THEORETICAL RESEARCH ON CHATTER VIBRATION OF LATHE TOOLS

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Chapter I. General Remarks

1. Introduction

The cutting of metals is frequently accompanied by vibrations of cutting tool or workpiece, known as "chatter," the action of which is at times extremely violent and produces undulations on a work surface. The existence of chatter is a serious problem because it is detrimental to the life of tool, to the surface finish and to the accuracy of the machined parts. In these respect, an elucidation of the problem is very important. In recent years, chatter has become of increasing importance as a result of the increase in cutting speeds made possible by the development of the cemented carbide tools, because the ecconomic application of these tools are often prohibited owing to the risk of fatigue fractures in this material notwithstanding that it has high hardness at high temperature.

Chatter vibration occurring in machining metals are of two kinds, forced and self-excited.

Forced vibrations are those which occur under the action of a periodically varying force on the cutting tool or on the workpiece, the variations being due to mechanical causes such as unbalanced rotating members, rough spindle bearings, or headstock gears of insufficient accuracy. Such vibrations occur at a frequency dictated by the mechanical source of which frequency may be quite different from the natural frequencies of the vibrating members and will change in proportion to the speed of the spindle or machine.

Self-excited vibrations occurring in practical machining operation are classified into two types: One due to flexible cutting tools; the other to deflection of main spindle of lathe or workpiece. The frequency of self-excited chatter vibration is independent of the spindle speed but is very close to the natural frequency of the vibrating member.

Further, these self-excited vibrations are devided into the following two groups because of the circumstances under which chatter vibration occurs, *i.e.*, primary and regenerative. The so-called primary chatter is that occurring in the cutting conditions as where there is no interaction between the vibratory motion of the system and any undulatory surface produced in the preceding revolution of workpiece. The so-called regenerative chatter is that occurring in such cutting conditions as where the vibratory motion of the system is subject to the effect of undulatory surface produced during the preceding revolution. This effect of undulatory surface on the vibrationl system is called "feed-back."

Of these two vibrations, *i.e.*, forced and self-excited, the latter is generally more severe than the former, and ordinally chatter is commonly of the self-excited. In addition, the elimination of forced vibration is usually straightforward in comparison with that of self-excited vibration. Then, many investigators who concerned themselves with chatter vibration have studied on chatter of self-excited type, which is also the subject of this paper.

2. *Work of previous investigators*

Even though the elucidation of chatter vibration is very important, many investigators have concerned themselves with the theory of the metal cutting process for many years but very little work has been done on chatter vibration because chatter is of a very complex nature. In recent years, however, many investigators of various countries have become to study on chatter vibration.

In 1946, R. N. Arnold[1] carried out experiments on chatter vibration of a cutting tool and reported that the chatter is a self-excited vibration caused by the falling characteristic of cutting force depending upon cutting speed, the same as is self-excited vibration due to dry friction. In 1949, A. J. Chisholm[2] stated the same theory.

By the author[3] and S. Doi, however, it was clarified experimentally that the chatter resembles the frictional vibration in character, and that the former can hardly be caused by the effect of cutting speed on cutting force. R. S. Hahn[4] in 1953 and E. Salje[5] in 1956 also reported that chatter is not caused or influenced by the falling characteristic of cutting force depending upon the cutting speed. R. S. Hahn said that chatter is believed to be caused by lag of temperature and stress in the vicinity of the shear plane. According to E. Salje, chatter is caused by the cutting-angle-variation effect and the chip-thickness-variation effect on the cutting force during vibration.

Further investigations on chatter vibration have been made by J. Thusty[6] (1955), A. P. Sokolowski[7] (1955), N. H. Cook[8] (1955), W. Hölken[9] (1956) and S. A. Tobias and W. Fishwick[10] (1958).

By these investigations, it can be said that many characteristics of occurring chatter vibration and the effect of various cutting conditions on chatter vibration have been ascertained fairly well, but that the fundamental cause of chatter vibration has not yet been made clear.

