A NUMERICAL SOLUTION OF ONE-DIMENSIONAL HEAT CONDUCTION PROBLEMS, THERMAL CONDUCTIVITY BEING FUNCTION OF TEMPERATURE AND SITUATION

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1. Introduction

The fundamental equation of one-dimensional heat conduction problems can be written in the following form,

$$\frac{\partial}{\partial x}\left(K\frac{\partial u}{\partial x}\right) = c\rho\frac{\partial u}{\partial t} \tag{1}$$

where u; temperature,
K; thermal conductivity,
c; specific heat,
ρ; density,
x; situation,
t; time,

and K and c depend on x in general. If $c\rho$ can be assumed to be constant, the equation above becomes

$$\frac{\partial}{\partial x}\left(\alpha\frac{\partial u}{\partial x}\right) = \frac{\partial u}{\partial t} \tag{2}$$

where α; diffusivity.

We can also treat the problems of a cylnider or a sphere as one-dimenstional, in the case where the temperature changes only in the direction of its radius. For a cylinder, e.g., the equation is

$$\frac{\partial}{\partial r}\left(\alpha\frac{\partial u}{\partial r}\right) + \frac{\alpha}{r}\frac{\partial u}{\partial r} = \frac{\partial u}{\partial t}. \tag{3}$$

In these cases, the general solutions can be obtained, assuming α to bc a constant, but the numerical computation from them, when wanted, requires often a great deal of labour. Moreover, the analytical method cannot give a solution easily when α depends on u or x, The method of finite difference will be greatly advantageous in such cases, and several reports have been published using this method. The author presents, in this paper, a more generalized method of finite difference usefull for these problems, and it will greatly save the labours in computations.

2. The case $\alpha=\alpha\ (u)$ in (2)

Now we define v as

$$\alpha_c v = \int \alpha(u) du$$

where α_c is a constant of the dimensions of heat conductivity. Then the equation (2) becomes

$$\alpha_c \frac{\partial^2 v}{\partial x^2} = \frac{\partial u}{\partial t} \tag{4}$$

Considering the following relations

$$u(01) - u(00) = \tau \frac{\partial u}{\partial t} + \frac{\tau^2}{2} \frac{\partial^2 u}{\partial t^2} + \dots$$

$$u(0\bar{1}) - u(00) = -\tau \frac{\partial u}{\partial t} + \frac{\tau^2}{2} \frac{\partial^2 u}{\partial t^2} - \dots$$

$$u(10) + u(\bar{1}0) - u(1\bar{1}) - u(\bar{1}\bar{1}) = 2\tau \frac{\partial u}{\partial t} - \tau^2 \frac{\partial^2 u}{\partial t^2} + \tau h^2 \frac{\partial^3 u}{\partial t \partial x^2} + \dots$$

$$v(10) + v(\bar{1}0) - 2v(00) = h^2 \frac{\partial^2 v}{\partial x_2} + \frac{h^4}{13} \frac{\partial^4 v}{\partial x^4} + \dots$$

$$v(1\bar{1}) + v(\bar{1}\bar{1}) - 2v(0\bar{1}) = h^2 \frac{\partial^2 v}{\partial x^2} - \tau h^2 \frac{\partial^3 v}{\partial t \partial x^2} + \frac{h^4}{12} \frac{\partial^4 v}{\partial x^4} + \dots$$

and writing $h^2 = k\alpha_c\tau$, the expression for Eq. (4) can be obtained, neglecting the terms of order h^6,

	u		
$Cu-(6C+1)\frac{v}{k}$	$(10C-2)u+2(6C+1)\frac{v}{k}$	$Cu-(6C+1)\frac{v}{k}$	(a)
$-Cu-(6C-1)\frac{v}{k}$	$(1-10C)u+2(6C-1)\frac{v}{k}$	$-Cu-(6C-1)\frac{v}{k}$	

where C is an arbitrary constant. Hereupon we think $C=1/10$ fit, to avoid the accumulation of error.

When $\alpha(u) = \text{const.} = \alpha_c$, consequently, $u=v$, we arrive at the well-known expression (b) taking $k=4$. If $C=1/12$ and $k=6$ are assumed, the expression is reduced to (c), which is nothing but the expression previously used by Takahashi.

