ON A NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATION, $\varDelta X = Y$, BY MEANS OF FINITE DIFFERENCE METHOD

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1. The method of finite difference has been applied to various problems of partial differential equations by many authors, however in choosing this method in practice, several troubles must be overcome; The one of them is to obtain an expression of finite difference that has as high accuracy as possible, and the next is to get an expression of the boundary condition with sufficient accuracy.

Here let us consider an equation of the following type,

$$\varDelta X = Y \qquad \varDelta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{1}$$

where X is a function of x and y, and it may be assumed that Y is a known function of x, y and X in general. For instance, problems of torsion of prismatical bars can be solved by taking X the stress function and Y a pure constant. In the case of vibration of membrane, Y is equal to $-\lambda X$ where λ is a constant which may be decided.

For the equation (1) present a new expression to apply the satisfactorily. Our method will be especially convenient to such problems where a given form of the boundary can be translated into a simple figure by means of conformal translation, since the becomes to the same form as (1).

2. From relations between functional values at the intersecting points of square network (interval h), we can write the following equations.

$$X(10) + X(\bar{1}0) + X(01) + X(0\bar{1}) - 4X(00) =$$
$$h^2 \varDelta X + \frac{h^4}{12}\left(\varDelta\varDelta X - 2\frac{\partial^4 X}{\partial x^2 \partial y^2}\right) + \frac{h^6}{360}\left(\varDelta\varDelta\varDelta X - 3\frac{\partial^4 \varDelta X}{\partial x^2 \partial y^2}\right) + 0(h^8),$$

$$X(11) + X(1\bar{1}) + X(\bar{1}1) + X(\bar{1}\bar{1}) - 4X(00) =$$
$$2h^2 \varDelta X + \frac{h^4}{6}\left(\varDelta\varDelta X + 4\frac{\partial^4 X}{\partial x^2 \partial y^2}\right) + \frac{h^6}{180}\left(\varDelta\varDelta\varDelta X + 12\frac{\partial^4 \varDelta X}{\partial x^2 \partial y^2}\right) + 0(h^8),$$

where

$$X(m, n) = X(x + mh, y + nh). \tag{2}$$

Then we arrive to

$$4\,X(10)+4(\bar{1}0)+4(01)+4(0\bar{1})+X(11)+X(1\bar{1})+X(\bar{1}1)+X(\bar{1}\bar{1})-20X(00)$$
$$=6\,h^2\varDelta X+\frac{1}{2}h^4\varDelta\varDelta X+\frac{1}{60}h^6\varDelta\varDelta\varDelta X+\frac{1}{30}h^6\frac{\partial^4\varDelta X}{\partial x^2\partial y^2}+0(h^8),$$
$$=6\,h^2Y+\frac{1}{2}h^4\varDelta Y+\frac{1}{60}h^6\varDelta\varDelta Y+\frac{1}{30}h^6\frac{\partial^4 Y}{\partial x^2\partial y^2}+0(h^8). \tag{3}$$

The above expression, however, cannot be used at points near by the boundary because some points around (00) will be forced out of the boundary. Then we write the following equation from relations between values at the points around,

$$\begin{aligned}X(\xi,\eta)=\;&X(00)(1-\xi^4-\eta^4-\xi^2\eta^2)\\
&+X(10)(4\xi+3\xi^2-3\eta^2+2\xi^3-6\xi\eta^2+3\xi^4-6\xi^2\eta^2+3\eta^4)\frac{1}{12}\\
&+X(\bar{1}0)(-4\xi+3\xi^2-3\eta^2-2\xi^3+6\xi\eta^2+3\xi^4-6\xi^2\eta^2+3\eta^4)\frac{1}{12}\\
&+X(01)(4\eta-3\xi^2+3\eta^2+2\eta^3-6\xi^2\eta+3\xi^4-6\xi^2\eta^2+3\eta^4)\frac{1}{12}\\
&+X(0\bar{1})(-4\eta-3\xi^2+3\eta^2-2\eta^3+6\xi^2\eta+3\xi^4-6\xi^2\eta^2+3\eta^4)\frac{1}{12}\\
&+X(11)(\xi+\eta+3\xi\eta-\xi^3+3\xi\eta^2-\eta^3+3\xi^2\eta+3\xi^2\eta^2)\frac{1}{12}\\
&+X(\bar{1}1)(-\xi+\eta-3\xi\eta+\xi^3-3\xi\eta^2-\eta^3+3\xi^3\eta+3\xi^2\eta^2)\frac{1}{12}\\
&+X(1\bar{1})(\xi-\eta-3\xi\eta-\xi^3+3\xi\eta^2+\eta^3-3\xi^2\eta+3\xi^2\eta^2)\frac{1}{12}\\
&+X(\bar{1}\bar{1})(-\xi-\eta+3\xi\eta+\xi^3-3\xi\eta^2+\eta^3-3\xi^2\eta+3\eta^2\xi^2)\frac{1}{12}\\
&+h^4\frac{1}{12}(\xi^3\eta-\xi\eta^3)\left(\frac{\partial^4X}{\partial x^3\partial y}-\frac{\partial^4X}{\partial x\partial y^3}\right)+\frac{1}{4}(\xi^3+\eta^2-\xi^4-\eta^4)\,Yh^2\\
&+\frac{1}{6}(-\xi+\xi^3)h^3\frac{\partial Y}{\partial x}+\frac{1}{6}(-\eta+\eta^3)h^3\frac{\partial Y}{\partial y}+\frac{1}{48}(-\xi^2+\eta^2+\xi^4-\eta^4)h^4\frac{\partial^2Y}{\partial x^2}\\
&+\frac{1}{48}(\xi^2-\eta^2-\xi^4+\eta^4)h^4\frac{\partial^2Y}{\partial y^2}+\frac{1}{12}(-2\xi\eta+\xi^3\eta+\xi\eta^3)h^4\frac{\partial^2Y}{\partial x\partial y}+0(h^5).\end{aligned}\tag{4}$$

Considerig the condition at the intersectiong points between the boundary and the straight line through (00) and (mn) as shown in Fig. 1, we arrive to an expression where are used functional values only at the internal points.

