BENDING OF A THIN ELLIPTIC PLATE OF AN ORTHOTROPIC MATERIAL SUPPORTED AT ITS PERIPHERY AND SUBMITTED TO A UNIFORM LATERAL LOAD

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We shall assume the plate to be of uniform thickness h, and take x-, y-axes in the plane of the plate before bending, the directions of which are chosen in the directions of the symmetrical axes of the material.

Then the deflection of the plate w satisfies the differential equation[1)]

$$D_1\frac{\partial^4 w}{\partial x^4}+2(D_2+D_4)\frac{\partial^4 w}{\partial x^2\partial y^2}+D_3\frac{\partial^4 w}{\partial y^4}=q \tag{1}$$

where q is the intensity of a uniform lateral load, D_1, D_2, D_3 and D_4 are constants derived from the elastic moduli.

If x', y' and ϑ represent the axes of the ellipse and the angle between x and x', respectively, then

$$x'+iy'=e^{i\vartheta}(x+iy). \tag{2}$$

Transform the coordinates into the following systems of curvilinear coordinates

$$\left.\begin{aligned}
x+iy&=ce^{-i\vartheta}\cosh(\alpha+i\beta)\\
x+ik_1y&=c'e^{-i\vartheta'}\cosh(\alpha'+i\beta'+i\beta_1)\\
x+ik_2y&=c''e^{-i\vartheta''}\cosh(\alpha''+i\beta''+i\beta_2)
\end{aligned}\right\} \tag{3}$$

where c', c'', ϑ', ϑ'', β_1 and β_2 are arditrary constants, and k_1 and k_2 are as follows,

$$k_1=\frac{D_1^{1/2}}{\sqrt{D_2+D_4+\{(D_2+D_4)^2-D_1D_3\}^{1/2}}},\qquad k_2=\frac{D_1^{1/2}}{\sqrt{D_2+D_4-\{(D_2+D_4)^2-D_1D_3\}^{1/2}}}.$$

Assume that the curves $\alpha=\alpha_0$ and $\alpha'=\alpha_0'$ represent the periphery of the plate and that $\beta=\beta'$ at the boundary, and we have

$$\left.\begin{aligned}
c\cosh\alpha_0\cos\vartheta&=c'\cosh\alpha_0'\cos\vartheta'\cos\beta_1+c'\sinh\alpha_0'\sin\vartheta'\sin\beta_1,\\
c\sinh\alpha_0\sin\vartheta&=-c'\cosh\alpha_0'\cos\vartheta'\sin\beta_1+c'\sinh\alpha_0'\sin\vartheta'\cos\beta_1,\\
k_1c\sinh\alpha_0\cos\vartheta&=c'\sinh\alpha_0'\cos\vartheta'\cos\beta_1+c'\cosh\alpha_0'\sin\vartheta'\sin\beta_1,\\
k_1c\cosh\alpha_0\sin\vartheta&=-c'\sinh\alpha_0'\cos\vartheta'\sin\beta_1+c'\cosh\alpha_0'\sin\vartheta'\cos\beta_1.
\end{aligned}\right\} \tag{4}$$

from which

$$(k_1^2-1)\sinh 2\alpha_0 \sin 2\vartheta \tan^2\beta_1 + 4\{(k_1^2-1)(\cosh^2\alpha_0 \sin^2\vartheta - \sinh^2\alpha_0 \cos^2\vartheta)+1\}\tan\beta_1 - (k_1^2-1)\sinh 2\alpha_0 \sin 2\vartheta = 0 \quad (5)$$

Solving the equation (5), we have two solutions for $\tan\beta_1$, but the difference of the corresponding values of β_1 is $\pi/2$, and so it is sufficient for further calucu-lations if we take only one value of the two.

