THE THREE DIMENSIONAL BUCKLING PROBLEMS OF STRAIGHT RODS UNDER AXIAL AND/OR TORSIONAL LOADS

ISAMU IMACHI

Department of Chemical Engineering

(Received April 20, 1954)

Summary

The following problems of elastic instability are theoretically discussed with some calculations and experiments:

(a) The buckling under compression of a rod with ends hinged in directions not coincident with its principal axes.

(b) The buckling under compression of a rod consisting of two parts, their directions of principal axes not coinciding with each other.

(c) General discussion on bucklings of a straight uniform rod having some initial twist under axial, torsional or combined loads.

(d) The same problem as (c) when two principal bending rigidities are equal.

(e) The buckling under compression of a rod with some initial twist when two principal bending rigidities are unequal.

(f) The same problem as (c) when there is no initial twist, especially of a strip.

I. Introduction

It appears that the three dimensional buckling problems of rods have not yet been fully investigated and we believe that some of the problems discussed herein are new and fairy interesting in themselves in respect of the theory of elastic instability. The buckling problem of a strip under torsion combined with a tension has been solved by A. E. Green[1] by expanding its deformation in Fourier's series, but his treatment is much too complicated to become familiar to engineers. As a matter of fact, if the effects of sectional deformations are neglected, exact solutions can be obtained theoretically even when large torsional deformations exists. We shall develop our own series of theoretical investigations on problems belonging to this same category.

II. Notations

- x, y, z: rectangular coordinates fixed to the space, x coincident with the initial direction of the center line of the rod,
- s, ρ, n: rectangular coordinates showing directions of the tangent, principal normal, and bi-normal respectively relating to the center line of the rod,
- 1, 2: directions of sectional principal axes,
- λ_i, μ_i, ν_i: direction-cosines of i-axis relating to the x, y, z-coordinate system,

y, z : flexural deformations,

$\frac{1}{\rho}, \frac{1}{\rho_1}, \frac{1}{\rho_2}$: principal curvature and curvatures with respect to 1 and 2-axes,

$\frac{1}{\tau}$: rate of twist of the center line curve,

α : angular direction of l-axis measured from ρ-axis,

ψ : angular direction of ρ-axis measured (approximately) from y-axis (In Secs. III and IV this symbol is used for a somewhat different meaning.),

$\phi = \varphi' l$: one-half total angle of twist between both ends,

l : one-half length of the bar,

EI_1, EI_2 : principal bending rigidities with respect to 1 and 2-axes,

GJ_e : torsional rigidity,

$X = -P, Y, Z$: tension and shears in directions x, y and z,

T, S_1, S_2 : tension in s, shearing forces in directions 1 and 2-axes,

M_i : moment, suffix i denoting the axis to which the moment is refered,

$Q = M_s$: torque,

$$m = EI_2/EI_1, \quad r = GJ_e/EI_2, \quad k^2 = k_2^2 = -X/EI_2, \quad k_1^2 = -X/EI_1, \quad h = Q/GJ_e.$$

In general, [$'$] denotes the derivative with respect to s or approximately with respect to x.

III. The buckling under compression of a uniform non-twisted rod when the ends are hinged in directions not coincident with any of its principal axes

The general from of lateral deflection of a uniform, non-twisted rod buckled under axial compression can always be expressed as:

$$\left.\begin{aligned} y &= A_2 \cos k_2 x + B_2 \sin k_2 x + C_2 k_2 x + D_2, \\ z &= A_1 \cos k_1 x + B_1 \sin k_1 x + C_1 k_1 x + D_1, \end{aligned}\right\} \tag{1}$$

where y denotes the deflection in direction 1 and z in direction 2. When the end-conditions are asymmetric with respect to any of the principal axes, y- and z-deflections cannot exist independently but they form a space curve.

1. When both ends are hinged in direction with an angle φ from the principal axis 1

Let $2l$ be the length of the bar and take the origin of x at mid-point. The end-conditions are:

$$\text{At} \quad x = \pm l; \qquad \left.\begin{aligned} y = z = 0, \quad z' \sin\varphi + y' \cos\varphi = 0, \\ M_1 \cos\varphi + M_2 \sin\varphi = 0. \end{aligned}\right\} \tag{2}$$

Applying these conditions to Eqs. (1) we obtain the following critical conditins of buckling:

$$k_1 \tan k_1 l \sin^2\varphi + k_2 \tan k_2 l \cos^2\varphi = 0, \tag{3}$$

$$(1 - k_1 l \cot k_1 l)\sin^2\varphi + (1 - k_2 l \cot k_2 l)\cos^2\varphi = 0. \tag{4}$$

The calculated results of Eq. (3) together with our experimental results are

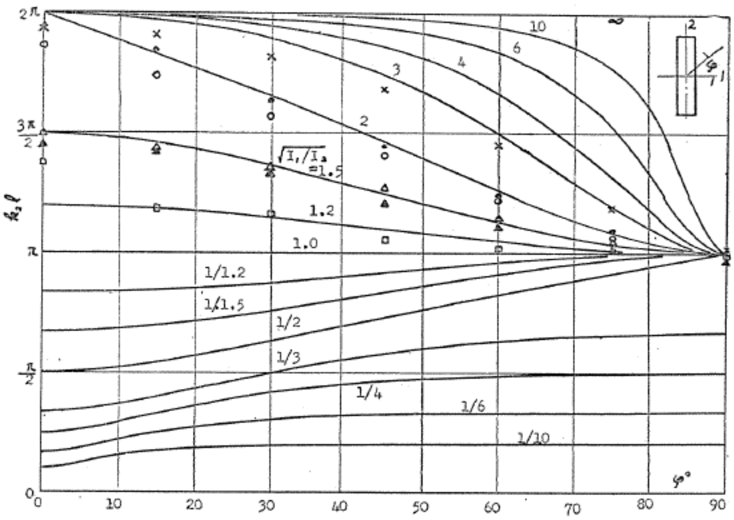


FIG. 1. Buckling under Compression, Ends Obliquely Hinged, $\varphi_1=\varphi_2$. The marks show the experimental results on wooden (Hinoki) struts, by K. Konishi.

plotted in Fig. 1. Eq. (4) always gives greater values of critical loads than Eq. (3).

In special cases where $\varphi=0$, Eqs. (3) and (4) give:

$$\sin k_2 l=0, \quad \cos k_1 l=0, \quad \sin k_1 l=0, \quad \text{or} \quad \cos k_2 l-\sin k_2 l/k_2 l=0.$$

These conditions relate to ordinary problems of a column.

If one of the bending rigidities becomes very large, $EI_1=\infty$ e.g., the deflection z cannot exist and buckling conditions are obtained by putting $k_1=0$. Thus, $\sin k_2 l=0$ regardless of φ except when $\varphi=90°$. In this case a slight deviation of φ from 90° causes an abrupt change in the buckling load to 4 P_e from P_e, which is the value when $\varphi=90°$.

2. When the hinge line at one end is perpendicular to the hinge line at the other

The end-conditions then are:

$$\text{At} \quad x=\pm l: \quad y=z=0, \quad y'\cos\varphi \pm z'\sin\varphi=0, \quad M_1\cos\varphi \pm M\sin\varphi=0, \tag{5}$$

which lead to the following critical conditions:

$$\begin{aligned}&(1-k_2 l\cot k_2 l+k_1 l\tan k_1 l)(1-k_1 l\cot k_1 l+k_2 l\tan k_2 l)\\ &=\cos^2 2\varphi(k_1 l\cot k_1 l-k_2 l\cot k_2 l)(k_2 l\tan k_2 l-k_1 l\tan k_1 l).\end{aligned} \tag{6}$$

When $\varphi=0$ or $\pi/2$ Eq. (6) is reduced to:

$$(1-2\,k_1 l\cot 2k_1 l)(1-2\,k_2 l\cot 2k_2 l)=0 \tag{7}$$

indicating that the buckling occurs independently in either of the planes of symmetry as an ordinary strut with one end hinged and the other end fixed.

