theory, constant temperature approxi. and adiabatic approxi.

FIG. 6. Potential distribution, for $j/j_0 = 10^{-3}$, $j_0 = 0.16\,A/cm^2$, $T_0 = 2{,}400°K$ and $d = 0.5$ cm and the direction of forces.

TABLE 1. The Minimum Electric Potential V_m and the Distance from the Cathode to the Point of the Minimum Electric Potential x_m.

(a) V_m

$\log_{10} j/j_0$	L	C.T.A.	A.A.
0	0	0	0
−1	0.48	0.40	0.25
−2	0.96	0.85	0.31
−3	1.44	1.33	〃
−4	1.91	1.80	〃
−5	2.38	2.28	〃
−6	2.86	2.75	〃

(b) x_m

$\log_{10} j/j_0$	L	C.T.A.	A.A.
0	0	0	0
−1	6.19×10^{-3}	6.65×10^{-3}	4.38×10^{-3}
−2	2.25×10^{-2}	2.58×10^{-2}	6.14×10^{-3}
−3	7.51×10^{-2}	8.61×10^{-2}	$7.0\ \times10^{-3}$
−4	2.38×10^{-1}	2.79×10^{-1}	7.63×10^{-3}
−5	7.53×10^{-1}	8.82×10^{-1}	8.08×10^{-3}
−6	2.39	2.79	8.76×10^{-3}

4. With the constant temperature approximation, an electric potential distribution is shown in Fig. 6 under the same set of conditions as in the paragraph 3. In the same figure, the acting direction of forces is indicated. In the region between the position of the minimum electric potential and the cathode, *i.e.* in a α region, the electric force and the pressure gradient are oriented in opposite directions. In a residual region, *i.e.* in a β region, both forces are oriented in the same direction.

If $j/j_0=1$, the electrons on the cathode surface can be accelerated only by the pressure gradient. If $j/j_0<1$, the electrons overflow towards the anode unless they are decelerated by the retarding electric field. For $j/j_0 \ll 1$, therefore, the retarding electric field must be near the cathode. The pressure gradient, however, becomes smaller with further separation from cathode, so that the electrons must be transported to the anode by the electric field as well as by the pressure gradient. Now the direction of electric field is reversed.

5. Finally, as the last stage, we discuss the relation between V/V_0 and x/x_0 for $j/j_0=1$. The simple 3/2 power's law is shown by the curve a in Fig. 7. Using the constant temperature approximation, the curve c shows the calculated result. When the pressure gradient in Eq. (11) is neglected, under the conditions that the initial velocity and the electric field at cathode respectively are equal to v_0 and zero, the solution becomes

$$\frac{V}{V_0}=8\left(\frac{j}{j_0}\right)^2\sinh^2\theta\cdot\left(\sinh^2\theta+\frac{1}{2}\right), \tag{59}$$

where $\sinh 3\theta=3x/4x_0$. The curve b in Fig. 7 shows the calculated result using Eq. (59). The difference between the curves b and c is associated with the pressure gradient. The curve b coincides with the results calculated by Langmuir's approximate equation

$$j=\frac{1}{9\pi}\left(\frac{2e}{m}\right)^{1/2}\frac{(V+V_m)^{3/2}}{(x-x_m)}\times\left\{1+2.66\left(\frac{V_0}{V}\right)^{1/2}\right\}, \tag{60}$$

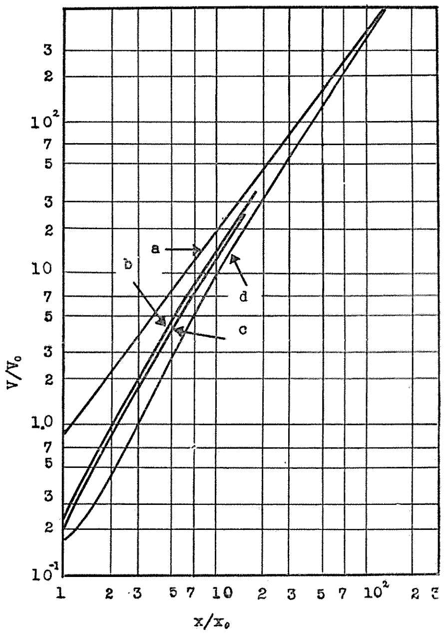

FIG. 7. V/V_0 *vs.* x/x_0, for $j/j_0=1$. Curves a, b, c and d show calculated results by simple 3/2 power's law, Eq. (60) neglected pressure gradient, constant temperature approxi. and adiabatic approxi.

where x_m and V_m are both equal to zero for $j/j_0=1$. The difference between the curves c and the result by Eq. (60) is quantitatively associated with the difference in the initial velocity at cathode, *viz.*, $v_0=(2kT_0/\pi m)^{1/2}$ in Langmuir's theory; but in our theory $v_0=(kT_0/m)^{1/2}$. The curve d shows the result of adiabatic approximation. The difference between the curves d and c and the difference between the curves d and b are not always small. For large x/x_0 all the curves in Fig. 7

coincide because the effects of the pressure gradient and the initial velocity disappear.

VII. Conclusion

In this paper, we have developed a space charge theory in vacuum diode tube on the basis of hydrodynamics. From the macroscopic point of view, the motion of many thermal electrons can be described by equation of the motion of a particle only if the velocity of flow is very much larger than the thermal velocity. The initial velocity, however, does not satisfy the condition that the equation of the motion of a particle is applicable. Consequently Euler's hydrodynamic equation must be used.

With a hydrodynamic interpretation of space charge, we can remove the difficulty in Jaffe's theory, *i.e.* that the field strength at cathode is not easily determined, and then Boltzmann's density distribution and Langmuir's electric current distribution can be made general.

Under the two extreme conditions that temperature is nearly constant in the system and that temperature changes adiabatically, we have solved the space charge problem. Using the constant temperature approximation, the results are in good agreement with those of Langmuir's theory. With the adiabatic approximation, however, the results for currents maller than the saturated current are not good, because under this condition the temperature may be constant due to the large thermal conductivity of electron gas. We have carried out numerical integrations in as much as adiabatic approximation is valid under such conditions as will be dicussed in the part II. The conditions where the above approximations are applicable, also will be discussed in the part II on the basis of thermodynamics.

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