NOTE ON MANY VALUED LOGIC

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I. Introduction

The electronic computers based on 2-valued logic (Boolean algebra) have been largely developed up to the present. Some researches for many valued logic and logical machine were attempted,[1)-3)] but it appears that this subject is still left unexploited. In this paper, we shall deliberate on the basic theory of many valued logical machines with view-points different from the studies above-mentioned. The results obtained in this paper are essentially the generalizations of 2-valued logic.

II. Conventions

Let propositional variables in n-valued logic be $X_1, \dots, X_m$ (where n is not necessarily prime but any positive integer). The set of n ordered truth values, $t_0, \dots, t_{n-1}$ on which $X_1, \dots, X_m$ are defined is represented by T.

$$T = \{t_i\}, \qquad i = 0, 1, \dots, n-1.$$

In the subsequent descriptions, the following basic logical operators are used: The binary operators; the disjunction $\vee$, the conjunction $\cdot$, the implication $\rightarrow$ and the equivalence $\sim$. The monomial operators; the negations of the first kind (cyclic permutation)

$$X_1^k \overset{D}{\equiv} t_l, \quad l = i \oplus k, \quad (\text{mod } n), \quad i, k = 0, 1, \dots, (n-1), \tag{D-1}$$

the negations of the second kind ($\urcorner_0$ is simply complementary operation)

$$\urcorner_k X_1 \overset{D}{\equiv} \urcorner_0 X_1^k \overset{D}{\equiv} t_l, \; l = (n-1)(i \oplus k \oplus 1), \quad (\text{mod } n), \tag{D-2}$$

where $\overset{D}{\equiv}$ denotes equality by definition and we assume that X_1 is in t_i. In the 2-valued logic, the above two negations degenerate into the single complement $\overline{X}$.

III. Fundamental Laws in n-valued Logic

The commutative, associative, distributive and absorption laws are also valid in n-valued logic as in 2-valued logic.

Generalized complementary law: We have

$$\left.\begin{array}{l} \sum_{i=0}^{n-1} \prod_{i=i}^{i \oplus j} X_1^i \sim t_p, \\ \sum_{i=0}^{n-1} \prod_{i=i}^{i \oplus j} \urcorner_i X_1 \sim t_p, \end{array}\right\} \quad p = (n-1)(j \oplus 1), \quad (\text{mod } n), \quad j = 0, \dots, n-1 \tag{F-1}$$

where $\sum$ or $\prod$ implies the disjunction or the conjunction of finite number of terms or factors.

Multiple negation law: We have

$$\{\{X^i\}^j\}^k \sim X^l, \quad l = i \oplus j \oplus k, \quad \text{(F-2.1)}$$

$$\urcorner_i X^i \sim \urcorner_l X^k, \quad l \oplus k = i \oplus j, \quad (\text{mod } n) \quad \text{(F-2.2)}$$

$$\urcorner_k \{\urcorner_j X^i\} \sim X^l, \quad l = i \oplus j \oplus (n-1)k, \quad \text{(F-2.3)}$$

$$\{\urcorner_j X^i\}^k \sim \urcorner_l X, \quad l = i \oplus j \oplus (n-1)k. \quad \text{(F-2.4)}$$

Generalized recurrence law: We have

$$\{\{X^i\}^j\}^k \sim X, \quad i \oplus j \oplus k = 0, \quad (\text{mod } n) \quad \text{(F-3.1)}$$

$$\urcorner_k \{\urcorner_j X^i\} \sim X, \quad i \oplus j \oplus (n-1)k = 0. \quad \text{(F-3.2)}$$

De Morgan's law: This law is also valid in n-valued logic,

$$\urcorner_0(X_1 \vee X_2) \sim \urcorner_0 X_1 \cdot \urcorner_0 X_2, \quad \text{(F-4.1)}$$

$$\urcorner_0(X_1 \cdot X_2) \sim \urcorner_0 X_1 \vee \urcorner_0 X_2. \quad \text{(F-4.2)}$$

Generalization of De Morgan's law: We have

$$\urcorner_l \{X_1^i \vee X_2^j\}^k \sim \{(\urcorner_i X_1) \cdot (\urcorner_j X_2)\}^{(n-1)(k \oplus l)}, \quad \text{(F-5.1)}$$

$$\urcorner_l \{X_1^i \cdot X_2^j\}^k \sim \{(\urcorner_i X_1) \vee (\urcorner_j X_2)\}^{(n-1)(k \oplus l)}. \quad \text{(F-5.2)}$$