	1		(b)
$-\frac{3}{10}$	$-\frac{2}{10}$	$-\frac{3}{10}$	
	$-\frac{2}{10}$		

	1		(c)
$-\frac{1}{6}$	$-\frac{4}{6}$	$-\frac{1}{6}$	

Returning to our subject, we assume

$$\alpha(u) = \alpha_0 + (\alpha_1 - \alpha_0)u, \qquad 0 \leqq u \leqq 1 \tag{5}$$

where α_0; α at the lower limit of u considered, $u=0$,
α_1; α at the upper limit, $u=1$,

then
$$\alpha_C v = \alpha_0 u + \frac{1}{2}(\alpha_1 - \alpha_0)u^2. \tag{6}$$

i) If $\alpha_0 > \alpha_1$. taking $\alpha_C = \alpha_0$, we have

$$v = u - f, \qquad f = \frac{\alpha_0 - \alpha_1}{2\,\alpha_0}u^2. \tag{7}$$

ii) If $\alpha_0 < \alpha_1$, taking $\alpha_C = \dfrac{\alpha_0 + \alpha_1}{2}$. we have

$$v = u - f, \qquad f = \frac{\alpha_1 - \alpha_0}{\alpha_1 + \alpha_0}u(1-u). \tag{8}$$

Therefore the expression (a) becomes

(d)

	$10\,u$	
$-3\,u+4f$	$-2\,u-8f$	$-3\,u+4f$
$-f$	$-2\,u+2f$	$-f$

3. The case $\alpha = \alpha(x)$, in (2).

The equation (2) can be easily transformed into

$$\frac{\partial^2(Au)}{\partial x^2} + Bu = \frac{\partial(Cu)}{\partial t}, \tag{9}$$

where $A(x)$, $B(x)$ and $C(x)$ are determined from the following relations.

$$\begin{aligned} &\frac{2}{A}\frac{\partial A}{\partial x} = \frac{1}{\alpha}\frac{\partial \alpha}{\partial x}, \\ &B + \frac{\partial^2 A}{\partial x^2} = 0, \\ &\alpha C - A = 0. \end{aligned} \tag{10}$$

Then considering the relations,

$$Au(10) + Au(\bar{1}0) - 2\,Au(00) = h^2\frac{\partial^2(Au)}{\partial x^2} + \frac{h^4}{12}\frac{\partial^4(Au)}{\partial x^4} + \cdots$$

$$Au(1\bar{1}) + Au(\bar{1}\bar{1}) - 2\,Au(0\bar{1}) = h^2\frac{\partial^2(Au)}{\partial x^2} - h^2\tau\frac{\partial^3(Au)}{\partial x^2\partial t} + \frac{h^4}{12}\frac{\partial^4(Au)}{\partial x^4} + \cdots$$

$$Bu(10) + Bu(\bar{1}0) - 2\,Bu(00) = h^2\frac{\partial^2(Bu)}{\partial x^2} + \cdots$$

$$Bu(0\bar{1}) - Bu(00) = -\tau\frac{\partial(Bu)}{\partial t} + \cdots$$

$$Cu(1\bar{1}) + Cu(\bar{1}\bar{1}) - Cu(10) - Cu(\bar{1}0) - 2\,Cu(01) + 2\,Cu(00) = -h^2\tau\frac{\partial^3(Cu)}{\partial x^2\partial t} + \cdots$$

$$Cu(0\bar{1}) - Cu(00) = -\tau\frac{\partial(Cu)}{\partial t} + \frac{\tau^2}{2}\frac{\partial^2(Cu)}{\partial t^2} - \ldots$$

$$Cu(01) - Cu(00) = \tau\frac{\partial(Cu)}{\partial t} + \frac{\tau^2}{2}\frac{\partial^2(Cu)}{\partial t^2} + \ldots$$

with the equation (9) and putting $k=4$, we arrive at the expression with $O(h^6)$,

	Cu	
$\left(\frac{C}{10} - \frac{4}{10}A - \frac{Bh^2}{160}\right)u$	$\left(-C + \frac{8}{10}A - \frac{14}{160}Bh^2\right)u$	$\left(\frac{C}{10} - \frac{4}{10}A - \frac{Bh^2}{160}\right)u$
$\left(-\frac{C}{10} + \frac{A}{10}\right)u$	$\left(-\frac{2}{10}A + \frac{4}{160}Bh^2\right)u$	$\left(-\frac{C}{10} + \frac{A}{10}\right)u$

(e)

4. Heat conduction in a cylinder

For simplification, the case α=constant will be considered, here. Putting $u=r^{-1/2}Z$, Eq. (3) becomes

$$\frac{\partial Z}{\partial t} = \alpha\left(\frac{\partial^2 Z}{\partial r^2} + \frac{1}{4}\frac{Z}{r^2}\right). \tag{11}$$

Therefore we have the expression with $O(h^6)$,

	Z	
$-\frac{3}{10}Z - \frac{1}{160}\frac{Z}{r^2}$	$-\frac{2}{10}Z - \frac{14}{160}\frac{Z}{r^2}$	$-\frac{3}{10}Z - \frac{1}{160}\frac{Z}{r^2}$
	$-\frac{2}{10}Z + \frac{4}{160}\frac{Z}{r^2}$	

(f)

As $\frac{Z}{r^2} \to \infty$ at $r \to 0$, we cannot use the above expression at $r=0$, h, so the other form of the expression must be obtained.

