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STRESS CONCENTRATION FACTORS FOR A CIRCUMFERENTIAL NOTCH IN A CYLINDRICAL SHAFT

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The object of this paper is to present curves of stress concentration factors for circumferential notches in cylindrical torsion members, for convenience of practical use. The formula used in plotting the curves has been developed in a previous paper. This paper presents theoretical and experimental verifications for this formula and establishes its useful range. Another approximate formula for extending the useful range of the curves is also presented.

In a previous paper, an approximate formula of the torsional maximum stress has been obtained on the basis of hydrodynamical analogy.[1] The formula is

$$\tau_{\max} = \frac{2T(1+\sqrt{h/\rho})}{\pi r(r-h)^2}, \tag{1}$$

where T, r, h and ρ represent the twisting moment, the radius of the cylindrical part of the shaft, and the depth and radius of notch, respectively. The validity of the formula had been confirmed only in the particular case of $h=\rho$, though the formula was given in a form covering all cases of the ratio h/ρ. Some years ago, Sonntag obtained a similar formula, as is well known in the field of engineering mechanics.[2] His formula is

$$\tau_{\max} = \frac{2T\{(r-h+\rho)^2(h+\rho)+\rho^2(h-\rho)\}}{\pi r\rho(r-h)^3(r-h+2\rho)}. \tag{2}$$

The values of $\tau_{\max}$ calculated from the two formulae, (1) and (2), are both in good agreement with exact ones when $h=\rho$, but they become considerably different from each other with increase of the ratio h/ρ. This discrepancy between the two, at large values of h/ρ, is too large for practical use. Ample theoretical justification for the two formulae, however, would not be easily found, since they are approximate ones derived from somewhat different assumptions respectively. Fortunately, theoretical and experimental data, valid for the justification of the two, have been published subsequently, and the validity of the author's formula has also been confirmed in cases of $h \neq \rho$.

The theoretical data are ones recently obtained by Momma for the torsional maximum stress in a shaft with multiple-notches.[3] He obtained the ratio of the maximum stress for a single-notch to that for multiple-notches, using two different formulae for a single-notch, (1) and (2), respectively. The ratio derived

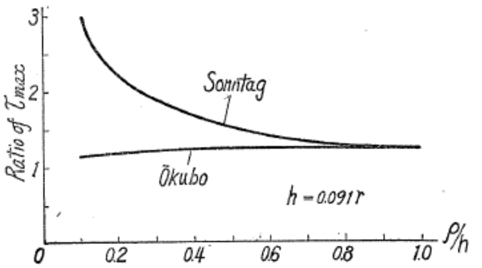

FIG. 1. Ratio of the maximum stress for a single-notch to that for multiple-notches of the pitch $0.273\,r$ (after Momma).

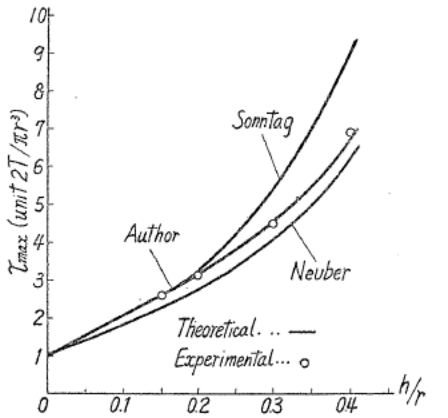

FIG. 2. Relation between the maximum stress and the depth of notch, when $\rho = 0.2\,r$.

from (1) remains almost constant by the variation of ρ, Fig. 1, while if the formula (2) be used instead of (1), the ratio considerably increases with the decrease of ρ. For sharp notches, however, it has been shown by Neuber that the ratio is given by[4]

$$\left(\frac{\pi h}{2a} \coth \frac{\pi h}{2a}\right)^{\frac{1}{2}}, \tag{3}$$

provided with the notches are vary small, where a represents the half pitch of notches along the length of the shaft. From (3) we have the ratio 1.16 for $2a = 0.273\,r$, $h = 0.091\,r$. As is shown in the figure, with decrease of ρ the curve derived from (1) approaches a value slightly smaller than 1.16. The notches employed by Momma, however, are not constant in form, but their form varies in such a manner that with the decrease of ρ the maximum stress becomes slightly larger as compared to that for notches of constant form but with varying ρ. Accordingly, if we take this into account, the curve of the ratio derived from (1), is amended into a more straightened one, and the ratio will reach a value very near 1.16. This shows that the author's formula is more comprehensible for the Neuber's result described.