Proof: In $l = i \oplus j \oplus (n-1)k$ of (F-2.3), let $i = j = 0$, and if the two sides of this equality is multiplied by $(n-1)$, the following relation is obtained because of $(n-1)^2 = 1$, (mod n).

$$(X_1^i \vee X_2^j)^k \sim \urcorner_{(n-1)k}\{\urcorner_0(X_1^i \vee X_2^j)\} \sim \urcorner_{(n-1)k}\{\urcorner_0 X_1^i \cdot \urcorner_0 X_2^j\}, \quad \text{by (F-4.2)},$$

$$\sim \urcorner_{(n-1)k}\{\urcorner_i X_1 \cdot \urcorner_j X_2\}, \quad \text{by (D-2)},$$

$$\therefore \quad \urcorner_l \{X_1^i \vee X_2^j\}^k \sim \urcorner_l \urcorner_{(n-1)k}\{\urcorner_i X_1 \cdot \urcorner_j X_2\}$$

$$\sim \{\urcorner_i X_1 \cdot \urcorner_j X_2\}^{(n-1)(l \oplus k)}, \quad \text{by (F-2.3)}.$$

(F-5.2) is also proved analogously.

IV. The Principal Canonical Forms of n-valued Logical Polynomials

There exist n^{n^m} m-ary logical functions in n-valued logic and they are prescribed uniquely by their truth-tables. To each function, many logical polynomials equivalent to it correspond. The principal canonical form of polynomial implies that every function is represented uniquely in that form of polynomial, that is, the different polynomials in the principal canonical form should constitute the different logical functions.

Lemma 1: The following monomial logical funtion (m. l. f.) $\mathfrak{x}_{\nu_1}(X_1)$ is in t_{n-1} of the maximum element in T if and only if $X_1 \sim t_{\nu_1}$, and always when $X_1 \nsim t_{\nu_1}$, is in t_0 of the minimum element in T.

$$\mathfrak{x}_{\nu_1}(X_1) \sim \{t_1 \cdot X_1^{(n-1)\nu_1}\}^{n-1} \sim \{\sum_{\mu_1=0}^{n-1}{}^{*} X_1^{\mu_1}\}^1,$$

where $\sum^*$ denotes the disjunction with respect to $\mu_1 \neq (n-1)(\nu_1 \oplus 1)$.

Proof: Let X_1 be in t_k. By (D-1), $X_1^{(n-1)\nu_1} \sim t_l$, $l = k \oplus (n-1)\nu_1$. Therefore,

$$t_1 \cdot X_1^{(n-1)\nu_1} \sim \begin{cases} t_0, & k = \nu_1 \\ t_1, & k \neq \nu_1. \end{cases}$$

Again using (D-1), we have

$$\{t_1 \cdot X_1^{(n-1)\nu_1}\}^{n-1} \sim \begin{cases} t_{n-1} & k = \nu_1 \\ t_0 & k \neq \nu_1. \end{cases}$$

The analogous proof is possible as to $\{\sum_{\mu_1=0}^{n-1}{}^{*} X_1^{\mu_1}\}^1$.

Theorem 1: Any m-ary function in n-valued logic is represented uniquely in the following principal disjunctive canonical form (p. d. c. f.) only using $\cdot$, $\vee$ and the negation of the first kind:

$$\mathfrak{A}(X_1, \ldots, X_m) \sim \sum_{\nu_1=0}^{n-1} \cdots \sum_{\nu_m=0}^{n-1} \mathfrak{A}(t_{\nu_1}, \ldots, t_{\nu_m}) \cdot \mathfrak{X}_{\nu_1 \cdots \nu_m}(X_1, \ldots, X_m), \qquad (1)$$
$$\mathfrak{X}_{\nu_1 \cdots \nu_m}(X_1, \ldots, X_m) \sim \prod_{k=1}^{n} \mathfrak{x}_{\nu_k}(X_k).$$

Proof: By lemma 1, it is trivial that $\mathfrak{X}_{\nu_1 \cdots \nu_m}(X_1, \ldots, X_m)$ is in t_{n-1} if and only if $X_1 \sim t_{\nu_1}, \ldots, X_m \sim t_{\nu_m}$ and otherwise, is in t_0. Let us prove (1) by induction on the number m of variables.