To ascertain the cause of chatter vibration, the lateral vibration of rotating workpiece and the fluctuations in cutting force must be measured simultaneously. In this respect, S. Doi's work[11][12] is most significant. He measured the laterel vibration of rotating workpiece and the fluctuations in cutting force in each period of 0.001 sec. using a well-devised experimental apparatus, and reported that the fluctuation in horizontal cutting force lags slightly behind that of horizontal vibration of workpiece, and that because of this lag some amounts of energies are available for maintaining vibration. As a result, it was proved experimentally that chatter is a kind of self-excited vibration caused by the lag in fluctuations of horizontal cutting force existing behind the horizontal vibration of workpiece, which finding seems to be in good agreement with Hahn's opinion.[4]

However, it is not yet clear whether this lag of cutting force appears as an effect of occurring chatter vibration or whether it is an essential characteristic of machining metals. In other words, there is no critical data to show whether this lag is a cause or an effect of occurring chatter vibration. For this reason, the author[13][14][15] together with S. Doi carried out the following experiments.

3. *Preliminary experiments on the cause of chatter vibration*

(3.1) Method of experiments

The scheme of experimental apparatus is illustrated in Fig. 1. In the figure, workpiece A, diameter 70 mm, length 500 mm, is held by bearing B supported only in a vertical position by means of a steady rest. The bearing, together with the workpiece, is deflected periodically by eccentric roller C, driven by shaft D, and the thickness of chip is thus changed periodically. To verify whether the fluctuations in cutting force do or do not lag behind the variations of chip thickness, the horizontal oscillation of rotating workpiece and the fluctuations in cutting force must be measured simultaneously.

First, to measure precisely the horizontal oscillation of workpiece which is rotating, a ring of mild steel is fitted on the workpiece near the place at which the workpiece is to be cut. On this ring, mirror 4 (see Fig. 1) is glued and its surface is sooted. Rotating the workpiece, the soot is scratched so as to draw precisely a fine lined circle. A vertical bright line in the matrix can be observed when the small segment of the circle most laterally situated is lighted by vertical illuminater 3 and is magnified by a microscope. The center of the circle drawn on the mirror coincides exactly with the axis of rotation of the workpiece, so that the vertical line segment is stationary and is not affected in any way by rotation of the workpiece. When this vertical line segment is illuminated, the reflected light forms an image point on rotating film 6 passing through microscope and cylindrical lens 5. The microscope used in this experiment magnifies about 20 times.

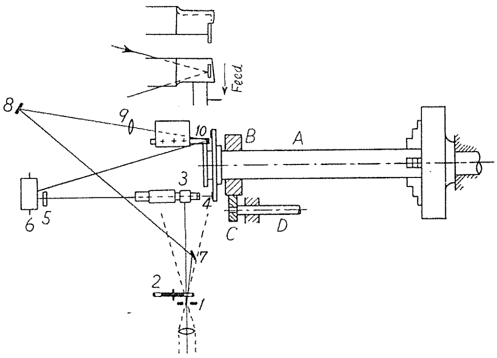

FIG. 1. Method of experiment.

Next, the cutting force is measured optically by the deflection angle of the tool end. A small mirror 10 is fitted to the lower nose of tool with glue, as shown in Fig. 1. The light, passing through small circular hole 1, reflecting on mirrors 7 and 8, and through condensing lens 9, is reflected on mirror 10, and this reflected light is recorded on rotating film 6. Lens 9 is adjusted so as to bring the focus of small hole on the rotating film.

To measure the horizontal displacement of workpiece and the cutting force simultaneously, two lights are derived from the same light source, as shown in Fig. 1. One is reflected by mirror 10 fitted to the tool end, and the other is reflected by the fine line of the ring mirror fitted on the workpiece; these lights are recorded simultaneously on the same rotating film 6. To find the relation between the fluctuations in cutting force and the oscillation of workpiece in each phase, the light source is interrupted by time-marker 2 at intervals of 0.04 sec. To record these two lights at the same instant, thick marks are made at intervals of 0.4 sec. The cutting force and the horizontal oscillation of workpiece in each phase are noted at each period of 0.04 sec.

(3.2) Experimental results

In order to simplify the cutting manner, an orthogonal cutting is operated on the flange of mild steel with a side lathe tool, as shown in Fig. 1.