Substituting the one value of β_1 into (4), we can find α_0', ϑ and c'. In the same manner, we can determine β_2, α_0'', ϑ'' and c'' so as to satisfy the cognate conditions. If we determine the constants in this way, $\alpha=\alpha_0$, $\alpha'=\alpha_0'$, and $\alpha''=\alpha_0''$ represent respectively the periphery of the plate and $\beta=\beta'=\beta''$ at the boundary.

Taking the deflection of the plate in the expression

$$\left.\begin{aligned} w &= R[f_1(x+ik_1y)+f_2(x+ik_2y)] \\ &\quad + C_1x'^4 + C_2x'^3y' + C_3x'^2y'^2 + C_4x'y'^3 + C_5y'^4 + C_6x'^2 + C_7x'y' + C_8y'^2 + C_9 \\ &\text{and} \\ f_1''(x+ik_1y) &= \sum_{n=2}^{\infty}(A_n+iA_n')\cosh 2n(\alpha+i\beta'+i\beta_1) \\ f_2''(x+ik_2y) &= \sum_{n=2}^{\infty}(B_n+iB_n')\cosh 2n(\alpha''+i\beta''+i\beta_2) \end{aligned}\right\} \quad (6)$$

we have

$$\begin{aligned} w &= \frac{c'^2}{4}\sum_{n=2}^{\infty}(A_n\cos 2\vartheta' + A_n'\sin 2\vartheta')\Big[\frac{\cosh(2n+2)\alpha'}{(2n+2)(2n+1)}\cos(2n+2)(\beta'+\beta_1) \\ &\quad - \left\{\frac{1}{(2n+1)2n}+\frac{1}{2n(2n-1)}\right\}\cosh 2n\alpha' \cos 2n(\beta'+\beta_1) \\ &\quad + \frac{\cosh(2n-2)\alpha'}{(2n-1)(2n-2)}\cos(2n-2)(\beta'+\beta_1)\Big] \\ &\quad + \frac{c'^2}{4}\sum_{n=2}^{\infty}(A_n\sin 2\vartheta' - A_n'\cos 2\vartheta')\Big[\frac{\sinh(2n+2)\alpha'}{(2n+2)(2n+1)}\sin(2n+2)(\beta'+\beta_1) \\ &\quad - \left\{\frac{1}{(2n+1)2n}+\frac{1}{2n(2n-1)}\right\}\sinh 2n\alpha' \sin 2n(\beta'+\beta_1) \\ &\quad + \frac{\sinh(2n-2)\alpha'}{(2n-1)(2n-2)}\sin(2n-2)(\beta'+\beta_1)\Big] \\ &\quad + \frac{c''^2}{4}\sum_{n=2}^{\infty}(B_n\cos 2\vartheta'' + B_n'\sin 2\vartheta'')\Big[\frac{\cosh(2n+2)\alpha''}{(2n+2)(2n+1)}\cos(2n+2)(\beta''+\beta_2) \\ &\quad - \left\{\frac{1}{(2n+1)2n}+\frac{1}{2n(2n-1)}\right\}\cosh 2n\alpha'' \cos 2n(\beta''+\beta_2) \\ &\quad + \frac{\cosh(2n-2)\alpha''}{(2n-1)(2n-2)}\cos(2n-2)(\beta''+\beta_2)\Big] \\ &\quad + \frac{c''^2}{4}\sum_{n=2}^{\infty}(B_n\sin 2\vartheta'' - B_n'\cos 2\vartheta'')\Big[\frac{\sinh(2n+2)\alpha''}{(2n+2)(2n+1)}\sin(2n+2)(\beta''+\beta_2) \\ &\quad - \left\{\frac{1}{(2n+1)2n}+\frac{1}{2n(2n-1)}\right\}\sinh 2n\alpha'' \sin 2n(\beta''+\beta_2) \end{aligned}$$

$$+\frac{\sinh(2n-2)\alpha''}{(2n-1)(2n-2)}\sin(2n-2)(\beta''+\beta_2)$$
$$+C_1x'^4+C_2x'^3y'+C_3x'^2y'^2+C_4x'y'^3+C_5y'^4+C_6x'^2+C_7x'y'+C_8y'^2+C_9 \tag{7}$$