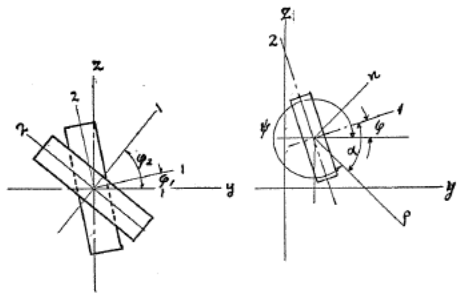
3. When the hinge angle is φ at one end and $-\varphi$ at the other end

With similar treatment as the foregoing, we get the two following critical conditions:

$$\tan^2\varphi\cdot k_1 l\tan k_1 l+(1-k_2 l\cot k_2 l)=0,$$
$$\cot^2\varphi\cdot k_2 l\tan k_2 l+(1-k_1 l\cot k_1 l)=0.$$

IV. The buckling under compression of a rod consisting of two parts, their directions of principal axes not coinciding with each other

Consider a case where a bar with a length of $2\,l=l_1+l_2$, constant sectional bending rigidities EI_1 and EI_2 throughout the length, the direction of $1-$axis however making an anfle φ_1 from y-direction for $-l_1<x<0$ and an angle φ_2 for $0<x<l_2$, is subjected to an axial compression. The jointed portion $x=0$ is assumed to have sufficient rigidity for continuity. If we use a symbol i in place of 1 or 2 which indicates the part of the bar, for convenience, the deflections in directions 1 and 2 must be expressed by the equations



(*a*) (*b*)

FIG. 2

$$\left.\begin{aligned} v_i&=A_i\cos k_1x+B_i\sin k_2x+E_ik_2x+F_i,\\ w_i&=C_i\cos k_1x+D_i\sin k_1x\ +G_ik_1x+H_i\end{aligned}\right\}\tag{9}$$

respectively, and those in directions y and z by the equations

$$y_i=v_i\cos\varphi_i-w_i\sin\varphi_i,\quad z_i=v_i\sin\varphi_i+w_i\cos\varphi_i.\tag{10}$$

The conditions of continuity at $x=0$ are such that $y,\ z,\ y',\ z',\ M_y,\ M_z,\ S_y$ and S_z are continuous.

1. When both ends are simply pivoted

With the end-conditions

$$v_1=w_1=M_{11}=M_{21}=S_{11}=S_{21}=0\qquad \text{at } x=-l_1$$

and
$$v_2=w_2=M_{12}=M_{22}=S_{12}=S_{22}=0\qquad \text{at } x=l_2,$$

the deflections are expressed:

$$\left.\begin{aligned} v_1&=A_1\sin k_2\,(x+l_1), & w_1&=B_1\sin k_1(x+l_1),\\ v_2&=A_2\sin k_2\,(x-l_2), & w_2&=B_2\sin k_1(x-l_2).\end{aligned}\right\}\tag{11}$$

The conditions of continuity of $y,\ z,\ y'$ and z' at $x=0$ lead to the critical condition:

$$\begin{aligned}\tan^2\varphi\,&(k_1\cos k_1l_2\sin k_2l_1+k_2\cos k_2l_1\sin k_1l_2)\,(k_1\cos k_1l_1\sin k_2l_2\\ &+k_2\cos k_2l_2\sin k_1l_1)+k_1k_2\sin k_1\,(l_1+l_2)\sin k_2\,(l_1+l_2)=0\end{aligned}\tag{12}$$

where $\varphi=\varphi_2-\varphi_1$. The condition for buckling in cases where $\varphi=0$ or $\varphi=90°$ can easily be obtained from Eq. (12).

In special cases where $l_1=l_2=l$, we obtain

$$\tan^2\varphi\,(k_1\cos k_1l\sin k_2l+k_2\cos k_2l\sin k_1l)^2+k_1k_2\sin 2\,k_1l\sin 2\,k_2l=0\tag{13}$$

and in special cases where $EI_1 \gg EI_2$, or $k_1 \to 0$,

$$\tan^2\varphi\,(\sin k_2l_1 + k_2l_2\cos k_2l_1)\,(\sin k_2l_2 + k_2l_1\cos k_2l_2) \\ + k_2(l_1+l_2)\sin k_2\,(l_1+l_2) = 0. \tag{14}$$

2. When both ends are fixed

With the end-conditions $v=w=v'=w'=0$ at $x=-l_1$ and $x=l_2$ the deflections are preferably expressed:

$$\left.\begin{aligned} v_1 &= A_1\{1-\cos k_2(x+l_1)\} + B_1\{k_2(x+l_1) - \sin k_2(x+l_1), \\ w_1 &= C_1\{1-\cos k_1(x+l_1)\} + D_1\{k_1(x+l_1) - \sin k_1(x+l_1), \\ v_2 &= A_2\{1-\cos k_2(x-l_2)\} + B_2\{k_2(x-l_2) - \sin k_2(x-l_2), \\ w_2 &= C_2\{1-\cos k_1(x-l_2)\} + D_2\{k_1(x-l_2) - \sin k_1(x-l_2). \end{aligned}\right\} \tag{15}$$

The eight conditions of continuity at $x=0$ applied to these expressions give the following critical condition for buckling where, for brevity, conventional notations c 12, s 12 etc. are used instead of $\cos k_1l_2$, $\sin k_1l_2$ etc.:

$$\begin{vmatrix} \mathrm{c}\,21-\mathrm{c}\,12 & \dfrac{\mathrm{s}\,21}{k_2}+\dfrac{\mathrm{s}\,12}{k_1}-2\,l\,\mathrm{c}\,12 & (\mathrm{c}\,22-\mathrm{c}\,12)\cos\varphi & \left(-\dfrac{\mathrm{s}\,22}{k_2}+\dfrac{\mathrm{s}\,12}{k_1}\right)\cos\varphi \\ -k_2\,\mathrm{s}\,21-k_1\,\mathrm{s}\,12 & \mathrm{c}\,21-\mathrm{c}\,12-2\,k_1l\,\mathrm{s}\,12 & (k_2\,\mathrm{s}\,22-k_1\,\mathrm{s}\,12)\cos\varphi & (\mathrm{c}\,22-\mathrm{c}\,12)\cos\varphi \\ (\mathrm{c}\,21-\mathrm{c}\,11)\cos\varphi & \left(\dfrac{\mathrm{s}\,21}{k_2}-\dfrac{\mathrm{s}\,11}{k_1}\right)\cos\varphi & \mathrm{c}\,22-\mathrm{c}\,11 & -\dfrac{\mathrm{s}\,22}{k_2}-\dfrac{\mathrm{s}\,11}{k_1}+2\,l\,\mathrm{c}\,11 \\ (-k_2\,\mathrm{s}\,21+k_1\,\mathrm{s}\,11)\cos\varphi & (\mathrm{c}\,21-\mathrm{c}\,11)\cos\varphi & k_2\,\mathrm{s}\,22+k_1\,\mathrm{s}\,11 & \mathrm{c}\,22-\mathrm{c}\,11-2\,k_1l\,\mathrm{s}\,11 \end{vmatrix} = 0. \tag{16}$$

In cases where $l_1=l_2=l$, Eq. (16) is reduced to:

$$(1-\cos k_1l\cos k_2l)\sin^2\varphi + \left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\sin k_1l\sin k_2l\,(1+\cos^2\varphi) \\ -2(k_1l\sin k_1l\cos k_2l + k_2l\sin k_2l\cos k_1l) \\ \pm 2\left(\frac{k_1}{k_2}\sin k_1l\cdot\mathfrak{B}_2 - \frac{k_2}{k_1}\sin k_2l\cdot\mathfrak{B}_1\right)\cos\varphi = 0, \tag{17}$$

where

$$\mathfrak{B}_1 = \sin k_1l - k_1l\cos k_1l, \qquad \mathfrak{B}_2 = \sin k_2l - k_2l\cos k_2l.$$

When $\varphi=0$, Eq. (16) is nalurally reduced to the four equations:

$$\sin k_1l = 0, \qquad \sin k_2l = 0, \qquad \mathfrak{B}_1 = 0, \text{ and } \mathfrak{B}_2 = 0.$$

When $\varphi=90°$, the following two conditions are reduced:

$$2(1-\cos k_2l_1\cos k_1l_2 - k_2l\sin k_2l_1\cos k_1l_2 - k_1l\sin k_1l_2\cos k_2l_1) \\ + \left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\sin k_2l_1\sin k_1l_2 = 0, \tag{18}$$

$$2(1-\cos k_2l_2\cos k_1l_1-k_2l\sin k_2l_2\cos k_1l_1-k_1l\sin k_1l_1\cos k_1l_2)$$
$$+\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\sin k_2l_2\sin k_1l_1=0. \tag{18}$$

If one of the principal bending rigidities becomes very large, or $k_1\to 0$, the conditions of instability are reduced to:

$$\left.\begin{aligned}\sin\frac{k_2l_1}{2}\left(\frac{k_2l_1}{2}\cos\frac{k_2l_1}{2}-\sin\frac{k_2l_1}{2}\right)=0,\\ \sin\frac{k_2l_2}{2}\left(\frac{k_2l_2}{2}\cos\frac{k_2l_2}{2}-\sin\frac{k_2l_2}{2}\right)=0.\end{aligned}\right\} \tag{19}$$

These are equivalent to those conditions for individual bars l_1 and l_2 clamped at $x=0$ in spite of the value of φ.

V. General discussion on the buckling of a straight uniform rod having some initial twist under axial and/or torsional loads

1. General expressions of lateral deflections

Let us assume that a rod uniform in its section, straight in its center line and having some initial twist about its longitudinal axis has been buckled to some small lateral deflections under an axial load, a torsional load or combined loads of both. Further, assume that the shear center line of the section coincides with the neutral line for bending and also that the ordinary elastic behaviors for bending and tortion, in other word, the relations

$$M_1=EI_1\left(\frac{1}{\rho_1}-\frac{1}{\rho_{10}}\right),\quad M_2=EI_2\left(\frac{1}{\rho_2}-\frac{1}{\rho_{20}}\right),\quad\text{and}\quad Q=GJ_e\left(\frac{d\varphi}{dx}-\frac{d\varphi_0}{dx}\right) \tag{20}$$

hold, neglecting the effects of higher terms of deformation.