Considering the relations

$$\frac{\partial u}{\partial r} = \frac{\partial^3 u}{\partial r^3} = 0, \qquad \alpha\frac{\partial^2 u}{\partial r^2} = \frac{1}{2}\frac{\partial u}{\partial t}, \qquad \alpha^2\frac{\partial^4 u}{\partial r^4} = \frac{1}{4}\frac{\partial^2 u}{\partial t^2}, \qquad \text{at } r=0,$$

we have the expression at the point $r=0$ using the values at $r=0$, $\sqrt{2}\,h$,

$$u(01) = \frac{1}{10}(2\,u(00) + 2\,u(0\bar{1}) + 6u(\sqrt{2}, 0)). \tag{g}$$

The value at $(\sqrt{2}-1)h$ is also given by the following expression.

$$u(\sqrt{2}-1,\ 0) = 0.887302\,u(00) + 0.126758\,u(\sqrt{2},\ 0) - 0.014059\,u(\sqrt{2}+1,\ 0). \tag{h}$$

To go a step of integration further, these expressions can be successfully applied, taking the points $r=0$, $(\sqrt{2}-1)h$, $\sqrt{2}\,h$, $(\sqrt{2}+1)h, \ldots$.

5. Numerical examples

i) Solve Eq. (2), giving the conditions;

$$u=0 \quad (x=0\sim l,\ t=0), \qquad u=1 \quad (x=0,\ t\geqq 0), \qquad \frac{\partial u}{\partial x}=0 \quad (x=l,\ t\geqq 0).$$

The results u at $x=1$ are shown in figs. 1 and 2, assuming (5) and

$$\frac{\alpha_1}{\alpha_0}=\frac{1}{2},\ 1,\ 2.$$

ii) Solve Eq. (3), giving the conditions;

$$u=0 \quad (r=0\sim a,\ t=0), \qquad u=1 \quad (r=a,\ t\geqq 0).$$

The results u at $r=0$ are shown in fig. 3, and the exact solution calculated by the analytical method is also plotted in the figure.

6. Conclusions

In this paper, the author has derived a numerical integration method for one-dimensional heat conduction problems, and shown several numerical examples. The results obtained have sufficient accuracy for practical purpose by dividing the variable range of x into only 2 intervals. The author believes that similar methods above mentioned will be available to the various complicated problems in transient phenomena.