To satisfy the differential equation (1), we have

$$\{3C_1D_1+C_3(D_2+D_4)+3C_5D_3\}\cos^4\vartheta+3\{C_2(D_1-D_2-D_4)-C_4D_3\}\cos^3\vartheta\sin\vartheta$$
$$+\{3C_1(D_1+D_2+D_3+D_4)+(6C_1-7C_3+6C_5)(D_2+D_4)\}\cos^2\vartheta\sin^2\vartheta$$
$$+3\{C_2(D_2-D_3+D_1)+C_4D_1\}\sin^3\vartheta\cos\vartheta$$
$$+\{3C_1D_3+C_3(D_2+D_4)+3C_5D_1\}\sin^4\vartheta=\frac{q}{8} \tag{8}$$

From the condition that $w=0$ at the boundary, we find

$$3C_1c^4\cosh^4\alpha_0+C_3c^4\sinh^2\alpha_0\cosh^2\alpha_0+3C_5c^4\sinh^4\alpha_0$$
$$+4C_6c^2\cosh^2\alpha_0+4C_8c^2\sinh^2\alpha_0+8C_9=0,$$
$$C_1c^4\cosh^4\alpha_0-C_5c^4\sinh^4\alpha_0+C_6c^2\cosh^2\alpha_0-C_8c^2\sinh^2\alpha_0$$
$$+\frac{1}{12}(A_2\cos2\vartheta'+A_2'\sin2\vartheta')c'^2\cosh2\alpha_0'\cos2\beta_1$$
$$+\frac{1}{12}(A_2\sin2\vartheta'-A_2'\cos2\vartheta')c'^2\sinh2\alpha_0'\sin2\beta_1$$
$$+\frac{1}{12}(B_2\cos2\vartheta''+B_2'\sin2\vartheta'')c''^2\cosh2\alpha_0''\cos2\beta_2$$
$$+\frac{1}{12}(B_2\sin2\vartheta''-B_2'\cos2\vartheta'')c''^2\sinh2\alpha_0''\sin2\beta_2=0,$$
$$C_2c^4\sinh\alpha_0\cosh^3\alpha_0+C_4c^4\sinh^3\alpha_0\cosh\alpha_0+2C_7c^2\sinh\alpha_0\cosh\alpha_0$$
$$-\frac{1}{6}(A_2\cos2\vartheta'+A_2'\sin2\vartheta')c'^2\cosh2\alpha_0'\sin2\beta_1$$
$$+\frac{1}{6}(A_2\sin2\vartheta'-A_2'\cos2\vartheta')c'^2\sinh2\alpha_0'\cos2\beta_1$$
$$-\frac{1}{6}(B_2\cos2\vartheta''+B_2'\sin2\vartheta'')c''^2\cosh2\alpha_0''\sin2\beta_2$$
$$+\frac{1}{6}(B_2\sin2\vartheta''-B_2'\cos2\vartheta'')c''^2\sinh2\alpha_0''\cos2\beta_2=0,$$
$$C_1c^4\cosh^4\alpha_0-C_3c^4\sinh^2\alpha_0\cosh^2\alpha_0+C_5c^4\sinh^4\alpha_0$$
$$+\left\{\left(\frac{A_3}{10}-\frac{4A_2}{15}\right)\cos2\vartheta'+\left(\frac{A_3'}{10}-\frac{4A_2}{15}\right)\sin2\vartheta'\right\}c'^2\cosh4\alpha_0'\cos4\beta_1$$
$$+\left\{\left(\frac{A_3}{10}-\frac{4A_2}{15}\right)\sin2\vartheta'-\left(\frac{A_3'}{10}-\frac{4A_2'}{15}\right)\cos2\vartheta'\right\}c'^2\sinh4\alpha_0'\sin4\beta_1$$
$$+\left\{\left(\frac{B_3}{10}-\frac{4B_2}{15}\right)\cos2\vartheta''+\left(\frac{B_3'}{10}-\frac{4B_2'}{15}\right)\sin2\vartheta''\right\}c''^2\cosh4\alpha_0''\cos4\beta_2$$
$$+\left\{\left(\frac{B_3}{10}-\frac{4B_2}{15}\right)\sin2\vartheta''-\left(\frac{B_3'}{10}-\frac{4B_2'}{15}\right)\cos2\vartheta''\right\}c''^2\sinh4\alpha_0''\sin4\beta_2$$
$$=0,$$