In general, the following analytical relations plainly hold:

$$M_1=\lambda_1M_x+\mu_1M_y+\nu_1M_z\quad\text{etc.},\qquad M_x'=\nu_iY-\mu_zZ-m_x\quad\text{etc.},$$
$$S_1=\lambda_1X+\mu_1Y+\nu_1Z\quad\text{etc.},\qquad X'=-q_x\quad\text{etc.},$$
$$\lambda_1=\cos\alpha\cdot\lambda_p+\sin\alpha\cdot\lambda_n\quad\text{etc.},\qquad \lambda_2=-\sin\alpha\cdot\lambda_p+\cos\alpha\cdot\lambda_n\quad\text{etc.},$$
$$\lambda_1'=\lambda_2\alpha'+\cos\alpha\cdot\lambda_p'+\sin\alpha\cdot\lambda_n'=\lambda_2\left(\alpha'+\frac{1}{\tau}\right)-\left(\frac{\lambda_s}{\rho}\right)\cos\alpha\quad\text{etc.},$$
$$\lambda_2'=-\lambda_1\left(\alpha'+\frac{1}{\tau}\right)+\left(\frac{\lambda_s}{\rho}\right)\sin\alpha\quad\text{etc.},$$
$$\lambda_s'=\frac{\lambda_p}{\rho}\quad\text{etc.},$$
$$\frac{1}{\rho_1}=\left(\frac{1}{\rho}\right)\sin\alpha,\quad\frac{1}{\rho_2}=\left(\frac{1}{\rho}\right)\cos\alpha.$$

Differentiating M_1, M_2, Q, S_1, S_2 and T with respect to s and substituting the above relations in these, we get the following six well-known equations of force balance:

$$M_1' = M_2\left(\alpha' + \frac{1}{\tau}\right) - \frac{Q}{\rho_2} + S_2 - m_1,$$

$$M_2' = -M_1\left(\alpha' + \frac{1}{\tau}\right) + \frac{Q}{\rho_1} - S_1 - m_2,$$

$$Q' = \frac{M_1}{\rho_2} - \frac{M_2}{\rho_1} - m_s,$$

$$S_1' = S_2\left(\alpha' + \frac{1}{\tau}\right) - \frac{T}{\rho_2} - q_1,$$

$$S_2' = -S_1\left(\alpha' + \frac{1}{\tau}\right) + \frac{T}{\rho_1} - q_2,$$

$$T' = \frac{S_\rho}{\rho} = \frac{S_1}{\rho_2} - \frac{S_2}{\rho_1} - q_s.$$

When the bending deflections y and z remain small, as is the case in our porblems, the direction cosines can be expressed as:

$$\left.\begin{array}{lll} \lambda_s \doteqdot 1, & \mu_s = y', & \nu_s = z', \\ \lambda_\rho = -\rho(y'y'' + z'z''), & \mu_\rho = \rho y'' = \cos\psi, & \nu_\rho = \rho z'' = \sin\psi, \\ \lambda_n = \rho(y'z'' - z'y''), & \mu_n = -\rho z'' = -\sin\psi, & \nu_n = \rho y'' = \cos\psi, \\ \lambda_1 = -y'\cos(\alpha+\psi) - z'\sin(\alpha+\psi), & \mu_1 = \cos(\alpha+\psi), & \nu_1 = \sin(\alpha+\psi), \\ \lambda_2 = y'\sin(\alpha+\psi) - z'\cos(\alpha+\psi), & \mu_2 = -\sin(\alpha+\psi), & \nu_2 = \cos(\alpha+\psi), \end{array}\right\} \quad (22)$$

and the rate of twist of the space curve of the center line is expressed by

$$\frac{1}{\tau} = \psi'.$$

The amount of twist of the actual material of the rod is now $\alpha+\psi$ which hereafter is replaced by φ (Fig. 2 a).

By putting q's$=m$'s$=0$ and omitting the higher orders of infinitesimals, Eqs. (21) are reduced to:

$$\left.\begin{array}{l} Q \doteqdot M_x = \text{const.}, \qquad T \doteqdot X = \text{cost.}, \\ M_1' - M_2\varphi' - S_2 + \dfrac{Q}{\rho_2} = 0, \\ M_2' + M_1\varphi' + S_1 - \dfrac{Q}{\rho_1} = 0, \\ S_1' - S_2\varphi' + \dfrac{X}{\rho_2} = 0, \\ S_2' + S_1\varphi' - \dfrac{X}{\rho_1} = 0, \end{array}\right\} \quad (23)$$

or eliminating S_1 and S_2,

$$M_1'' - M_1\varphi'^2 - 2M_2'\varphi' - M_2\varphi'' + \frac{Q\varphi'}{\rho_1} + Q\left(\frac{1}{\rho_2}\right)' - \frac{X}{\rho_1} = 0,$$

$$M_2'' - M_2\varphi'^2 - 2M_1'\varphi' + M_1\varphi'' + \frac{Q\varphi'}{\rho_2} - Q\left(\frac{1}{\rho_1}\right)' - \frac{X}{\rho_2} = 0.$$

If we assume the initial rate of twist to be constant, φ' must also be constant and the above equations become:

$$\left.\begin{aligned} M_1'' - M_1\Big(\varphi'^2 - \frac{Q\varphi'}{EI_1} + \frac{X}{EI_1}\Big) &= \Big(2\,\varphi' - \frac{Q}{EI_2}\Big)M_2', \\ M_2'' - M_2\Big(\varphi'^2 - \frac{Q\varphi'}{EI_2} + \frac{X}{EI_2}\Big) &= -\Big(2\,\varphi' - \frac{Q}{EI_1}\Big)M_1'. \end{aligned}\right\} \tag{24}$$

Eqs. (24) are the fundamental equations for our problems and the general solution of them is expressed as follows:

$$\left.\begin{aligned} M_2 &= EI_2\,\frac{1}{\rho}\cos\alpha = A\cos\beta_1 x + B\sin\beta_1 x + C\cos\beta_2 x + D\sin\beta_2 x, \\ M_1 &= EI_1\,\frac{1}{\rho}\sin\alpha = s_1(-A\sin\beta_1 x + B\cos\beta_1 x) \\ &\qquad\qquad + s_2(-C\sin\beta_2 x - D\cos\beta_2 x), \end{aligned}\right\} \tag{25}$$

where β_1, β_2, s_1 and s_2 are the characteristic constants determined by the equations

$$\begin{aligned} \beta^4 - \beta^2\{(1+m)(k^2 - hr\varphi') + 2\,\varphi'^2 + mr^2h^2\} \\ + (mk^2 - \varphi'^2 + mhr\varphi')(k^2 - \varphi'^2 + hr\varphi') = 0, \end{aligned} \tag{26 a}$$

$$s = \frac{k^2 - \beta^2 - \varphi'^2 + hr\varphi'}{\beta(2\,\varphi' - mhr)} = \frac{\beta(2\,\varphi' - hr)}{mk^2 - \beta^2 - \varphi'^2 + mhr\varphi'}, \tag{26 b}$$

where

$$k^2 = -\frac{X}{EI_2}, \quad h = \frac{Q}{GJ_e}, \quad m = \frac{EI_2}{EI_1}, \quad \text{and} \quad r = \frac{GJ_e}{EI_2}.$$

Eq. (26 a) gives two sets of solutions $\beta^2 = \beta_1^2$ and $\beta^2 = \beta_2^2$, thus giving two values s_1 and s_2 corresponding to β_1 and β_2 respectively.

The general expressions for deflection curvatures of the elastic center line can now be obtained from Eqs. (25) recalling definitions ψ and φ.

$$\left.\begin{aligned} y'' &= \Big(\frac{1}{\rho}\Big)\cos\psi = \Big(\frac{1}{\rho}\Big)\cos(\varphi - \alpha) \\ &= \Big(\frac{M_2}{EI_2}\Big)\cos\varphi' x + \Big(\frac{M_1}{EI_2}\Big)m\sin\varphi' x, \\ z'' &= \Big(\frac{M_2}{EI_2}\Big)\sin\varphi' x - \Big(\frac{M_1}{EI_2}\Big)m\cos\varphi' x. \end{aligned}\right\} \tag{27}$$

Further, the general expressions for y', z', y and z can be obtained by integrating Eqs. (27). But as the deflection curve, with regard to $1/\rho$ for instance, is clearly expected to be perfectly symmetric or perfectly antisymmetric with respect to the mid-point of the rod when both end-conditions are similar, it is always convenient to treat such problems in two seperate ways.

