RESEARCH REPORTS

STATIONARY MOTION OF VISCOUS FLUID AROUND A ROTATING SOLID SPHERE

Zyurô SAKADI

Department of Applied Physics

(Received January 10, 1958)

I. Introduction

The stationary motion of an incompressible viscous fluid around a solid sphere, which is rotating with a constant angular velocity about one fixed diameter, was treated by Lamb[1] and there it is shown that the second order terms in the equations of motion are of practical importance. Stokes[2] took the second order terms due to the acceleration into account and solved the problem. The corresponding problem for the case of oscillating motion was treated recently by Carrier and Di Prima.[3] In this work the author considered the stationary motion with the extended equations of motion, which the author developed previously.[4][5] The corresponding case of oscillating motion seems to be very troublesome.

II. Equations of the Motion

Let

x_i : rectangular coordinates,

u_i : components of velocity,

A_{ij}: components of stress tensor,

$$\sigma_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right).$$

The equations of motion are, when no body force acts on the fluid,

$$\rho\frac{du_i}{dt} = \rho\frac{\partial u_i}{\partial t} + \rho u_k\frac{\partial u_i}{\partial x_k} = \frac{\partial A_{ij}}{\partial x_j}, \tag{1}$$

$$\frac{\partial u_j}{\partial x_j} = 0. \tag{2}$$

Since the left hand side of (1) contains the first and second order terms of u_j, also for the expressions of A_{ij}, it is natural to take the second order terms of u_j into account as well. When we assume that A_{ij} contains u_j only in the shape of σ_{ij}, we have:

$$A_{ij} = -p\delta_{ij} + 2\mu\sigma_{ij} + 4a\sigma_{ik}\sigma_{jk}. \tag{3}$$

Inserting (3) into (1) we obtain:

$$\frac{\partial u_i}{\partial t}+u_k\frac{\partial u_i}{\partial x_k}=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\frac{\mu}{\rho}\Delta u_i+2\frac{a}{\rho}\sigma_{ik}\Delta u_k+4\frac{a}{\rho}\sigma_{jk}\frac{\partial\sigma_{ik}}{\partial x_j}. \qquad (4)$$

III. Equations for the Disturbed Motion

Let the solid sphere of the radius c and with the center at origin of co-ordinates rotate with the constant angular velocity ω about the x_3-axis. We put

$$r=\sqrt{x_j^2}=\sqrt{x_1^2+x_2^2+x_3^2},$$

$$\frac{3}{2}\omega c^3=\alpha,$$

$$u_j=u_j'+v_j, \qquad p=p'+q,$$

$$u_1'=-\frac{2}{3}\alpha\frac{x_2}{r^3}, \qquad u_2'=\frac{2}{3}\alpha\frac{x_1}{r^3}, \qquad u_3'=0, \qquad p'=0. \qquad (5)$$

Here (5) is the solution of the linearized equations:

$$-\frac{1}{\rho}\frac{\partial p'}{\partial x_i}+\frac{\mu}{\rho}\Delta u_i'=0, \qquad \frac{\partial u_j'}{\partial x_j}=0$$

with the boundary conditions:

$$u_1'=-\omega x_2,\ u_2'=\omega x_1,\ u_3'=0$$

at $r=c$.

The equations for v_j and q, when the higher order terms than the second order are neglected, become:

$$u_k'\frac{\partial u_i'}{\partial x_k}=-\frac{1}{\rho}\frac{\partial q}{\partial x_i}+\frac{\mu}{\rho}\Delta v_i+2\frac{a}{\rho}\sigma_{ik}'\Delta u_k'+4\frac{a}{\rho}\sigma_{jk}'\frac{\partial\sigma_{ik}'}{\partial x_j}, \qquad (6)$$

$$\frac{\partial v_j}{\partial x_j}=0 \qquad (7)$$

with

$$\sigma_{ij}'=\frac{1}{2}\left(\frac{\partial u_j'}{\partial x_i}+\frac{\partial u_i'}{\partial x_j}\right).$$

The boundary conditions are

$$v_j=0 \qquad (8)$$

at $r=c$.