$$
\left.\begin{aligned}
&C_2c^4 \sinh \alpha_0 \cosh^3 \alpha_0 - C_4c^4 \sinh^3 \alpha_0 \cosh \alpha_0 \\
&\quad - \left\{\left(\frac{A_3}{10} - \frac{4A_2}{15}\right)\cos 2\vartheta' + \left(\frac{A_3'}{10} - \frac{4A_2'}{15}\right)\sin 2\vartheta'\right\} c'^2 \cosh 4\alpha_0' \sin 4\beta_1 \\
&\quad + \left\{\left(\frac{A_3}{10} - \frac{4A_2}{15}\right)\sin 2\vartheta' - \left(\frac{A_3'}{10} - \frac{4A_2'}{15}\right)\cos 2\vartheta'\right\} c'^2 \sinh 4\alpha_0' \cos 4\beta_1 \\
&\quad - \left\{\left(\frac{B_3}{10} - \frac{4B_2}{15}\right)\cos 2\vartheta'' + \left(\frac{B_3'}{10} - \frac{4B_2'}{15}\right)\sin 2\vartheta''\right\} c''^2 \cosh 4\alpha_0'' \sin 4\beta_2 \\
&\quad + \left\{\left(\frac{B_3}{10} - \frac{4B_2}{15}\right)\sin 2\vartheta'' - \left(\frac{B_3'}{10} - \frac{4B_2'}{15}\right)\cos 2\vartheta''\right\} c''^2 \sinh 4\alpha_0'' \cos 4\beta_2 \\
&\quad = 0, \\
&\left\{\left(\frac{A_4}{21} - \frac{4A_3}{35} + \frac{A_2}{15}\right)\cos 2\vartheta'\right. \\
&\quad \left. + \left(\frac{A_4'}{21} - \frac{4A_3'}{35} + \frac{A_2'}{15}\right)\sin 2\vartheta'\right\} c'^2 \cosh 6\alpha_0' \cos 6\beta_1 \\
&\quad + \left\{\left(\frac{A_4}{21} - \frac{4A_3}{35} + \frac{A_2}{15}\right)\sin 2\vartheta'\right. \\
&\quad \left. - \left(\frac{A_4'}{21} - \frac{4A_3'}{35} + \frac{A_2'}{15}\right)\cos 2\vartheta'\right\} c'^2 \sinh 6\alpha_0' \sin 6\beta_1 \\
&\quad + \left\{\left(\frac{B_4}{21} - \frac{4B_3}{35} + \frac{B_2}{15}\right)\cos 2\vartheta''\right. \\
&\quad \left. + \left(\frac{B_4'}{21} - \frac{4B_3'}{35} + \frac{B_2'}{15}\right)\sin 2\vartheta''\right\} c''^2 \cosh 6\alpha_0'' \cos 6\beta_2 \\
&\quad + \left\{\left(\frac{B_4}{21} - \frac{4B_3}{35} + \frac{B_2}{15}\right)\sin 2\vartheta''\right. \\
&\quad \left. - \left(\frac{B_4'}{21} - \frac{4B_3'}{35} + \frac{B_2'}{15}\right)\cos 2\vartheta''\right\} c''^2 \sinh 6\alpha_0'' \sin 6\beta_2 = 0, \\
&- \left\{\left(\frac{A_4}{21} - \frac{4A_3}{35} + \frac{A_2}{15}\right)\cos 2\vartheta'\right. \\
&\quad \left. + \left(\frac{A_4'}{21} - \frac{4A_3'}{35} + \frac{A_2'}{15}\right)\sin 2\vartheta'\right\} c'^2 \cosh 6\alpha_0' \sin 6\beta_1 \\
&\quad + \left\{\left(\frac{A_4}{21} - \frac{4A_3}{35} + \frac{A_2}{15}\right)\sin 2\vartheta'\right. \\
&\quad \left. - \left(\frac{A_4'}{21} - \frac{4A_3'}{35} + \frac{A_2'}{15}\right)\cos 2\vartheta'\right\} c'^2 \sinh 6\alpha_0' \cos 6\beta_1 \\
&\quad - \left\{\left(\frac{B_4}{21} - \frac{4B_3}{35} + \frac{B_2}{15}\right)\cos 2\vartheta''\right. \\
&\quad \left. + \left(\frac{B_4'}{21} - \frac{4B_3'}{35} + \frac{B_2'}{15}\right)\sin 2\vartheta''\right\} c''^2 \cosh 6\alpha_0'' \sin 6\beta_2 \\
&\quad + \left\{\left(\frac{B_4}{21} - \frac{4B_3}{35} + \frac{B_2}{15}\right)\sin 2\vartheta''\right. \\
&\quad \left. - \left(\frac{B_4'}{21} - \frac{4B_3'}{35} + \frac{B_2'}{15}\right)\cos 2\vartheta''\right\} c''^2 \sinh 6\alpha_0'' \cos 6\beta_2 = 0, \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \quad (9)
$$