Basis: $m = 1$. $\mathfrak{A}(X_1)$ is expanded as follows:

$$\mathfrak{A}(X_1) \sim \sum_{\nu_1=0}^{n-1} C_{\nu_1} \cdot \mathfrak{x}_{\nu_1}(X_1). \qquad (2)$$

By lemma 1, $C_{\nu_1} \sim \mathfrak{A}(t_{\nu_1})$, $\nu_1 = 0, \ldots, n-1$ is easily seen. Then, $\mathfrak{A}(X)$ is represented uniquely by (2).

Induction step: $m > 1$. By (2),

$$\mathfrak{A}(X_1, \ldots, X_m) \sim \sum_{\nu_m=0}^{n-1} \mathfrak{A}(X_1, \ldots, X_{m-1}, t_{\nu_m}) \cdot \mathfrak{x}_{\nu_m}(X_m).$$

Now, by the hypothesis of the induction,

$$\mathfrak{A}(X_1, \ldots, X_{m-1}, t_{\nu_m}) \sim \sum_{\nu_1=0}^{n-1} \cdots \sum_{\nu_m=0}^{n-1} \mathfrak{A}(t_{\nu_1}, \ldots, t_{\nu_m}) \cdot \prod_{k=1}^{m-1} \mathfrak{x}_{\nu_k}(X_k),$$

Therefore, we have

$$\mathfrak{A}(X_1, \ldots, X_m) \sim \sum_{\nu_1=0}^{n-1} \cdots \sum_{\nu_m=0}^{n-1} \mathfrak{A}(t_{\nu_1}, \ldots, t_{\nu_m}) \cdot \prod_{k=1}^{m} \mathfrak{x}_{\nu_k}(X_k).$$

Lemma 2: The necessary and sufficient condition for a m.l.f. $\mathfrak{R}(X)$ to be homomorphic is that the following relation is valid:

$$\begin{aligned} &\mathfrak{R}(0) \to \mathfrak{R}(1) \to \cdots \to \mathfrak{R}(n-1), && \text{for homomorphic,} \\ &\mathfrak{R}(n-1) \to \mathfrak{R}(n-2) \to \cdots \to \mathfrak{R}(0), && \text{for dual homomorphic.} \end{aligned} \tag{3}$$

Proof: Let $X_1 \sim k_1$ and $X_2 \sim k_2$ (hereafter t_{k_1} is abbreviated as k_1).

Sufficient condition: Assume that (3) is valid.
Case 1): $k_1 = k_2$.

$$\mathfrak{R}(X_1 \vee X_2) \sim \mathfrak{R}(k_1) \sim \mathfrak{R}(k_2),$$
$$\mathfrak{R}(X_1) \vee \mathfrak{R}(X_2) \sim \mathfrak{R}(k_1) \vee \mathfrak{R}(k_2) \sim \mathfrak{R}(k_1) \sim \mathfrak{R}(k_2).$$

Case 2): $k_1 \neq k_2$, $(k_1 > k_2)$

$$\mathfrak{R}(X_1 \vee X_2) \sim \mathfrak{R}(k_1),$$
$$\mathfrak{R}(X_1) \vee \mathfrak{R}(X_2) \sim \mathfrak{R}(k_1) \vee \mathfrak{R}(k_2) \sim \mathfrak{R}(k_1).$$

Then,

$$\mathfrak{R}(X_1 \vee X_2) \sim \mathfrak{R}(X_1) \vee \mathfrak{R}(X_2),$$

similarly,

$$\mathfrak{R}(X_1 \cdot X_2) \sim \mathfrak{R}(X_1) \cdot \mathfrak{R}(X_2).$$

If (3) is satisfied for the case of dual homomorphic, then we have analogously,

$$\mathfrak{R}(X_1 \cdot X_2) \sim \mathfrak{R}(X_1) \vee \mathfrak{R}(X_2)$$
$$\mathfrak{R}(X_1 \vee X_2) \sim \mathfrak{R}(X_1) \cdot \mathfrak{R}(X_2).$$

Necessary condition: If (3) is not valid, then we can easily see that there exists at least one pair of truth values (k_1, k_2) which can't satisfy the case 2.

Theorem 1—Corollary 1: The p.d.c.f. (1) in n-valued logic can be written in the following reduced form:

$$\begin{gathered} \mathfrak{A}(X_1, \ldots, X_m) \sim \sum_{p=0}^{n^m-1} \mathfrak{A}_p(\nu_1, \ldots, \nu_m) \cdot \mathfrak{X}_p(X_1, \ldots, X_m), \\ \mathfrak{X}_p \sim \{1 \cdot \sum_{i=1}^{m} X_i^{(n-1)\nu_i}\}^{n-1}, \end{gathered} \tag{4}$$

where $p = \nu_1 + \nu_2 n + \nu_3 n^2 + \cdots + \nu_m n^{m-1}$.