The experimental data are ones recently obtained by the author by means of electroplating,[5] and are more valid for the justification of the two formulae. Only the final results are shown in Fig. 2, in order to dispense with the detailed description of the experiment, since it has been fully published in a previous paper. It may be seen from the figure that the values of maximum stress, calculated from (1), are in fairly good agreement with the measured ones.

Gathering from the theoretical and experimental results hitherto obtained, we may conclude that the author's formula is also valid in cases of $h \neq \rho$. In order to investigate further the safe range of the formula for the dimensions of notch, we shall rewrite the formula (1) into the form

$$K = \tau_{\max}\Big/\frac{16\,T}{\pi d^3} = \frac{1+\sqrt{h/\rho}}{1+2\,h/d}, \tag{4}$$

in which K is the stress concentration factor and d is the minimum diameter of the shaft. In a previous paper, the validity of the formula (4) was confirmed for a semi-circular notch in the range of $d \geqq 8\rho$.[1] In addition to this, K was subsequently measured for $d = 6\rho$ by means of electroplating. The measured value of K is 1.47, while the calculated value from (4) is 1.50. The discrepancy between the two remains within a few percent, so the formula is sufficiently accurate in a still wider range of ρ/d.

Now consider a notch, the depth of which is slightly larger than ρ. The stress concentration factor K_1 for this notch is always larger than K_0 for a semi-circular notch of radius ρ, namely, $K_1 > K_0$.

If the depth of the former be $\rho + \varepsilon$, from (4) we have the relation

$$K_1 - K_0 = \frac{\varepsilon(d - 6\rho)}{2\rho d(1 + 2\rho/d)^2},$$

since ε is assumed to be a very small quantity. To have the relation, $K_1 > K_0$, it follows that $d > 6\rho$.

Consequently the formula fails for either one of these notches, when $d \leqq 6\rho$. Formerly Jacobsen obtained the stress concentration factors in shafts of two diameters by using an electrical analogy.[6] His results show that when d and ρ are fixed, K increases with the increase of h, but it remains almost constant by the variation of h when $h \geqq \rho$ if ρ/d is not very small, e.g., the ratio between K for $h = \rho$ and that for large values of h is 0.87 when $\rho/d = 0.075$, but this ratio becomes 0.95 when $\rho/d = 0.125$. This shows that the ratio approaches unity when ρ/d becomes large enough. The conclusion for the shafts of two diameters may be extended to the present problem, and the value of K for large values of h can be replaced by the one for a semi-circular notch, when ρ/d is not very small. It has been shown that the formula (4) fails for a notch, the depth of which is slightly larger than ρ, when $d \leqq 6\rho$, but if we replace the value of K for the notch by the one for a semi-circular notch, the formula becomes valid for still larger values of ρ/d, namely, to the extent of which the validity of the formula has been confirmed for a semi-circular notch. This manner regarding the use of the formula may be extended to the cases of semi-elliptic notches. Plot the curves of K, taking h/ρ as a constant in each, and consider the envelope for the family of curves, then the formula (4) is always valid at all points on these curves to the extent of ρ/d, at which these curves touch the envelope, namely

$$\rho/d \leqq \frac{1}{2\sqrt{h/\rho}(2 + \sqrt{h/\rho})}. \qquad (5)$$

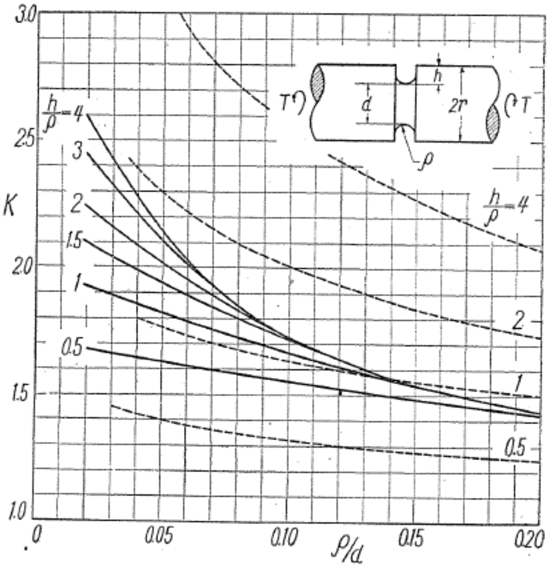

FIG. 3. Relation between the stress concentration factor and the depth and radius of the notch.