and from the condition that $M_\alpha = 0$ at the boundary, we have the relation

$$\{(D_1 + D_2) + (D_1 - D_2)\cos 2(\phi - \vartheta)\}\frac{\partial^2 w}{\partial x^2}$$
$$+ \{D_2 + D_3 + (D_2 - D_2)\cos 2(\phi - \vartheta)\}\frac{\partial^2 w}{\partial y^2}$$
$$+ 2\,D_4 \sin\,2(\phi - \vartheta)\frac{\partial^2 w}{\partial x \partial y} = 0 \tag{10}$$

where

$$\cos 2(\phi - \vartheta) = \frac{\cosh 2\,\alpha_0 \cos 2\,\vartheta \cos 2\,\beta + \sinh 2\,\alpha_0 \sin 2\,\vartheta \sin 2\,\beta - \cos 2\,\vartheta}{\cosh 2\,\alpha_0 - \cos 2\,\beta}$$

$$\sin\,2(\phi - \vartheta) = \frac{\sinh 2\,\alpha_0 \cos 2\,\vartheta \sin 2\,\beta - \cosh 2\,\alpha_0 \sin 2\,\vartheta \cos 2\,\beta + \sin 2\,\vartheta}{\cosh 2\,\alpha_0 - \cos 2\,\beta}$$

Mathematical difficulties are encountered in determining the constants $C_1, C_2, \ldots, C_9, A_2, B_2, A_2', B_2', \ldots$, to satisfy the equations (8), (9) and (10), under the condition that the functions f_1 and f_2 converge, so we shall find an approximate solution by the method shown in the following example.

As an example, we consider an oak plate of unit thickness cut parallel to the grain of wood and take the y-axis in the direction of the fibres, then it becomes[2]

$$D_1 = 8.582, \qquad D_2 = 4.341, \qquad D_3 = 50.65, \qquad D_4 = 13.02$$

from which

$$k_1^2 = 0.3428 - 0.2279\,i, \qquad k_2^2 = 0.3428 + 0.2279\,i.$$

Taking the ratio of major and minor axes of the ellipse as 3 : 1, we treat first the case of $\vartheta = 0°$, since the procedure is quite similar in other cases.

