ON A SIMPLIFIED METHOD OF STRESS ANALYSIS OF SWEPT BACK WING STRUCTURES

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1. A method of stress analysis of swept back wings with ribs parallel to the line of flight has been developed by W. S. Hemp.[1] However, his elabolate method based on the application of oblique coordinates is considerably complicated, and hardly may be applied practically to various loading conditions. This paper is to propose a simplified practical method for such problems.

The results obtained here, including some simple examples, have sufficient accuracy for practical procedures, and our method will be successfully used to actual swept back box beams, tapered from root to tip of the wing with ribs both normal to the spars and parallel to the line of flight.

2. The stress-strain relations of the orthogonal anisotropic plate, assuming that one of the axial rigidities can be neglected, are expressed by

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{T_x}{Et}, \qquad \gamma_z = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{S_z}{Gt}. \quad (1)$$

and it is convenient to consider them in terms of oblique coordinates ξ, η, direction ξ is parallel to x, when the plate forms a parallelogram (Fig. 1), then we obtain

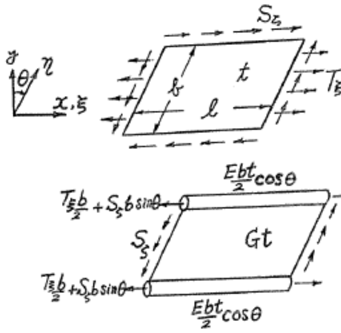


FIG. 1

$$\varepsilon_\xi = \frac{\partial U}{\partial \xi} = \frac{1}{Et\cos\theta}\,(T_\xi + 2\sin\theta S_\zeta),$$

$$\gamma_\zeta = \frac{\partial U}{\partial \eta} + \frac{\partial V}{\partial \xi} = \frac{1}{Et\cos\theta}\left\{2\sin\theta T_\xi + \left(4\sin^2\theta + \frac{Et}{Gt}\cos^2\theta\right)\times S_\zeta\right\}. \quad (2)$$

The expression of strain energy may be easily established in the following form,

$$W = \int\frac{(T_\xi + 2S_\zeta\sin\theta)^2}{2Et\cos\theta}\,d\xi\,d\eta + \int\frac{S_\zeta^2}{2Gt}\,d\xi\,d\eta\cos\theta. \quad (3)$$

For a special case where T_ξ, S_ζ are assumed to be independent of η, we have

$$W = \int\frac{(T_\xi b + 2S_\zeta b\sin\theta)^2}{2Ebt\cos\theta}\,d\xi + \int\frac{S_\zeta^2 b\cos\theta}{2Gt}\,d\xi. \quad (4)$$

From these relations we may substitute two axial members and a plate for the anisotropic plate, where each substituted rod undergoes $\frac{T_\xi b}{2} + S_\zeta b\sin\theta$, and the plate does not resist to the axial stress. Thus the beam where the frange rigidity is EA_0 and the axial rigidity and shear rigidity of the web is Et and Gt respectively, as shown in Fig. 2-*a*, will be transposed by the beam whose frange area is enlarged to $A_0+\frac{1}{2}bt\cos\theta$ as in Fig. 2-*b*.

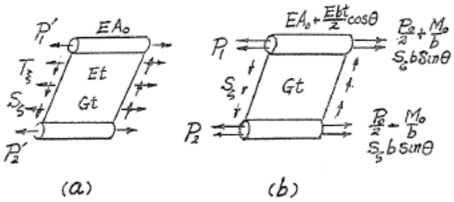


FIG. 2

When the cantilever beam undergoes its tip load, for instance, the axial force of the upper or lower frange is established to be

$$P_{\frac{1}{2}} = \pm F\xi/b + F\sin\theta. \tag{5}$$

So the force shared in the actual frange area is

$$P'_{\frac{1}{2}} = P_{\frac{1}{2}} \times \frac{EA_0}{EA_0+Ebt\cos\theta/2} = \frac{2\lambda}{1+2\lambda}(\pm\xi/b+\sin\theta)F, \quad \lambda \equiv \frac{EA_0}{Ebt\cos\theta}\cdot \tag{6}$$

and the deflection at the tip in the direction to the force will be obtained by the following procedures.

$$\delta/F = 1/F\cdot\partial W/\partial F = \frac{l\cos\theta}{Gtb} + \frac{2l}{EA'}\left(\frac{l^2}{3b^2}+\sin^2\theta\right)\cdot \tag{7}$$