For still larger values of ρ/d, however, from the reason mentioned above, the values must be taken on the envelope instead of on these curves. The equation of the envelope is readily found from (4), as

$$8K(K-1)=d/\rho. \tag{6}$$

Since the condition (5) fails for a very large ρ, K is found from (6) instead of (4), and it follows that the influence of the notch entirely disappears.

The stress concentration factors for various values of ρ and h are calculated from (4) and (6), and presented by curves in Fig. 3 for convenience of practical use. The corresponding values of K, based on Sonntag's results, which are still referred to in practice, are also shown by dotted lines in the same figure.[7] We find a considerable discrepancy between the values of K herein recorded and those now used in practice.

To verify the results shown in Fig. 3, two different notches are considered, the one is $h=2\rho$, $\rho=0.167\,d$ and the other $h=\rho$, $\rho=0.167\,d$. The measured values of K for the two, obtained by the author using the electroplating method, are 1.49 and 1.47, respectively, while the values of K taken from the curves in Fig. 3 are 1.50 for either notch, so the results shown in the figure are sufficiently accurate for practical use.

Sonntag's formula has been quoted in text books of former edition and still used for the computation of K in practice, but since his results are in poor agreement with those of other investigators Neuber's solution is now more generally used with few exceptions.

Neuber's formula for deep Hyperbolic notches is[8]

$$K=\frac{3(1+\sqrt{d/2\rho+1})^2}{4(1+2\sqrt{d/2\rho+1})}. \tag{7}$$

The above formula, however, often gives much smaller values as compared to those obtained here. For instance, in the case of a semi-circular notch of $\rho=d/8$, the exact value by Willers is $K=1.57$, and the experimental value by the author is $K=1.59$, while his formula gives much smaller value of $K=1.44$. Comparing these figures for K, a question arises in connection with the validity of applying his solution to the present problem, since it has been derived for $h=\infty$. His solution for moderate values of h, based on an interpolation between two limiting solutions gives still smaller values of K, as shown in Fig. 2.

Furthermore, his solution gives a small value of K which almost agrees with the one for a fillet obatined by means of electrical analogy,[6] e.g., his solution gives $K=1.58$ for $\rho/d=0.078$, $h/\rho=4$, while the corresponding value for a shaft containing a fillet is 1.53. Those theoretical and experimental results of other investigators all indicate that his solution gives a much smaller value than the substantial one. Exceptionally, photoelastic results using stress freezing method often agree with Neuber's results. From which, however, one can hardly conclude the validity of his solution, because photoelastic results in three dimensions are not so accurate as in two dimensions, and it is reported that a considerable measuring errors up to 30 per cent is found in some instances.[9]

His another solution for a flat tension bar with deep hyperbolic notches also gives, in some instances, slightly smaller values of K even compared to those for semi-circular notches, obtained by photoelasticity,[10] and his solution for semi-circular notches is always smaller than the observed values, as is shown by a dotted line in Fig. 4. While the measured values of K are smaller than the accurate ones[11] shown in the same figure.

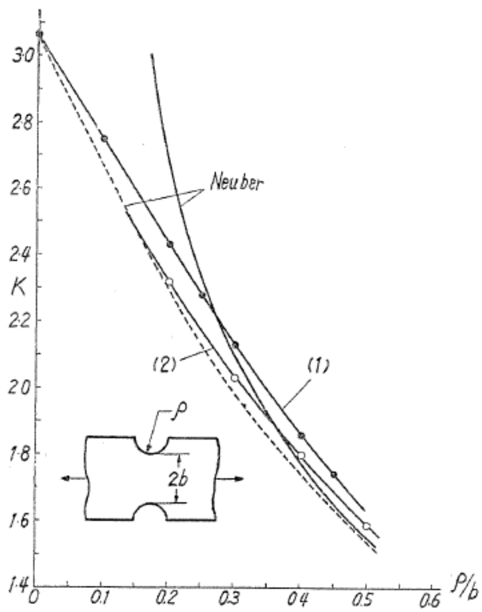

FIG. 4. Stress concentration factors for flat tension bars with semi-circular notches, (1) exact values by Isida, (2) measured values by Frocht.