Take the origin of x at mid-point of a bar with a length of $2\,l$. Then the expressions for a symmetric deformation are:

$$\frac{1}{\rho_2} \text{ or } M_2 = A\cos\beta_1 x + C\cos\beta_2 x,$$

$$\frac{1}{\rho_1} \text{ or } mM_1 = -ms_1 A \sin\beta_1 x - ms_2 C \sin\beta_2 x,$$

$$y'' = \frac{A}{2}\Big[(1+ms_1)\cos(\beta_1+\varphi')x + (1-ms_1)\cos(\beta_1-\varphi')x\Big] + \frac{C}{2}\Big[-2-\Big],$$

$$y' = \frac{A}{2}\Big[(1+ms_1)\frac{\sin(\beta_1+\varphi')x}{(\beta_1+\varphi')} + (1-ms_1)\frac{\sin(\beta_1-\varphi')x}{(\beta_1-\varphi')}\Big] + \frac{C}{2}\Big[-2-\Big],$$

$$y = \frac{A}{2}\Big[-(1+ms_1)\frac{\cos(\beta_1+\varphi')x}{(\beta_1+\varphi')^2} - (1-ms_1)\frac{\cos(\beta_1-\varphi')x}{(\beta_1-\varphi')^2}\Big] + \frac{C}{2}\Big[-2-\Big] + F,$$

$$z'' = \frac{A}{2}\Big[(1+ms_1)\sin(\beta_1+\varphi')x - (1-ms_1)\sin(\beta_1-\varphi')x\Big] + \frac{C}{2}\Big[-2-\Big],$$

$$z' = \frac{A}{2}\Big[-(1+ms_1)\frac{\cos(\beta_1+\varphi')x}{(\beta_1+\varphi')} + (1-ms_1)\frac{\cos(\beta_1-\varphi')x}{(\beta_1-\varphi')}\Big] + \frac{C}{2}\Big[-2-\Big] + G,$$

$$z = \frac{A}{2}\Big[-(1+ms_1)\frac{\sin(\beta_1+\varphi')x}{(\beta_1+\varphi')^2} + (1-ms_1)\frac{\sin(\beta_1-\varphi')x}{(\beta_1-\varphi')^2}\Big] + \frac{C}{2}\Big[-2-\Big] + Gx,$$

and for the antisymmetric deformation;

$$\frac{1}{\rho_2} \text{ or } M_2 = B\sin\beta_1 x + D\sin\beta_2 x,$$

$$\frac{1}{\rho_1} \text{ or } mM_1 = ms_1 B\cos\rho_1 x + ms_2 D\cos\beta_2 x,$$

$$y = \frac{B}{2}\Big[-\frac{\sin(\beta_1+\varphi')x}{(\beta_1+\varphi')^2}(1+ms_1) - \frac{\sin(\beta_1-\varphi')x}{(\beta_1-\varphi')^2}(1-ms_1)\Big] + \frac{D}{2}\Big[-2-\Big] + Ex,$$

$$z = \frac{B}{2}\Big[\frac{\cos(\beta_1+\varphi')x}{(\beta_1+\varphi')^2}(1+ms_1) - \frac{\cos(\beta_1-\varphi')x}{(\beta_1-\varphi')^2}(1-ms_1)\Big] + \frac{D}{2}\Big[-2-\Big] + H,$$

where [−2−] means an expression similar to its preceding term with suffix 2 instead of 1.

2. Boundary conditions and corresponding critical conditions for buckling

(a) A rod with both ends universally jointed

End conditions:

$$M_1 = M_2 = 0 \quad \text{at} \quad x = \pm l, \quad \text{i.e.,}$$

$$\left\{\begin{array}{l} A\cos\beta_1 l + C\cos\beta_2 l = 0, \\ s_1 A\sin\beta_1 l + s_2 C\sin\beta_2 l = 0, \end{array}\right. \qquad \text{or} \qquad \left\{\begin{array}{l} B\sin\beta_1 l + D\sin\beta_2 l = 0, \\ s_1 B\cos\beta_1 l + s_2 D\cos\beta_2 l = 0. \end{array}\right.$$

Critical conditions:

$$\left.\begin{array}{rl} & s_2\cos\beta_1 l\sin\beta_2 l - s_1\sin\beta_1 l\cos\beta_2 l = 0, \\ \text{or} & s_2\sin\beta_1 l\cos\beta_2 l - s_1\cos\beta_1 l\sin\beta_2 l = 0. \end{array}\right\} \tag{28}$$

(b) A rod with both ends clamped against bending

The end-conditions are now expressed by $y = z = y' = z' = 0$ at $x = \pm l$. Let $\varphi' l = \phi =$ one-half the total amount of twist between both ends and

$$a=\{(\beta l)^2-\phi^2\}\{\beta l\sin\beta l\cos\phi-\phi\cos\beta l\sin\phi\},$$
$$b=\{(\beta l)^2-\phi^2\}\{\phi\cos\beta l\cos\phi+\beta l\sin\beta l\sin\phi\},$$
$$c=2\,\phi\beta l\sin\beta l\cos\phi-\{(\beta l)^2+\phi^2\}\cos\beta l\sin\phi,$$
$$d=\{(\beta l)^2-\phi^2\}\{\beta l\cos\beta l\sin\phi-\phi\sin\beta l\cos\phi\},$$
$$e=\{(\beta l)^2-\phi^2\}\{-\phi\sin\beta l\sin\phi-\beta l\cos\beta l\cos\phi\},$$
$$f=2\,\phi\beta l\cos\beta l\sin\phi-\{(\beta l)^2+\phi^2\}\sin\beta l\cos\phi,$$

then the critical conditions reduced from above end-condition become to

$$\begin{aligned}\Delta_I=&\begin{vmatrix} a_1+ms_1d_1 & a_2+ms_2d_2 & 0\\ b_1+ms_1e_1 & b_2+ms_2e_2 & 1\\ c_1+ms_1f_1 & c_2+ms_2f_2 & 1\end{vmatrix}\\ =&\,\Delta_1+ms_1\Delta_2+ms_2\Delta_3+m^2s_1s_2\Delta_4=0\end{aligned}\tag{29}$$

for symmetric deformation, and

$$\Delta_{II}=\Delta_4+ms_1\Delta_3+ms_2\Delta_2+m^2s_1s_2\Delta_1=0\tag{30}$$

for the antisymmetric deformation, wherein

$$\Delta_1=\begin{vmatrix} a_1 & a_2 & 0\\ b_1 & b_2 & 1\\ c_1 & c_2 & 1\end{vmatrix},\qquad \Delta_2=\begin{vmatrix} d_1 & a_2 & 0\\ e_1 & b_2 & 1\\ f_1 & c_2 & 1\end{vmatrix},\qquad \Delta_3=\begin{vmatrix} a_1 & d_2 & 0\\ b_1 & e_2 & 1\\ c_1 & f_2 & 1\end{vmatrix},\qquad \Delta_4=\begin{vmatrix} d_1 & d_2 & 0\\ e_1 & e_2 & 1\\ f_1 & f_2 & 1\end{vmatrix}.$$

(*c*) *A rod with both ends hinged along its principal axis* 2 (*or* 1), *i.e., along its longer* (*or shorter*) *axis.*

The end-conditions at $x=\pm l$ are expressed as:

$$y=z=0,\qquad \frac{1}{\rho_2}=0,\qquad -y'\sin\phi+z'\cos\phi=0,$$

or

$$y=z=0,\qquad \frac{1}{\rho_1}=0,\qquad y'\cos\phi+z'\sin\phi=0.$$

The necessary critical conditions for buckling reduced from these are:

symmetrical, hinged along axis 2;

$$\Delta_{III}=(-\Delta_{a1}+\Delta_{a2}+ms_1\Delta_{b1}-ms_2\Delta_{b2})\cos\beta_1 l\cos\beta_2 l=0,\tag{31}$$

antisymmetrical, hinged along axis 2;

$$\Delta_{IV}=(\Delta_{c1}-\Delta_{c2}-ms_1\Delta_{d1}+ms_2\Delta_{d2})\sin\beta_1 l\sin\beta_2 l=0,\tag{32}$$

symmetrical, hinged along axis 1;

$$\Delta_V=\left(-\Delta_{c1}+\Delta_{c2}+\frac{\Delta_{d1}}{ms_1}-\frac{\Delta_{d2}}{ms_2}\right)\sin\beta_1 l\sin\beta_2 l=0,\tag{33}$$

antisymmetrical, hinged along axis 1;

$$\Delta_{VI} = \left(\Delta_{a1} - \Delta_{a2} - \frac{\Delta_{b1}}{ms_1} + \frac{\Delta_{b2}}{ms_2}\right) \cos \beta_1 l \cos \beta_2 l = 0, \tag{34}$$

where

$$\Delta_a = \left[\frac{\phi}{(\beta^2 l^2 - \phi^2)} - \frac{\cos \phi \{2\, \beta l \phi \cos \phi \tan \beta l - (\beta^2 l^2 + \phi^2) \sin \phi\}}{(\beta^2 l^2 - \phi^2)^2} \right],$$

$$\Delta_b = \left[\frac{\beta l}{(\beta^2 l^2 - \phi^2)} - \frac{\cos \phi \{ -2\, \beta l \phi \sin \phi + (\beta^2 l^2 + \phi^2) \cos \phi \tan \beta l\}}{(\beta^2 l^2 - \phi^2)^2} \right],$$

$$\Delta_c = \left[\frac{\phi}{(\beta^2 l^2 - \phi^2)} - \frac{\sin \phi \{ -2\, \beta l \phi \sin \phi \cot \beta l + (\beta^2 l^2 + \phi^2) \cos \phi\}}{(\beta^2 l^2 - \phi^2)^2} \right],$$

$$\Delta_d = \left[\frac{\beta l}{(\beta^2 l^2 - \phi^2)} - \frac{\sin \phi \{2\, \beta l \phi \cos \phi - (\beta^2 l^2 + \phi^2) \sin \phi \cot \beta l\}}{(\beta^2 l^2 - \phi^2)^2} \right],$$

the second suffix denoting the adoption β_1 or β_2 replacing β.