In order to compare with these results the solutions by Hemp's method are also obtained;

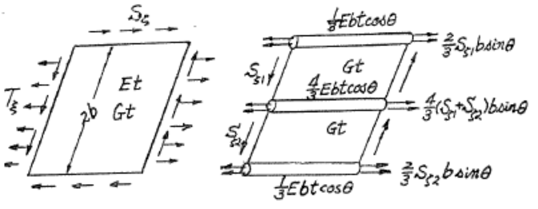
$$P'_{\frac{1}{2}} = \frac{6\lambda}{1+6\lambda}\left(\pm\xi/b+\frac{2(1+3\lambda)}{3(1+2\lambda)}\sin\theta\right)F,$$

$$\delta/F = \frac{l\cos\theta}{Gtb}\,\frac{\frac{6}{5}+12\lambda+36\lambda^2}{(1+6\lambda)^2} + \frac{2l}{EA_0}\left(\frac{2\lambda}{(1+6\lambda)}\left(\frac{l}{b}\right)^2 + \frac{\frac{12}{5}\lambda+\frac{144}{5}\lambda^2+72\lambda^3}{(1+6\lambda)^2(1+2\lambda)}\sin^2\theta\right)\cdot \tag{8}$$

The differences between our results and these expressions are small in usual, but our simplified solution may be only correct in the case where T_ξ does not depend on η as shown above, that is to say, the bending rigidity of the substituted beam does not coincide with the actual one, and it is desirable that one more axial member may be considered to improve our results as will be described in the next section.

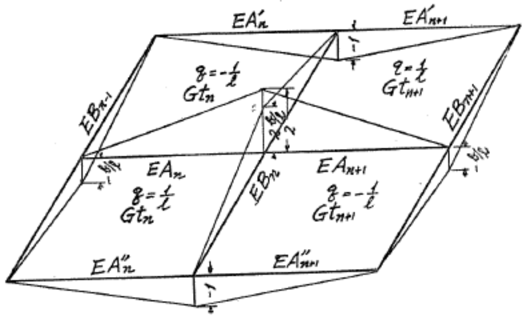
3. It must be noted that we may substitute another combination of plate and rods for the parallelogram plate as shown in Fig. 3, to have more accurate expressions, where each rod will be imposed by additional internal force respectively;

$$\frac{2}{3}S_{\zeta 1}b\sin\theta, \qquad \frac{4}{3}(S_{\zeta 1}+S_{\zeta 2})b\sin\theta, \qquad \frac{2}{3}S_{\zeta 2}b\sin\theta.$$



(a) (b)

FIG. 3



($X_n = -1$) Sytem

FIG. 4

According to such transposition, the beam shown in Fig. 2-*a* may be substituted similarly by the one with three axial members, and this structure will be easily solved by the usual method for statically indeterminate structures using self-equilibrating internal load systems Xn (Fig. 4). Then the elastic equations become

$$\delta_{n0} = \delta_{nn} X_n + \delta_{n\,n-1} X_{n-1} + \delta_{n\,n+1} X_{n+1} + \delta_{n\,n-2} X_{n-2} + \delta_{n\,n+2} X_{n+2}, \tag{9}$$

where

$$\delta_{nn} = \left(\frac{4}{9}\frac{b^2\sin^2\theta}{e} + \frac{l}{3}\right)\left(\frac{1}{EA_n'} + \frac{1}{EA_n''} + \frac{1}{EA_{n+1}'} + \frac{1}{EA_{n+1}''}\right)$$
$$+ \frac{2}{3} b\sin\theta\left(\frac{1}{EA_n'} - \frac{1}{EA_n''} - \frac{1}{EA_{n+1}'} + \frac{1}{EA_{n+1}''}\right) + \frac{4}{3} l\left(\frac{1}{EA_n} + \frac{1}{EA_{n+1}}\right)$$
$$+ \frac{2}{3}\frac{b^3}{l^2}\left(\frac{1}{EB_{n-1}} + \frac{1}{EB_{n+1}} + \frac{4}{EB_n}\right) + \frac{2b\cos\theta}{l}\left(\frac{1}{Gt_n} + \frac{1}{Gt_{n+1}}\right),$$