If the boundary is, particularly, given as a straight line through the points (00) and (10), ($\bar{1}$0) or (01), (0$\bar{1}$), the following relation will be available,

$$2\,X(01)+2\,X(0\bar{1})+4\,X(10)+X(11)+X(1\bar{1})-10X(00)$$
$$=6h\frac{\partial X}{\partial x}+3\,h^2Y+h^3\frac{\partial Y}{\partial x}+\frac{h^4}{4}\frac{\partial^2Y}{\partial x^2}+\frac{h^4}{4}\frac{\partial^2Y}{\partial y^2}+0(h^5). \tag{5}$$

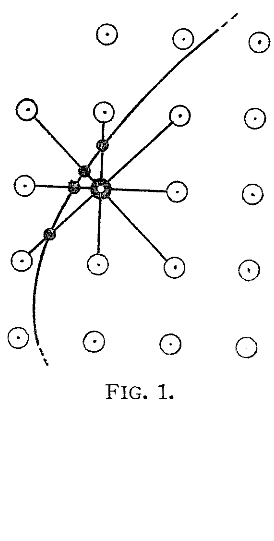

FIG. 1.

In the case where Y is a function of X, the terms of Y included in the above expressions may be rewritten using the relations;

$$Y(10)-Y(00)=h\frac{\partial Y}{\partial x}+\frac{h^2}{2}\frac{\partial^2 Y}{\partial x^2}+\frac{h^3}{6}\frac{\partial^3 Y}{\partial x^3}+\dots \tag{6}$$

Then we have for Eq. (3)

$$\begin{array}{|c|c|c|}\hline 1 & 4 & 1 \\ \hline 4 & -20 & 4 \\ \hline 1 & 4 & 1 \\ \hline \end{array}\,X = \begin{array}{ccc} & \frac{1}{2} & \\ \frac{1}{2} & 4 & \frac{1}{2} \\ & \frac{1}{2} & \end{array}\,h^2Y \;\; 0(h^6) \quad \text{or} \quad \begin{array}{ccccc} & & -\frac{1}{40} & & \\ & \frac{1}{15} & \frac{7}{15} & \frac{1}{15} & \\ -\frac{1}{40} & \frac{7}{15} & \frac{119}{30} & \frac{7}{15} & -\frac{1}{40} \\ & \frac{1}{15} & \frac{7}{15} & \frac{1}{15} & \\ & & -\frac{1}{40} & & \end{array}\,h^2Y \;\; 0(h^8), \tag{7}$$

and

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline -10 & 4 \\ \hline 2 & 1 \\ \hline \end{array}\,X = \begin{array}{cc} \frac{1}{4} & \\ 2 & \frac{1}{2} \\ \frac{1}{4} & \end{array}\,h^2Y + 6h\frac{\partial X}{\partial x}+\frac{1}{2}h^3\frac{\partial Y}{\partial x}+0(h^5) \tag{8}$$

for Eq. (5).

3. Applications

i) Torsion of a prismatical bar; the fundamental equation is

$$\Delta_{xy}F=C,$$

and the boundary condition is $F=0$.
By means of conformal translation the equation becomes

$$\Delta_{\alpha\beta}F=\frac{c}{f^2}, \tag{9}$$

where

$$f^2=\frac{\partial(\alpha,\beta)}{\partial(x,y)}.$$

ii) Torsion of a circular shaft of variable diameter; the following equation must be solved considering the boundary condition, $\phi=\text{const.}$

$$\frac{\partial^2\phi}{\partial r^2}-\frac{3}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial z^2}=0.$$

Putting $\phi = r^{3/2}\psi$, we have $\dfrac{\partial^2\psi}{\partial r^2} - \dfrac{15}{4}\dfrac{\psi}{r^2} + \dfrac{\partial^2\psi}{\partial z^2} = 0,$

and it may be transformed into

$$\Delta_{xy}\psi = \frac{15}{4}\frac{\psi}{r^2 f^2} \tag{10}$$

iii) Vibration problem of a membrane; the equation $\Delta w + \lambda w = 0$ and the boundary condition $w = 0$. So we get by the same procedure as above,

$$\Delta_{\alpha\beta} w + \frac{\lambda w}{f^2} = 0. \tag{11}$$

iv) Oscillation of water; the equation is

$$\frac{\partial}{\partial x}\left(H\frac{\partial\zeta}{\partial x}\right) + \frac{\partial}{\partial y}\left(H\frac{\partial\zeta}{\partial y}\right) + \lambda\zeta = 0$$

where $H(x, y)$ is depth of water, being a known function of x and y.

Substituting $z = \sqrt{H}\,\zeta$, we have

$$\Delta z + \left(\frac{\lambda}{H} - \frac{1}{\sqrt{H}}\Delta\sqrt{H}\right)z = 0. \tag{12}$$

Hereupon, as the boundary condition,

$$H\frac{\partial\zeta}{\partial n} = 0, \quad \text{or} \quad \sqrt{H}\,\frac{\partial z}{\partial n} = \frac{\partial\sqrt{H}}{\partial n}z.$$

These problems may be successfully solved by our method.