VI. The cases where $m=1$, or $EI_1 = EI_2 = EI$

This is the most simple case and might already have been investigated by other authors.

In this case the initial twist, if any, has no meaning analytically and, with the relation $\varphi' = h$, Eqs. (26) give

$$\left.\begin{aligned} \beta_{1,2} &= \sqrt{k^2 + \frac{(hr)^2}{4}} \pm \left(\frac{hr}{2}\right)\left(1 - \frac{2}{r}\right), \\ s_1 &= 1, \qquad s_2 = -1. \end{aligned}\right\} \tag{35}$$

1. With both ends universally jointed.

The critical conditions (28) are both reduced to:

$$\sin (\beta_1 + \beta_2)\, l = 0$$

or

$$k^2 + \frac{(hr)^2}{4} = \frac{\pi^2}{4\, l^2}. \tag{36}$$

This means that

$$\text{when} \quad Q = 0, \qquad -X_{cr} = EI\left(\frac{\pi}{2\, l}\right)^2 = P_e,$$

$$\text{when} \quad X = 0, \qquad Q_{cr} = EI\left(\frac{\pi}{l}\right) = Q_e,$$

and in general,

$$\left\{ \frac{P}{P_e} + \left(\frac{Q}{Q_e}\right)^2 \right\}_{cr} = 1. \tag{37}$$

The relation between P and Q satisfying Eq. (37) is shown in Fig. 3.

2. With both ends clamped against bending

The condition (29) or (30) is now simplified and expressed by

$$k^2l^2 \sin 2\sqrt{k^2+\left(\frac{hr}{2}\right)^2}\, l - \sqrt{k^2+\left(\frac{hr}{2}\right)^2}\, l\left\{\cos hrl - \cos 2\sqrt{k^2+\left(\frac{hr}{2}\right)^2}\, l\right\} = 0. \qquad (38)$$

When $X=0$, this gives

$$Q_{cr} = 1.43\, EI\left(\frac{\pi}{l}\right) = 1.43\, Q_e.$$

The results of calculating Eq. (38) are shown in Table 1 and in Fig. 3.

TABLE 1

hrl	$(kl)^2$	hrl	$(kl)^2$
$\pi/6\times 0$	$\pi^2\times 1.000$	$\pi/6\times 6$	$\pi^2\times$ 0.431
1	0.974	7	0.276
2	0.908	8	0.094
3	0.813	9	-0.096
4	0.699	10	-0.305
5	0.578		

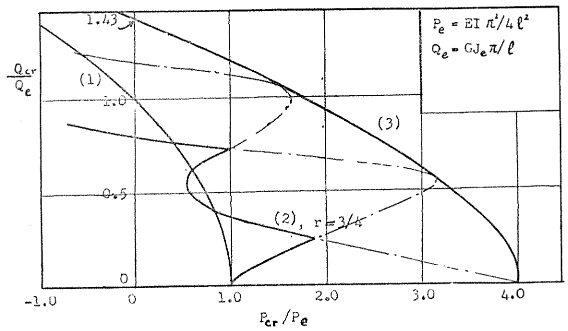


FIG. 3. Buckling under compression and torque, $EI_1=EI_2$, (1): Ends universally jointed, (2): Ends hinged parallel, $r=GJ_e/EI_2=3/4$, (3): Ends fixed.

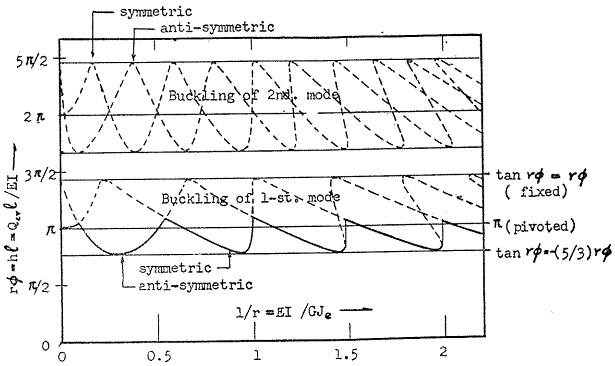


FIG. 4. Buckling under tortion, $EI_1=EI_2$, with ends hinged parallel.

3. With both ends hinged in directions initially parallel to each other

When torque Q exists, the hinge angle ϕ at the moment when buckling occurs cannot remain zero because of the twisting deformation due to Q. If we introduce new notations $B=\sqrt{k^2l^2+\Gamma^2}$ and $\Gamma=\frac{lhr}{2}$, or $\beta_1 l=B+\Gamma-\phi$ and $\beta_2 l=B-\Gamma+\phi$, the critical conditions (31) and (32) are now reduced to

$$\Delta_{III}=\begin{vmatrix} 2B\cos\beta_1 l\cos\beta_2 l & \cos\phi \\ \frac{(B-\Gamma)\cos\beta_2 l\sin(B+\Gamma)}{(B+\Gamma)}+\frac{(B+\Gamma)\cos\beta_1 l\sin(B-\Gamma)}{(B-\Gamma)} & 1 \end{vmatrix}=0, \tag{39}$$

$$\Delta_{IIII}=\begin{vmatrix} 2B\sin\beta_1 l\sin\beta_2 l & -\sin\phi \\ \frac{(B-\Gamma)\sin\beta_2 l\sin(B+\Gamma)}{(B+\Gamma)}-\frac{(B+\Gamma)\sin\beta_1 l\sin(B-\Gamma)}{(B-\Gamma)} & 1 \end{vmatrix}=0. \tag{40}$$

When $Q=0$, i.e. $\Gamma=\phi=0$ and $B=kl$, we get (i) $\cos kl=0$ and (ii) $\sin kl-kl\cdot\cos kl=0$ from Eq. (39) and (iii) $\sin kl=0$ from Eq. (40). The condition (ii) means the buckling in the plane containing hinge lines.

When $P=0$, i.e. $kl=0$, $B-\Gamma$ vanishes. Calculating the limiting case where $B-\Gamma$ tends to zero, we get following two conditions:

$$\left(\cos r\phi-\frac{\sin r\phi}{r\phi}\right)\left(\frac{1}{r\phi}-\tan\phi\right)=\sin r\phi\tan^2\phi, \tag{41}$$

$$\left(\cos r\phi-\frac{\sin r\phi}{r\phi}\right)\left(\frac{1}{r\phi}-\cot\phi\right)=\sin r\phi\cot\phi. \tag{42}$$

The critical values of $r\phi=\frac{Q}{EI}l$ calculated from Eq. (41) and (42) are plotted against $r=\frac{GJ_e}{EI}$ in Fig. 4. It must be noted that in most ranges of the value of r, the critical torques computed for this end-condition are somewhat smaller than those of a bar with both ends universally jointed. This result, which looks curious at a glance, may be explained by the fact that the end restrictions can in some cases be expected to facilitate the lateral deflection as the twist is increased. In Fig. 3 the relation between Q and P at the buckling state when $r=\frac{GJ_e}{EI}=\frac{3}{4}$ is plotted.

VII. The buckling under compression of a rod with some initial twist when $I_1 \neq I_2$

Let the rate of the initial twist be φ' which is constant throughout the length, and put $Q=h=0$. Eq. (26 a) then gives:

$$\beta_{1,2}^2=\frac{(1+m)k^2+2\varphi'^2}{2}\pm\sqrt{\frac{(1-m)^2}{4}k^4+2(1+m)k^2\varphi'^2}. \tag{43}$$

If φ' is very small compared with k, the above expression can be reduced (except when $1-m$ is also very small) to

$$\beta_1^2=k^2+\left\{1+2\frac{(1+m)}{(1-m)}\right\}\varphi'^2,\quad \beta_2^2=mk^2+\left\{1-2\frac{(1+m)}{(1-m)}\right\}\varphi'^2+4\frac{(1+m)^2}{(1-m)^2}\frac{\varphi'^4}{k^2}. \tag{44}$$

If, on the contrary, φ' is very large compared with k, it can be reduced to

$$\beta_{1,2}^2 = \left\{\varphi' \pm \sqrt{\frac{(1+m)}{2}} \cdot k\right\}^2 \tag{45}$$

and the coefficients s's corresponding to Eq. (45) are

$$s_{1,2} = -1 + (1-m)\left(\frac{k}{2\varphi'}\right)^2\left\{1 \mp \sqrt{\frac{(1+m)}{2}}\left(\frac{k}{\varphi'}\right)\right\}.$$

1. With both ends pivoted.

Here Eqs. (28) are the buckling conditions.