$$\delta_{n\,n-1} = \left(-\frac{4}{9}\frac{b^2\sin^2\theta}{l} + \frac{l}{6}\right)\left(\frac{1}{EA_n'} + \frac{1}{EA_n''}\right)$$
$$+ \frac{4}{6}\frac{l}{EA_n} - \frac{4}{3}\frac{b^3}{l^2}\left(\frac{1}{EB_{n-1}} + \frac{1}{EB_n}\right) - \frac{2b\cos\theta}{Gt_n l},$$

$$\delta_{n\,n+1} = \left(-\frac{4}{9}\frac{b^2\sin^2\theta}{l} + \frac{l}{6}\right)\left(\frac{1}{EA_{n+1}'} + \frac{1}{EA_{n+1}''}\right)$$
$$+ \frac{4}{6}\frac{l}{EA_{n+1}} - \frac{4}{3}\frac{b^3}{l^2}\left(\frac{1}{EB_{n+1}} + \frac{1}{EB_n}\right) - \frac{2b\cos\theta}{Gt_{n+1} l,},$$

$$\delta_{n\,n-2} = \frac{2}{3}\frac{b^3}{l^2}\frac{1}{EB_{n-1}},$$

$$\delta_{n\,n+2} = \frac{2}{3}\frac{b^3}{l^2}\frac{1}{EB_{n+1}}.$$

$$\delta_{n0} = -\frac{4}{9} q_{0n} b^2 \sin^2\theta\left(\frac{1}{EA_n'} - \frac{1}{EA_n''}\right) + \frac{1}{3} q_{0n} bl\sin\theta\left(\frac{8}{EA_n} - \frac{1}{EA_n'} - \frac{1}{EA_n''}\right)$$
$$+ \frac{4}{9} q_{0n+1} b^2 \sin^2\theta\left(\frac{1}{EA_{n+1}'} - \frac{1}{EA_{n+1}''}\right)$$
$$+ \frac{1}{3} q_{0n+1} bl\sin\theta\left(\frac{8}{EA_{n+1}} - \frac{1}{EA_{n+1}'} - \frac{1}{EA_{n+1}''}\right)$$

$$+\frac{l}{EA'_n}P'_{0n-1}\Big(-\frac{1}{3}\frac{b\sin\theta}{l}-\frac{1}{6}\Big)+\frac{l}{EA'_{n+1}}P'_{0n+1}\Big(\frac{b\sin\theta}{3l}-\frac{1}{6}\Big)$$

$$+\frac{l}{EA'_n}P'_{0n}\Big(-\frac{b\sin\theta}{3l}-\frac{1}{3}\Big)+\frac{l}{EA'_{n+1}}P'_{0n}\Big(\frac{b\sin\theta}{3l}-\frac{1}{3}\Big)$$

$$+\frac{l}{EA''_n}P''_{0n-1}\Big(\frac{b\sin\theta}{3l}-\frac{1}{6}\Big)+\frac{l}{EA''_{n+1}}P''_{0n+1}\Big(\frac{-b\sin\theta}{3l}-\frac{1}{6}\Big)$$

$$+\frac{l}{EA''_n}P''_{0n}\Big(\frac{b\sin\theta}{3l}-\frac{1}{3}\Big)+\frac{l}{EA''_{n+1}}P''_{0n}\Big(-\frac{b\sin\theta}{3l}-\frac{1}{3}\Big),$$

In the case where the uniform section of the beam is assumed and let l tend to zero, the equation (9) becomes to the following differential equation.

$$\frac{2}{3}\frac{b^3 l}{EB}\frac{d^4X}{d\xi^4}-\Big(\frac{4}{9}b^2\sin^2\theta\Big(\frac{1}{EA'}+\frac{1}{EA''}\Big)+\frac{2b\cos\theta}{Gt}\Big)\frac{d^2X}{d\xi^2}$$
$$+\Big(\frac{4}{EA}+\frac{1}{EA'}+\frac{1}{EA''}\Big)X=\frac{2}{9}b\sin^2\theta\Big(\frac{1}{EA'}-\frac{1}{EA''}\Big)\frac{dF_0}{d\xi}$$
$$-\Big(\frac{1}{EA'}+\frac{1}{EA''}-\frac{8}{EA}\Big)\frac{\sin\theta}{3}F_0+\Big(\frac{1}{EA'}-\frac{1}{EA''}\Big)\Big(\frac{b}{3}\sin\theta\frac{dP_0}{d\xi}-\frac{M_0}{2b\sin\theta}\Big)$$
$$+\Big(\frac{1}{EA'}+\frac{1}{EA''}\Big)\Big(\frac{\tan\theta}{3}\frac{dM_0}{d\xi}-\frac{P_0}{2}\Big), \qquad (10)$$