References

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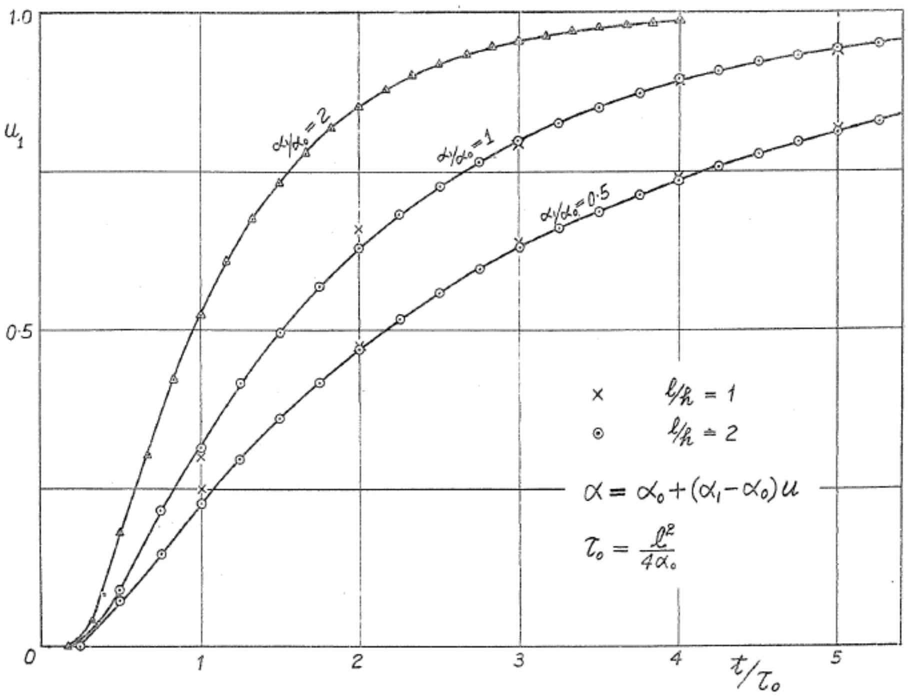
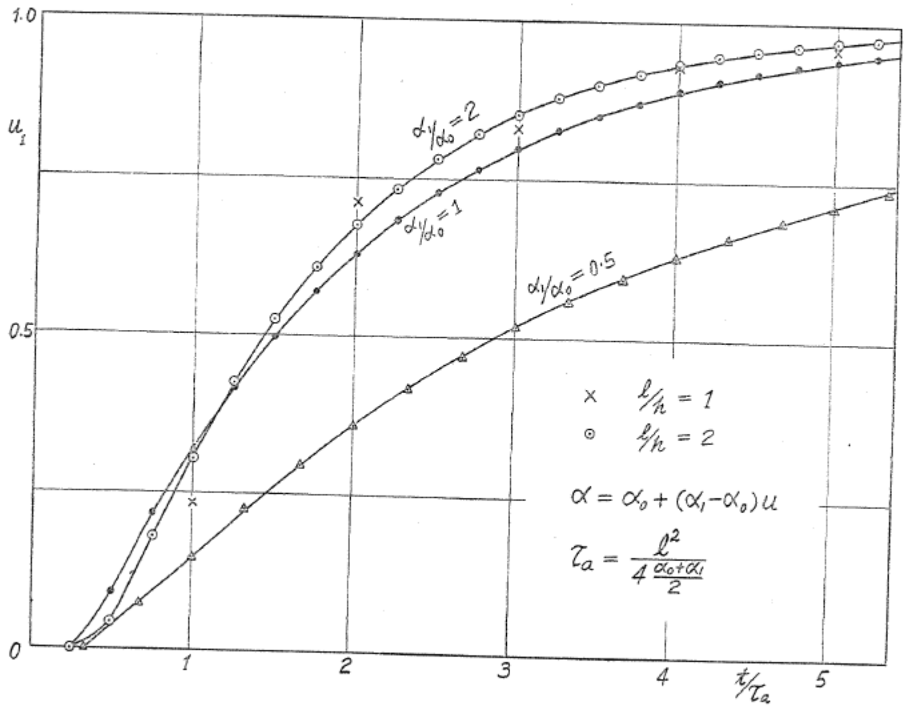
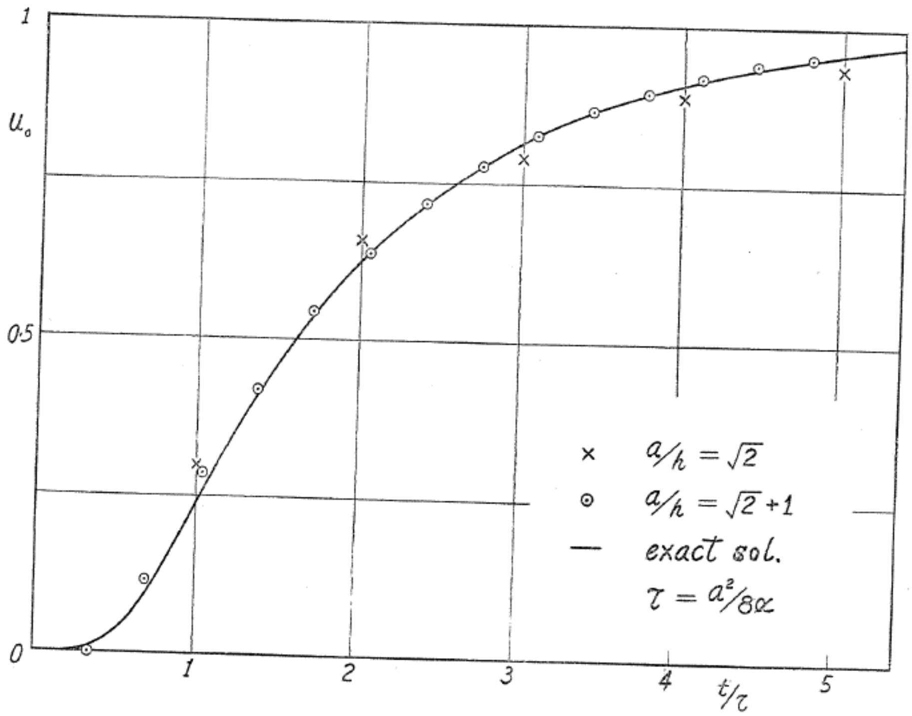


FIG. 1


1.0
0.5
0
u_1
$\alpha_1/\alpha_0=2$
$\alpha_1/\alpha_0=1$
$\alpha_1/\alpha_0=0.5$
1
2
3
4
5
t/τ_a
× $l/h = 1$
⊙ $l/h = 2$
$\alpha = \alpha_0 + (\alpha_1 - \alpha_0)u$
$\tau_a = \frac{l^2}{4\frac{\alpha_0+\alpha_1}{2}}$

FGI. 2


1
0.5
0
u_0
1
2
3
4
5
t/τ
× $a/h = \sqrt{2}$
⊙ $a/h = \sqrt{2}+1$
— exact sol.
$\tau = a^2/8\alpha$

FGI. 3