Consider first a case where $\phi = \varphi' l$ is very large corresponding to a tightly twisted bar. With the expression (45), Eqs. (28) are reduced to:

$$\pm (s_2 - s_1) \sin 2\phi - (s_1 + s_2) \sin 2\sqrt{\frac{(1+m)}{2}} \cdot kl = 0,$$

or

$$(1-m)\sqrt{\frac{(1+m)}{2}}\left(\frac{kl}{2\phi}\right)^2 \sin 2\phi - \sin 2\sqrt{\frac{(1+m)}{2}} \cdot kl$$

$$\fallingdotseq \sin 2\sqrt{\frac{(1+m)}{2}} \cdot kl = 0.$$

Thus

$$2kl\sqrt{\frac{(1+m)}{2}} \fallingdotseq n\pi. \qquad (n = 1, 2, \ldots) \tag{46}$$

$$P_{cr} = n^2 P_e \frac{2I_1}{I_1 + I_2},$$

where $P_e = \dfrac{\pi^2 EI_2}{(2l)^2}$. This result means that

$$P_{cr} = 2P_e \text{ when } m = \frac{I_2}{I_1} = 0 \text{ and } P_{cr} = P_e \text{ when } m = 1,$$

and in general the bar is equivalent to a flat bar with its sectional moment of inertia $\dfrac{2I_1I_2}{I_1+I_2}$.

When $\phi = 0$, Eqs. (44) show that $\beta_1 = k$, $\beta_2 = \sqrt{m} \cdot k$ and thus $s_1 = 0$, $s_2 = \infty$. The buckling conditions become $\cos \beta_2 l = 0$ or $\sin \beta_2 l = 0$, which naturally coincides with those of non-twisted bars.

The computed critical values $c = \dfrac{P_{cr}}{P_e}$ are plotted against 2ϕ in Fig. 5.

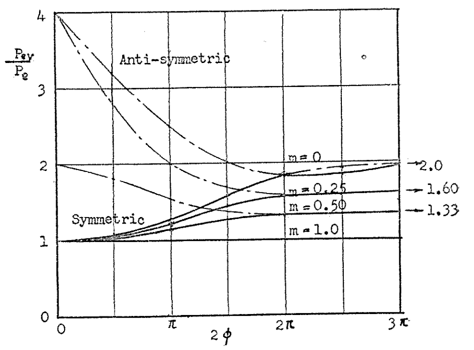


FIG. 5. Buckling under compression of twisted bars, pivoted, $m = \dfrac{I_2}{I_1}$, 2ϕ = amount of initial twist between the two ends.

2. With both ends hinged in the direction of axis 2 (pins parallel to the longer side).

Eqs. (31) and (32) are the characteristic equations for this problem. It is interesting to discuss the case of a strip where m is very small. In this case approximate expressions

$$\Delta_{III} \doteqdot \Delta_{a1} + \Delta_{a2} = 0 \quad \text{and} \quad \Delta_{IIII} \doteqdot \Delta_{c1} + \Delta_{c2} = 0 \tag{47}$$

give sufficiently correct critical loads for considerably large values of ϕ ($\phi >$ ca. $\pi/2$). When, however, ϕ becomes smaller m must not be unconditionally omitted.

From the expression (44), under the assumption that both m and ϕ are very small,

$$(\beta_1 l)^2 = (kl)^2 + 3\,\phi^2 + \ldots, \qquad (\beta_2 l)^2 = m(kl)^2 - \phi^2 + \ldots,$$
$$s_1 = \frac{-\,2\,\phi}{(1-m)\beta_1 l}, \qquad s_2 = (1-m)\frac{(kl)^2}{4\,\beta_2 l\phi} + \frac{2\,m\phi}{(1-m)\beta_2 l},$$

and expanding all the terms except $\tan\beta_1 l$ and $\cot\beta_1 l$ in power series in respect to ϕ, we get:

$$\Delta_{a1} + m s_1 \Delta_{b1} = \frac{2\,\beta_1^2 l^2 \phi^3}{(\beta_1^2 l^2 - \phi^2)^2}\left[\frac{1 - m\phi^2/\beta_1^2 l^2}{1-m}\left\{\frac{\tan\beta_1 l}{\beta_1 l}(1-\phi^2) - 1\right\} + \frac{\phi^2}{3}\right], \tag{48}$$

$$\begin{aligned}\Delta_{a2} + m s_2 \Delta_{b2} = \frac{\phi^3}{3}\Big[&\left\{-2 + \frac{1-m}{4}\,\frac{m}{\phi^2} k^2 l^2\right\} \\ &+ \frac{\beta_2^2 l^2 \phi^4}{(\beta_2^2 l^2 - \phi^2)^2}\left(2\frac{\beta_2^2 l^2}{\phi^2} - \frac{1-m}{4}\,\frac{m}{\phi^2}\,k^2 l^2\right)\left(1 - \frac{2}{5}\,\frac{\beta_2^2 l^2}{\phi^2}\right)\Big],\end{aligned} \tag{49}$$

$$\begin{aligned}\Delta_{c1} - m s_1 \Delta_{d1} = \frac{2\,\phi^5}{(\beta_1^2 l^2 - \phi^2)^2}\Big[&\frac{1 + 2\,m\phi^2/\beta_2^2 l^2}{1-m}\,\beta_1 l \cot\beta_1 l \\ &+ \frac{\beta_1^2 l^2}{3} - 1 + \frac{m}{1-m}\left(\frac{\beta_1^2 l^2}{\phi^2} - 3\right)\Big],\end{aligned} \tag{50}$$

$$\begin{aligned}\Delta_{c2} - m s_2 \Delta_{d2} = \frac{\phi^5}{(\beta_2^2 l^2 - \phi^2)^2}\Big[&\frac{1}{9}\,\beta_2^2 l^2 \\ &- \frac{1-m}{4}\,\frac{m}{\phi^2} k^2 l^2 \left\{-\frac{(\beta_2^2 l^2 - \phi^2)^2}{\beta_2^2 l^2 \phi^2} - \phi^2 + \frac{\beta_2^2 l^2}{3}\right\}\Big].\end{aligned} \tag{51}$$

Now, if we put $m=0$ preferentially to ϕ, Eqs. (31) and (32) give the conditions (47) which are further reduced to:

$$\frac{\tan\beta_1 l}{\beta_1 l}(1-\phi^2) = 1 + \frac{1}{3}(\beta_1 l)^2,$$

and

$$\beta_1 l \cot\beta_1 l = 1 - \frac{(\beta_1 l)^2}{3}, \tag{52}$$

giving the critical values of kl

$$(kl)^2 = 8.9\left(\frac{\pi}{2}\right)^2 \quad \text{and} \quad (kl)^2 = 13.4\left(\frac{\pi}{2}\right)^2$$

respectively in the limiting case when ϕ tends to zero.

On the contrary, if m is finite, however small it may be, the amount $\frac{m}{\phi^2}$ grows increasingly important when ϕ becomes smaller. As the expressions (49) and (51) are then reduced to:

$$\Delta_{a2} + ms_2\Delta_{b2} = \phi m(1-m)\frac{(\beta_1 l)^2}{12}$$

and

$$\Delta_{c2} - ms_2\Delta_{d2} = \frac{\phi(1-m)}{4},$$

the conditions (31) and (32) are transformed into

$$\left.\begin{aligned} \tan\beta_1 l &= -m(1-m)^2\frac{(\beta_1 l)^5}{24\,\phi^2} \\ \text{and}\qquad \cot\beta_1 l &= -(1-m)^2\frac{(\beta_1 l)^3}{8\,\phi^4}, \end{aligned}\right\} \tag{53}$$

giving the critical values

$$kl = \frac{\pi}{2} \quad \text{and} \quad kl = \pi$$

at the limit when $\phi=0$. These agree with wellknown results for an untwisted bar $2\,l$ in length.

The results of our calculations are tabulated in Table 2 and plotted in Fig. 6. The curve $m=1$ can naturally be obtained from Eqs. (8), Sec. III, when we put $k_1=k_2$. The most notable features obtained through our investigations are as follows:

a) For a bar with the small value of $m=\frac{I_2}{I_1}$, the existence of a slight initial

TABLE 2. The Buckling Coefficients of Twisted Bars under Compression, Both Ends Hinged along the Longer Axis

Mode of buckling	ϕ	$c=P_{cr}/P_e$					
		$m=0$	$m=0.01$	$m=0.04$	$m=0.25$	$m=0.50$	$m=1.0$
Symmetric	0	1∼8.9	1	1	1	1	1
	$\pi/12$	8.9	5.1	1.45	1.25		
	$\pi/6$	8.7	8.6	5.6	2.5		
	$\pi/4$	8.3		8.15	4.4		2.045
	$\pi/2$	6.4		6.25	5.8	5.15	4.00
	$3\pi/4$	3.75			3.3		2.045
	π	1.8			1.6		1.00
	$5\pi/4$	3.45					2.045
	$3\pi/2$	7.5		7.0	6.25	5.3	4.00
Anti-symmetric	0	4∼13.4	4	4	4	4	4
	$\pi/4$			3.5			2.045
	$\pi/3$			3.6			
	$5\pi/12$			5.5			
	$\pi/2$			10.2			1.0
	$3\pi/4$	10.6					2.045
	π	7.2		6.9	6.25	5.3	4.0
	$5\pi/4$	3.65					2.045
	$3\pi/2$	1.9					1.0

twist increases the buckling load remarkably. The buckling form for this range is symmetrical.

b) When ϕ becomes larger exceeding a limit, the buckling form is replaced by one which is antisymmetrical. The buckling load, after some stagnation at about $4\,P_e$, again increases up to a value several times that of P_e at $\phi = \text{ca.}\ \frac{\pi}{2}$ or thereabouts.

c) For still larger values of ϕ, the buckling form again becomes symmetric and the buckling load diminishes showing a minimum at $\phi = \text{ca.}\ \pi$.