The presumed cause of failure of his solution in those cases described is that his solution for deep notches has been derived without taking any consideration concerning the stress distribution at the terminal sections which will immediately affect the maximum stress. While, for a shaft containing a very deep notch, little is definitly known about the stress distribution at the section from which the change in section starts. In order to conclude the stress distribution at terminal sections given by his formula to be the same as descibed above, namely, his solution to be an exact one for $h=\infty$, it requires separate justifications. The discrepancy of his solution from the substantial one in those cases described, however, apparently shows that the stress distribution at the terminal sections, given by his formula, is not always adequate to the present problem.

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Appendix

Consider a shaft containing multiple-notches, and take for the stress function the expression

$$\phi = Cr^2 I_2\left(\frac{\pi r}{a}\right) \cos \frac{\pi z}{a} + \frac{1}{4}\theta r^4, \tag{8}$$

where a denotes the half pitch of notches along the length of the shaft, θ the mean twist per one pitch and C a constant. We assume that the minimum radius of the shaft is unity and that the curve $\phi = \text{const.}$, passes the point $r=1$, $z=0$. Then, the equation

$$C\left\{r^2 I_2\left(\frac{\pi r}{a}\right) \cos \frac{\pi z}{a} - I_2\left(\frac{\pi}{a}\right)\right\} + \frac{1}{4}\theta(r^4-1) = 0, \tag{9}$$

represents the contour line for multiple-notches of very large ρ.[12] From (8) and (9), we have

$$\left.\begin{array}{c} \tau_{\max} = G\left\{\dfrac{\pi C}{a} I_1\left(\dfrac{\pi}{a}\right) + \theta\right\}, \qquad T = 2\pi G C I_2\left(\dfrac{\pi}{a}\right) + \dfrac{1}{2}\pi G\theta, \\ \dfrac{\pi^2 \rho C}{a^2} I_2\left(\dfrac{\pi}{a}\right) = \dfrac{\pi C}{a} I_1\left(\dfrac{\pi}{a}\right) + \theta, \end{array}\right\} \tag{10}$$

where ρ is the radius at the base of notch. Eliminating C and θ from (10), and assuming ρ is very lage in comparison with unity, we have

$$K = \tau_{\max} \Big/ \frac{2T}{\pi} = 1 + s\frac{d}{\rho}, \tag{11}$$

where $d=2$ and $s = \dfrac{a^2}{\pi^2}\left\{\dfrac{\pi}{2a} I_1\left(\dfrac{\pi}{a}\right) \Big/ I_2\left(\dfrac{\pi}{a}\right) - 2\right\}$.

When $a > \pi$, $I_1\left(\frac{\pi}{a}\right)$ and $I_2\left(\frac{\pi}{a}\right)$ can be approximated with sufficient accuracy by

$$I_1\left(\frac{\pi}{a}\right) = \frac{\pi}{2a} + \frac{\pi^3}{16a^3} + \frac{\pi^5}{384a^5},$$

and

$$I_2\left(\frac{\pi}{a}\right) = \frac{\pi^2}{8a^2} + \frac{\pi^4}{96a^4} + \frac{\pi^6}{3072a^6}.$$

Neglecting small quantities of the higher order, it follows that

$$K = 1 + \frac{d}{12\rho}. \tag{12}$$

The stress concetration in such case is of little practical interest, since the influence of a notch practically disappears. However, the result is useful for the investigation of the formulae, (4) and (7), in the extreme case of a very shallow

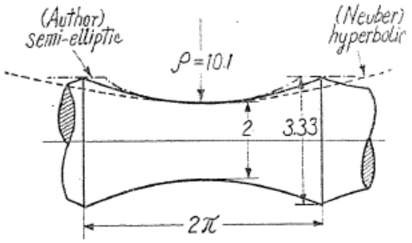

FIG. 5. The substantial form of a multiple-notch (after Kodama).

notch. Since the condition (5) fails for a notch of large ρ, K must be found form (6) instead of (4). From (6) it follows that

$$K=1+\frac{d}{8\rho}. \tag{13}$$

While the corresponding formula based on Neuber fairly agrees with (12). This shows that in the extreme case of large ρ, Neuber solution of hyperbolic notches agrees with the very one for a multiple-notch. For example, we consider a multiple-notch of $a=\pi$, $\rho=10.1$. The stress concentration factor is 1.016 both for this multiple-notch and for the hyperbolic notch of Neuber. While, (13) gives a slightly larger value of $K=1.025$. The comparison of the figures herein obtained with the forms of the notches, will probably give something useful for the justification of the formulae, (4) and (7), in the extreme case of large ρ. The substantial forms of the notches are shown in Fig. 5.

Reference

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