As the first approximation, assuming

$$A_2 = B_2 = A_2' = B_2' = A_3 = B_3 = A_3' = B_3' = \ldots = 0$$

from the equations (8), (9), and (10)

$$C_1 = 0.1868 \times 10^{-4} q, \qquad C_2 = -0.7559 \times 10^{-4} q, \qquad C_5 = 8.3443 \times 10^{-4} q$$
$$C_6 = -2.9830 \times 10^{-4} q b^2, \quad C_8 = -50.3108 \times 10^{-4} q b^2, \quad C_9 = 41.9802 \times 10^{-4} q b^4$$

The residual bending moment at the boundary is

$$M_\alpha = -\frac{0.02056 \cos 4\beta}{1.250 - \cos 2\,\beta}\,q b^2,$$

as the second approximation, assuming

$$A_3 = B_3 = A_3' = B_3' = A_4 = B_4 = A_4' = B_4' = \ldots = 0,$$

we have

$$C_1 = 0.3986\times10^{-4}q,\qquad C_3 = 4.4502\times10^{-4}q,\qquad C_5 = 7.6533\times10^{-4}q,$$
$$C_6 = -7.7790\times10^{-4}qb^2,\quad C_8 = -45.4191\times10^{-4}qb^2,\quad C_9 = 37.7367\times10^{-4}qb^4,$$
$$A_2 = -(0.03698-0.11734\,i)\times10^{-4}qb^2,\quad B_2 = -(0.03698+0.11734\,i)\times10^{-4}qb^2$$

and the residual moment is

$$M_\alpha = -\frac{0.00020\cos 6\beta}{1.250-\cos 2\beta}\,qb^2.$$

This shows that the residual moment decreases rapidly according as we take more terms in the series in (7), and so if we take only a few terms of the series we can find an approximate solution with sufficient accuracy.

The deflections on the axes are calculated using the results for second approximation. They are plotted in Fig. 1.

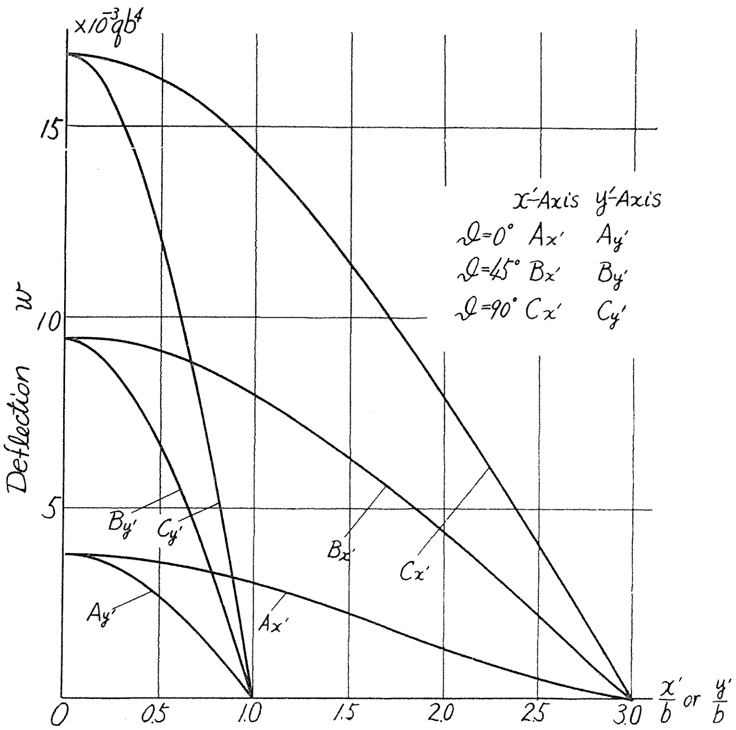

FIG. 1. Deflection curves on the axes of the elliptic plate.

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References

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