If $\mathfrak{A}_p(\nu_1, \ldots, \nu_m) \sim \mathfrak{A}_{p'}(\nu_1', \ldots, \nu_m')$, then the reduction of terms in (4) is possible. That is,

$$\begin{gathered} \mathfrak{A}_p \cdot \mathfrak{X}_p \vee \mathfrak{A}_{p'} \cdot \mathfrak{X}_{p'} \sim \mathfrak{A}_p \mathfrak{Y}_p, \\ \mathfrak{Y}_p \sim \{1 \cdot (\sum_{i=1}^{m} X_i^{(n-1)\nu_i}) \cdot (\sum_{i=1}^{m} X_i^{(n-1)\nu'_i})\}^{n-1}, \end{gathered} \tag{5}$$

Proof: In (1), $n^m \mathfrak{X}_{\nu_1 \cdots \nu_m}$ are contained and we denote these by $\mathfrak{X}_p$, $p = 0, \ldots, n^m - 1$. p is the conversion to decimal number of n-ary number $\nu_1 \nu_2 \cdots \nu_m$. It is easily seen that

$$\mathfrak{x}_0(X_k) \sim \{1 \cdot X_k\}^{n-1},$$
$$\mathfrak{x}_0(0) \sim n-1,\ \mathfrak{x}_0(1) \sim \mathfrak{x}_0(2) \sim \cdots \mathfrak{x}_0(n-1) \sim 0.$$

Therefore, $\mathfrak{x}_0(X_k)$ is dual homomorphic and also we have

$$\mathfrak{x}_{\nu_k}(X_k) \sim \mathfrak{x}_0\{X_k^{(n-1)\nu_k}\}.$$

Then, by lemma 2, (4) is proved as follows:

$$\mathfrak{X}_p(X_1, \ldots, X_m) \sim \prod_{k=1}^{m} \mathfrak{x}_{\nu_k}(X_k) \sim \mathfrak{x}_0\{\sum_{k=1}^{m} X_k^{(n-1)\nu_k}\}. \tag{6}$$

Analogously, since $\mathfrak{A}_p \sim \mathfrak{A}_{p'}$ and $\mathfrak{X}_p$ is also dual homomorphic by (6), it follows that $\mathfrak{A}_p \cdot \mathfrak{X}_p \vee \mathfrak{A}_{p'} \cdot \mathfrak{X}_{p'} \sim \mathfrak{A}_p \cdot \{\mathfrak{X}_p \vee \mathfrak{X}_{p'}\}$ and

$$\mathfrak{X}_p \vee \mathfrak{X}_{p'} \sim \mathfrak{Y}_p \sim \mathfrak{x}_0\{\sum_{i=1}^{m} X_i^{(n-1)\nu_i}) \cdot (\sum_{i=1}^{m} X_i^{(n-1)\nu'_i})\}.$$

Then, (5) is proved.

V. The Duality Principle in n-valued Logic

$\mathfrak{A}\{\mathfrak{B}(X)\}$ is called the functional multiplication and written as $\mathfrak{A}\mathfrak{B}$, and further, if $\mathfrak{A} \sim \mathfrak{B}$, is named the power of function and written as $\mathfrak{A}^2$.

Definition 3: We call the following the polynomials of depth 1 and 2 respectively.

$$C_1 \cdot X_1^i \vee C_2 \cdot X_2^j \vee \cdots, \qquad \text{depth 1,}$$
$$C_1 \cdot X_1^i \vee C_2 \cdot \{C_3 \cdot X_2^j\}^k \vee \cdots, \qquad \text{depth 2.}$$

Generally speaking, the greatest number of successive times at which reversible monomial operators are used in construction of partial formula of a polynomial is named the depth of it.

Theorem 2: Let any m-ary function $\mathfrak{H}$ be constructed with constants, variables, $\cdot$, $\vee$ and reversible monomial logical function (r. m. l. f.). We can get $\urcorner_0 \mathfrak{H}$ as follows. First, all $\cdot$ and $\vee$ which are contained in $\mathfrak{H}$, are interchanged mutually. Then, in the partial formulas of depth 1 in $\mathfrak{H}$, constants and r. m. l. f. are replaced with their $\urcorner_0$-negations. Further, we should substitute for partial formulas of more than depth 2 their conjugate functions with respect to $\urcorner_0 - \urcorner_0$ transformation.