IV. Solution of the Differential Equations

Using the expressions (5), we obtain from (6) the following equations:

$$\Delta v_i-\frac{1}{\mu}\frac{\partial q}{\partial x_i}=\frac{\partial S}{\partial x_i}+\delta_{3i}\frac{\partial}{\partial x_3}\left(\frac{4a\alpha^2}{3\mu}\frac{1}{r^6}-\frac{\rho\alpha^2}{9\mu}\frac{1}{r^4}\right),$$

where

$$S=\frac{2a\alpha^2}{\mu}\left(-\frac{5}{3}\frac{1}{r^6}+\frac{x_3^2}{r^8}\right)+\frac{\rho\alpha^2}{9\mu}\frac{1}{r^4}.$$

Or, putting

$$\frac{1}{\mu}q+S=s,$$

we have:

$$\Delta v_i-\frac{\partial s}{\partial x_i}=\delta_{3i}\frac{\partial}{\partial x_3}\left(\frac{4a\alpha^2}{3\mu}\frac{1}{r^6}-\frac{\rho\alpha^2}{9\mu}\frac{1}{r^4}\right). \tag{9}$$

We have to solve (7) and (9) with the conditions (8).

One particular solution of the equations (7) and (9) can be given by:

$$v_i'=\frac{\partial^3 U}{\partial x_i\partial x_3^2}+\delta_{3i}\frac{\partial U}{\partial x_3},\qquad s'=\Delta\frac{\partial^2 U}{\partial x_3^2},$$

$$U=-\frac{a\alpha^2}{2\cdot 3^2\mu}\frac{1}{r^2}+\frac{\rho\alpha^2}{2\cdot 3^2\mu}\log r,$$

$$V=-\Delta U=\frac{a\alpha^2}{3^2\mu}\frac{1}{r^4}-\frac{\rho\alpha^2}{2\cdot 3^2\mu}\frac{1}{r^2}.$$

The homogeneous equations (7) and

$$\Delta v_i-\frac{\partial s}{\partial x_i}=0$$

have the following solution:

$$v_i''=\frac{\partial W}{\partial x_i}+\delta_{3i}\frac{\partial}{\partial x_3}\left(-\frac{B}{r}\right),\qquad s''=\frac{\partial^2}{\partial x_3^2}\left(\frac{B}{r}\right),$$

$$W=\frac{B}{2}\left(\frac{1}{r}-\frac{x_3^2}{r^3}\right)+A\frac{\partial^2}{\partial x_3^2}\left(\frac{1}{r}\right)$$

with the arbitrary constants A and B.

The boundary conditions (8) for

$$v_i=v_i'+v_i'',\qquad s=s'+s''$$

can be satisfied by:

$$A=\frac{2}{3^2}\frac{a\alpha^2}{\mu}\frac{1}{c}+\frac{1}{2\cdot 3^3}\frac{\rho\alpha^2}{\mu}c,$$

$$B=\frac{2^2}{3^2}\frac{a\alpha^2}{\mu}\frac{1}{c^3}-\frac{1}{3^2}\frac{\rho\alpha^2}{\mu}\frac{1}{c}.$$

The moment of force about x_3-axis, which the sphere receives from the surrounding fiuid, is the same as in the linear case.

$$N=\int\left(-A_{1j}\frac{x_jx_2}{c}+A_{2j}\frac{x_jx_1}{c}\right)d\sigma=-\frac{2^4}{3}\pi\mu\alpha=-8\pi\omega c^3\mu.$$

References

1) Lamb, Hydrodynamics, Sixth Ed. Cambridge 1932, p. 588.
2) Stokes, Trans. of the Cambr. Phil. Soc., Vol. **9**, 1856, p. 8 (the author could not refer to this paper).
3) Carrier and Di Prima, J. of Applied Mechanics, Vol. **23**, 1956, p. 601.
4) Sakadi, Proc. of the Phys.-Math. Soc. of Jap., Ser. 3, Vol. **23**, 1941, p. 27.
5) Sakadi, l. c., Ser. 3, Vol. **24**, 1942, p. 717.