In Fig. 7. the results of our experiments are shown.

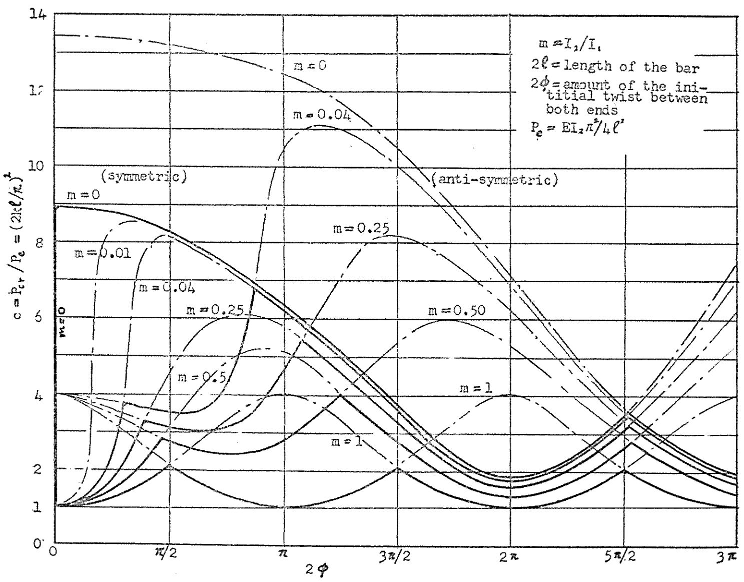


FIG. 6. Buckling under compression of twisted bars, ends hinged along the longer axis.

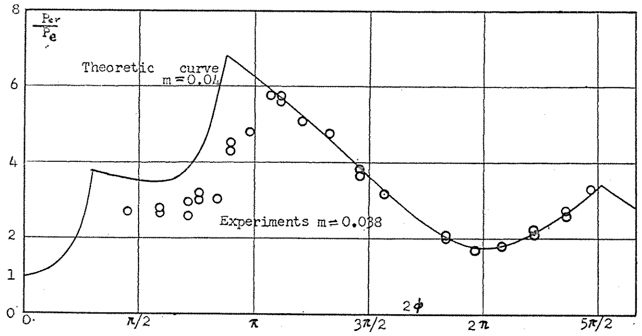


FIG. 7. Experimental results on the compressive buckling of twisted bars. (by S. Sekiya and K. Tatebe). With 2.95 mm × 15.15 mm steel bar, and $\varphi' = 380°$ per metre. The critical loads were determined by Donnell's method.

3. *With both ends fixed*

The characteristic equations in this case are Eq. (29) and (30). The partial determinants Δ_1 and Δ_4 in these equations are expanded to:

$$
\begin{aligned}
\Delta_1 = & -K_1\phi\{\beta_1 l\tan\beta_1 l-\beta_2 l\tan\beta_2 l\} \\
& +\beta_1 l\tan\beta_2 l\cos\phi\left\{(K_1-K_4)\sin\phi+\frac{K_4\cos\phi\cdot\beta_1 l\tan\beta_1 l}{\phi}\right\} \\
& +\sin\phi\{-(K_1+K_4)\cos\phi\cdot\beta_1 l\tan\beta_1 l+K_4\phi\sin\phi\},
\end{aligned}
\tag{54}
$$

$$
\begin{aligned}
\Delta_4 = & K_1\phi\{\beta_1 l\tan\beta_1 l-\beta_2 l\tan\beta_2 l\} \\
& +\beta_2 l\left(\frac{\sin\phi}{\phi}\right)\{(K_1-K_4)\phi\cos\phi\tan\beta_1 l+K_4\beta_1 l\sin\phi\} \\
& +\tan\beta_2 l\cos\phi\{-(K_1+K_4)\beta_1 l\sin\phi+K_4\phi\cos\phi\tan\beta_1 l\},
\end{aligned}
\tag{55}
$$

where $\quad K_1=(\beta_1^2 l^2-\phi^2)(\beta_2^2 l^2-\phi^2), \qquad K_4=2\phi^2(\beta_1^2 l^2-\beta_2^2 l^2).$

If m is very small, the critical loads can reasonably be calculated by the approximate equations

$$\Delta_I \fallingdotseq \Delta_1=0 \quad \text{and} \quad \Delta_{II}\fallingdotseq\Delta_4=0,$$

in-so-far as the value of ϕ shows a considerable amount. When, however, ϕ is reduced in size, a special consideration is required somewhat similar to that given in the previous paragraph. When ϕ is very large, the bar becomes equivalent to a uniform non-twisted bar with its sectional moment of inertia $I=\dfrac{2I_1I_2}{(I_1+I_2)}$. Results of calculations for $m=0.04$ are shown in Fig. 8.

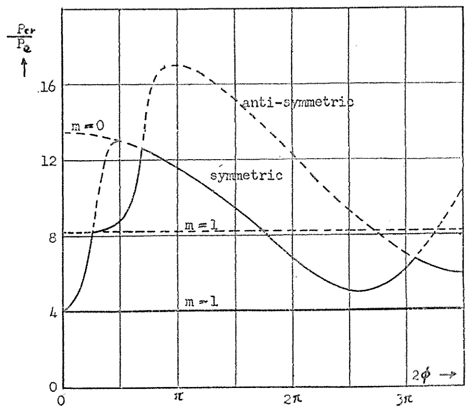


FIG. 8. Buckling under Compression of Twisted Bars, Fixed ends, $m=\dfrac{I_2}{I_1}=\dfrac{1}{25}$.

VIII. The buckling of flat strips under axial and/or torsional loads

Consider a rod subjected to both a compression P and a torque Q. The initial twist is assumed to be zero, or $\varphi'=\dfrac{Q}{GJ_e}=h$. If the bar is of a rectangular section $b\times t$, where $t\ll b$, the value of r can be estimated by the formula

$$r=\frac{GJ_e}{EI_2}=4\left(1-0.63\,\frac{t}{b}\right)\frac{G}{E}$$

and, with a typical value $\frac{G}{E}=\frac{3}{8}$ for practical use, we can put $r=\frac{3}{2}$.

Hence, from Eq. (26 a)

$$\beta_{1,2}^2=\frac{1}{2}\Big[\Big\{(1+m)k^2+\Big(\frac{1}{2}+\frac{3}{4}m\Big)\varphi'^2\Big\}$$
$$\pm\sqrt{(1-m)^2k^4+\frac{(2-m)(5-3m)}{2}k^2\varphi'^2+\Big(\frac{3}{2}-\frac{3}{4}m\Big)^2\varphi'^4} \qquad (56)$$

or, neglecting the small terms concerning to $m=\left(\frac{t}{b}\right)^2$ except when φ' is extremely small,

$$\beta_{12}^2=\frac{1}{2}\Big[\Big(k^2+\frac{1}{2}\varphi'^2\Big)\pm\sqrt{k^4+5k^2\varphi'^2+\frac{9}{4}\varphi'^4}\Big]. \qquad (57)$$

When ϕ becomes small enough as compared with mk^2l^2, β's and s's are to be written:

$$\beta_1^2\doteqdot k^2+\frac{3\varphi'^2}{2},\qquad \beta_2^2=mk^2-\varphi'^2\Big(1-\frac{m}{2}\Big)=m\beta_1^2-\varphi'^2, \qquad (58)$$
$$ms_1\doteqdot-\frac{m\phi}{2\beta_1 l},\qquad ms_2=\frac{\beta_2 l}{2\phi}.$$

1. With both ends universally jointed.

For brevity, we consider only the case where $X=0$. From (58)

$$\beta_1^2=\varphi'^2,\qquad \beta_2^2=\varphi'^2(1-r)(1-mr),$$
$$s_1=\frac{-(2-r)}{(2-mr)},\qquad s_2=-\sqrt{\frac{(r-1)}{(1-mr)}},$$

and thereby the buckling conditions (28) at the limit where $m\to 0$ are reduced to:

$$\tan\phi=\frac{\sqrt{r-1}}{1-r/2}\tanh\sqrt{r-1}\,\phi=2\sqrt{2}\tanh\Big(\frac{\phi}{\sqrt{2}}\Big),$$
$$\tan\phi=\frac{1-r/2}{\sqrt{r-1}}\tanh\sqrt{r-1}\,\phi=\Big(\frac{\sqrt{2}}{4}\Big)\tanh\Big(\frac{\phi}{\sqrt{2}}\Big) \qquad (59)$$