Proof: If any m-ary function is represented by the principal cononical form, as seen from theorem 1, it is constructed with constants, variables, $\vee$, $\cdot$ and the negations of first kind or second kind. This is extended so that $n!$ r. m. l. f. may be contained in $\mathfrak{H}$ generally. Let $\mathfrak{H}$ be the logical formula of depth k. Apply (F-4) to $\mathfrak{H}$. As for the partial formulas of depth 1, we should interchange $\cdot$ and $\vee$ each other and substitute $\urcorner_i X$ for X. Let one of the partial formulas of depth k

in $\mathfrak{H}$ be $\mathfrak{J}_k$. Then, we replace $\mathfrak{J}_k$ with $\rceil_0 \mathfrak{J}_k$. Next, let the conjugate function of $\mathfrak{J}_k$ with respect to $\rceil_0 - \rceil_0$ transformation be $\mathfrak{J}_k^*$. Then, $\mathfrak{J}_k \sim \rceil_0 \mathfrak{J}_k^* \rceil_0$. Therefore, we have $\rceil_0 \mathfrak{J}_k \sim \mathfrak{J}_k^* \rceil_0$ because $\rceil_0 \rceil_0 X \sim X$. If $\mathfrak{J}_k \sim \mathfrak{A} \mathfrak{J}_{k-1}$, where $\mathfrak{A}$ is a r. m. l. f. of depth 1, then $\mathfrak{J}_k^* \sim \mathfrak{A}^* \mathfrak{J}_{k-1}$. Therefore, $\mathfrak{J}_k^* \rceil_0 \sim \mathfrak{A}^* \{\rceil_0 \mathfrak{J}_{k-1}\}$. As for $\rceil_0 \mathfrak{J}_{k-1}$, we should repeat analogous operations until we get $\rceil_0 \mathfrak{J}_1$. Finally, (F-4) is applied to $\rceil_0 \mathfrak{J}_1$. Since the same arguments can be made for other partial formulas of more than depth 2, the theorem can be proved.

Theorem 3 (Generalized duality principle): Let any two m-ary functions be $\mathfrak{G}$ and $\mathfrak{H}$. Assume that both $\mathfrak{G}$ and $\mathfrak{H}$ are constructed with constants, r. m. l. f., $\vee$ and $\cdot$. If the proposition $\mathfrak{G} \sim \mathfrak{H}$ is tautology (in t_{n-1}), a proposition that is obtained as follows, is also tautology. That is, all $\cdot$ and $\vee$ should be interchanged each other and constants should be $\rceil_0$-negated in the two sides of the equivalent formula simultaneously. Moreover, all r. m. l. f. of every depth (up to depth 1) should be replaced with their conjugate functions with respect to $\rceil_0 - \rceil_0$ transformation.

Proof: Let the equivalent formula that is obtained by $\rceil_0$-negating the tow sides of $\mathfrak{G} \sim \mathfrak{H}$ simultaneously be $\mathfrak{G}' \sim \mathfrak{H}'$. Then, $\mathfrak{G}' \sim \mathfrak{H}'$ is still tautology. To obtain $\mathfrak{G}' \sim \mathfrak{H}'$, according to theorem 2, all $\cdot$ and $\vee$ should be interchanged each other and the partial formulas of more than depth 2 should be replaced with the conjugate functions with respect to $\rceil_0 - \rceil_0$ transformation. Moreover, constants and the partial formulas of depth 1 should be $\rceil_0$-negated, that is, truth values i and X^i are replaced with $(n-1)(i \oplus 1)$ and $\rceil_i X$ respectively. Next, if we make the substitutions of $\rceil_0 X_1 \sim \xi_1$, etc., then we have $\rceil_i X_1 \sim \xi_1^{(n-1)i}$, $X_1^i \sim \rceil_{(n-1)i} \xi_1$, etc. Let the equivalent formula which was subjected to the above substitution be $\mathfrak{G}'' \sim \mathfrak{H}''$. $\mathfrak{G}'' \sim \mathfrak{H}''$ is of course still tautology. In $\mathfrak{G}'' \sim \mathfrak{H}''$, the partial formulas of every depth (even the depth 1 by the above substitution) in its two sides are replaced with their conjugate functions in consquence.

References

1) A. Rose: Proc. Cambridge Philos. Soc., **54**, 307-321 (1958).
2) J. Lukasiewicz and A. Tarski: Comptes rendus Warsaw, Classe III, **23**, 1-21 (1930).
3) M. Itō: Technology reports of the Kyushu Univ., **28**, No. 2, 96-101 (1955).