where

$$F_0=2bq_0,\qquad P'_0+P''_0=P_0,$$
$$P'_0-P''_0=M_0/b\cos\theta.$$

and the boundary conditions are

i) at the free end ($\xi=0$); $X=\dfrac{dX}{d\xi}=0$

ii) at the fixed end ($\xi=L$);

$$\Big(\frac{2b\cos\theta}{Gt}+\Big(\frac{1}{EA'}+\frac{1}{EA''}\Big)\frac{4}{9}b^2\sin^2\theta\Big)\frac{dX}{d\xi}+\frac{2}{3}b\sin\theta\Big(\frac{1}{EA'}-\frac{1}{EA''}\Big)X$$
$$=-\Big(\frac{1}{EA'}-\frac{1}{EA''}\Big)\Big(\frac{2}{9}\sin^2\theta\, bF_0+P_0 b\sin\theta\Big)-\Big(\frac{1}{EA'}+\frac{1}{EA''}\Big)M_0\tan\theta,$$
$$\frac{d^2X}{d\xi^2}=0. \qquad (11)$$

Considering the particular case $EB/l\to\infty$, $EA'=EA''$, the above equation will be simplified to

$$-\Big(\frac{8}{9}\frac{b^2\sin^2\theta}{EA'}+\frac{2b\cos\theta}{Gt}\Big)\frac{d^2X}{d\xi^2}+\Big(\frac{4}{EA}+\frac{2}{EA'}\Big)X$$
$$=-\Big(\frac{2}{EA'}-\frac{8}{EA}\Big)\frac{\sin\theta}{3}F_0+\frac{2}{EA'}\Big(\frac{\tan\theta}{3}\frac{dM_0}{d\xi}-\frac{P_0}{2}\Big), \qquad (12)$$

and the boundary conditions to

$$\xi=0\,; \quad X=0, \quad \xi=L\,; \quad \left(\frac{8}{9}\frac{b^2\sin^2\theta}{EA'}+\frac{2b\cos\theta}{Gt}\right)\frac{dX}{d\xi}=-\frac{2}{EA'}M_0\tan\theta. \quad (13)$$

4. In this section, some simple examples are considered;

i) A cantilever undergoing its tip load

If the whole length of the beam L is assumed to be sufficiently large, the solution of Eq. (12) becomes

$$X=\frac{\frac{8}{3}\frac{\sin\theta}{EA}F}{\frac{4}{EA}+\frac{2}{EA'}}(1-e^{-\alpha\xi}), \qquad \alpha^2=\frac{\frac{4}{EA}+\frac{2}{EA'}}{\frac{8}{9}\frac{b^2\sin^2\theta}{EA'}+\frac{2b\cos\theta}{Gt}}. \quad (14)$$

The partition of the load in the actual frange is

$$P'_{\frac{1}{2}}=\left\{\pm\frac{F\xi}{2b}+\frac{F}{3}\sin\theta+\frac{\frac{8\sin\theta}{3EA}F}{\frac{4}{EA}+\frac{2}{EA'}}\left(1-e^{-\alpha\xi}\left(1\mp\frac{2}{3}b\alpha\sin\theta\right)\right)\frac{6\lambda'}{1+6\lambda'}\right\},$$

$$\lambda'=\frac{EA_0}{2bt\cos\theta}. \quad (15)$$

ii) A shear lag problem

The following solution can be obtained assuming L is large enough.

$$X=-\frac{P_0}{2+4\frac{EA'}{EA}}(1-e^{-\alpha\xi}), \quad (16)$$

$$P'_{\frac{1}{2}}=P_0\frac{6\lambda'}{1+6\lambda'}\left\{\frac{1+6\lambda'}{6(1+2\lambda')}+\frac{e^{-\alpha\xi}}{3(1+3\lambda')}\left(1\mp\frac{2}{3}b\alpha\sin\theta\right)\right\}. \quad (17)$$

Conclusion

In order to analyse the stress distribution of swept back wing structures with ribs parallel to the direction of flight, parallelogram plates surrounded with stringers and ribs may be substituted by idealized shear plates and the stringers, considering that their equivalent sectional area and the stringers are to undergo additional internal force.

Then we can deal with the structure under consideration by means of the usual method of analysis.

Reference

1) W. S. Hemp. Rep. No. 31. The College of Aeronautics. Cranfield.