The smallest critical torque is calculated as

$$\frac{Q_{cr}l}{GJ_e}=1.10\quad\text{or}\quad\frac{Q_{cr}}{EI_2}=0.525\frac{\pi}{l}.$$

2. With both ends fixed.

Eqs. (29) and (30) are the buckling conditions and $\Delta_1=0$ and $\Delta_4=0$ (Eqs. 54 and 55) can replace them for the most range of ϕ as previously seen. When ϕ is extremely small, however, Eqs. (58) show that $ms_1\to 0$ and $ms_2\to\infty$, and it is necessary to solve the equations:

$$\Delta_I \doteqdot \Delta_1 + ms_2\Delta_3 = 0 \quad \text{and} \quad \Delta_{II} \doteqdot \Delta_4 + ms_2\Delta_2 = 0. \tag{60}$$

We see from Eq. (57) that:

β_1 is real and β_2 is imaginary when $2k^2 \geqq -\varphi'^2$,
both β_1 and β_2 are complex when $-\varphi'^2 \geqq 2k^2 \geqq -9\varphi'^2$, and
both β_1 and β_2 are imaginary when $-9\varphi'^2 \geqq 2k^2$.

Our examination shows that the buckling occurs only in the first range above or the loading conditions such that $\dfrac{X}{EI_2} > \dfrac{(Q/GJ_e)^2}{2}$ do not produce any buckling.

The buckling conditions when $P=0$, or due only to torque Q, are reduced to:

$$\Delta_1 = \frac{1}{2}\tanh\frac{\phi}{\sqrt{2}}\left\{2\phi^2\sin\phi + \frac{11}{3}\phi\cos\phi + \frac{5}{6}\cos\phi\sin 2\phi\right\}$$
$$-\left\{2\phi^2\cos\phi + \frac{1}{3}\phi\sin\phi + \frac{1}{6}\sin\phi\sin 2\phi\right\} = 0, \tag{61}$$

$$\Delta_4 = \frac{1}{\sqrt{2}}\tanh\frac{\phi}{\sqrt{2}}\left\{\phi^2\sin\phi - \frac{1}{6}\phi\cos\phi + \frac{1}{12}\cos\phi\sin 2\phi\right\}$$
$$+\left\{\frac{1}{2}\phi^2\cos\phi - \frac{11}{12}\phi\sin\phi + \frac{5}{24}\sin\phi\sin 2\phi\right\} = 0. \tag{62}$$

When ϕ or Q is very small, we get from Eqs. (58):

$$\Delta_1 = 2\phi\,(\beta_2^2l^2 - \phi^2)^2\beta_1 l\left\{\sin\beta_1 l - \beta_1 l\cos\beta_1 l - \frac{1}{3}\beta_2^2 l^2\sin\beta_1 l\right\},$$

$$\Delta_3 = \beta_2 l\,(\beta_2^2 l^2 - \phi^2)^2\left\{\frac{1}{3}\beta_1^3 l^3\sin\beta_1 l - \ldots\right\},$$

and the condition for symmetrical buckling becomes:

$$\Delta_I = \frac{2\phi\beta_1 l}{\beta_1^2 l^2 - \phi^2}\Big[\sin\beta_1 l - \beta_1 l\cos\beta_1 l - \frac{1}{3}\beta_1^2 l^2\sin\beta_1 l + \frac{(m\beta_1^2 l^2 - \phi^2)}{12\phi^2}\beta_1^2 l^2\sin\beta_1 l = 0,$$

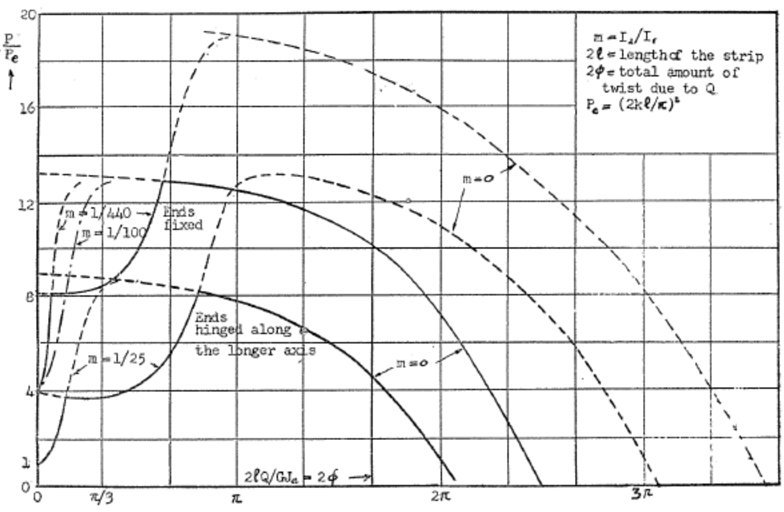


FIG. 9. Buckling under torque and compression of strips.

or

$$\frac{4\,\phi^2}{m\beta_1^2 l^2 - \phi^2} = \frac{-\,\beta_1^2 l^2 \sin\beta_1 l}{3\sin\beta_1 l - 3\,\beta_1 l\cos\beta_1 l - \beta_1^2 l^2 \sin\beta_1 l}\,. \tag{63}$$

For antisymmetrical buckling, however, $\Delta_{11} \fallingdotseq \Delta_2 = 0$ is sufficient for very small value of ϕ and we get

$$\phi^2 = \frac{6\,\beta_1^2 l^2(\beta_1 l\cos\beta_1 l - \sin\beta_1 l)}{\beta_1^2 l^2(\beta_1 l\cos\beta_1 l - \sin\beta_1 l) - (\beta_1^2 l^2 - \phi^2)\sin\beta_1 l - \beta_1 l\cos\beta_1 l}\,. \tag{64}$$

The results of computations are given in Fig. 9, and the results of our experiments are plotted in Fig. 10, showing good agreement with our theory.

3. With both ends hinged in directions parallel to the longer side

The characteristic equations (31) and (32) can be calculated under precautions similar to those previously discussed and the results are shown in Fig. 9.

The conditions for buckling in a special case where $X=0$ are reduced to

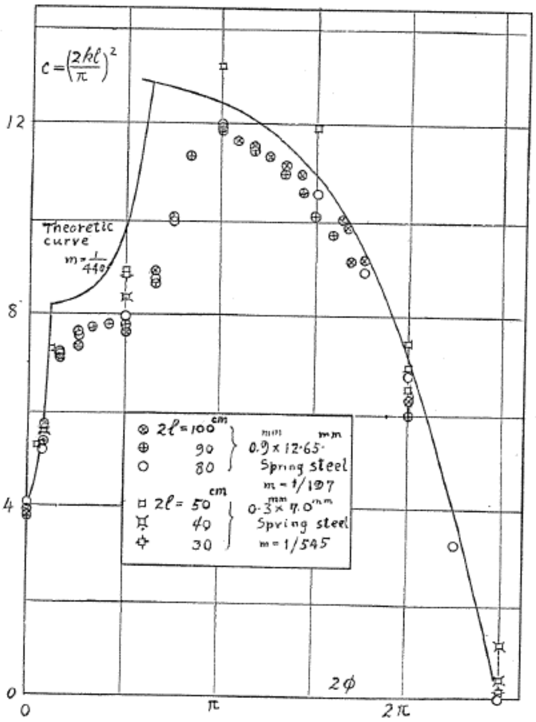


FIG. 10. Experimental Results on the buckling of strips under torque and compression (by K. Ohnishi).

$$\cos^2\phi\left\{-\frac{1}{36}\sin\phi + \frac{4}{9}\sqrt{2}\,\tanh\frac{\phi}{\sqrt{2}}\cos\phi\right\} - \frac{5}{12}\,\phi\cos\phi + \frac{1}{2}\,\phi^2\sin\phi = 0,$$

$$\sin^2\phi\left\{-\frac{1}{36}\cos\phi - \frac{4}{9}\sqrt{2}\,\tanh\frac{\phi}{\sqrt{2}}\sin\phi\right\} + \frac{5}{12}\phi\sin\phi + \frac{1}{2}\,\phi^2\cos\phi = 0.$$

IX. General remarks

The outstanding features described at the end of Paragraph 2, Section VII, are also applicable to the problems in Section VIII with an exception that there is a minimum value of P in the former case, but not in the latter. The most conspicuous point of interest is the fact that an existence of twist to some extent, either due to the initial state or due to the torque load, greatly increases the buckling compressive load.

The autuor expresses his appreciation to Messrs. T. Nishimura, K. Konishi, K. Ohnishi and to those undergraduate students who contributed valuable suggestions, labourious computations and carefully executed experiments.

References

1) A. E. Green: "The elsatic stability of a thin twisted strip," Proc. Roy, Soc. London, A **161**, 197, 1937.