Fig. 2 is an example of experimental records obtained in cutting conditions where the cutting speed is 1.0 m/min., cutting angle of tool 80°, thickness of flange 3 mm, and the frequency of oscillation of workpiece is 1.5/sec. with an amplitude 0.14 mm. In the figure, curve (*a*) represents the horizontal oscillation of workpiece. The upper displacement of each dot corresponds to the approaching movement of workpiece toward the cutting edge. Curve (*b*) represents the fluctuations of cutting force. The displacement of each dot in perpendicular direction corresponds

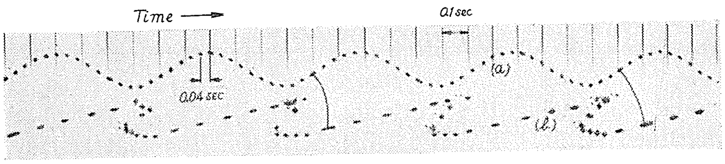

FIG. 2. Experimental record.

to the fluctuations of cutting force in horizontal direction, and transverse displacement corresponds to cutting force in vertical direction. The heavy dots spaced every 10 dots in each curve indicate the same instant and are connected by lines to make them quickly identifiable in the figure.

Measuring the position of each dot in curves (a), (b) in Fig. 2 by a sensitive comparator, we obtain Fig. 3. Curves (a), (b) and (c) in Fig. 3 represent the horizontal oscillation of workpiece, fluctuations of cutting force in horizontal direction and those in vertical direction respectively. Each dot on the same vertical line is the same instant.

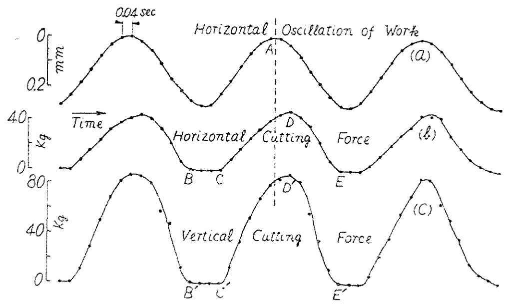

FIG. 3. Horizontal oscillation of workpiece and fluctuations of cutting force.

The upward displacement in curve (a) corresponds to the approaching movement toward the cutting edge. At point A the workpiece moves nearest to the cutting edge and the thickness of chip must be a maximum value. In curves (b) and (c), B-C-D-E and B'-C'-D'-E' represent one cycle of the fluctuations of cutting force in horizontal and vertical directions respectively. At intervals B-C and B'-C', the cutting forces remain zero so that the workpiece in this period is completely apart from the cutting edge. As the workpiece approaches the cutting edge, cutting forces in both directions increase along curves CD and $C'D'$, and reach maximum values at points D and D' lagging about 0.02–0.04 sec. behind point A at which the magnitude of chip thickness is considered to be maximum.

Now, Fig. 4 shows the relationship between the horizontal displacement of workpiece and the horizontal cutting force obtained from Fig. 3. The solid line shows a previous cycle and the broken line, the following cycle. The period 1-2 indicates the approaching stroke of oscillating workpiece toward the cutting edge, and 2-3-1 is the recess stroke. It is seen in the figure that the horizontal cutting force does not reach a maximum value at point 2 where the workpiece moves horizontally nearest to the cutting edge, and that when the workpiece rotates a few times after this nearest point, that is at point 3, cutting force becomes maximum. This lag of cutting force exists not only at point 2 or 3, but also at any point of a cycle. Therefore, the closed curves in the figure show somewhat elliptical forms, and the action of cutting force in recess stroke is considerably larger than that in approaching stroke.

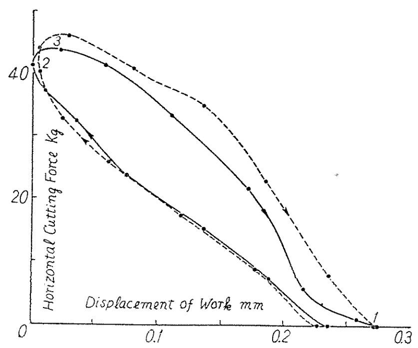

FIG. 4. Force-displacement curve.

From the foregoing, it can be considered that even in cuttings without chatter vibration, the fluctuations in horizontal cutting force also lag behind the variations of undeformed chip thickness in the same way as that observed in chatter vibration by S. Doi.[11][12]

The cause of this lag of cutting force is here considered. The force system acting on the tool-chip-work system in a cutting producing a flow-type chip without built up edge is shown schematically in Fig. 5. This force system is obtained by considering the chip as a separate body held in equilibrium by the action of frictional force F along the tool face, a normal force N perpendicular to the tool face, and force R' equal and opposite to the resultant R of forces F and N, acting on the shear plane. The force components acting on the shear plane are F_s and F_n. F_s represents the force required to shear the metal on the shear plane and is known as the shearing force. F_n acts normal to the shear plane and results in a compressive stress being applied to the shear plane. The shear strength of metal depends upon the compressive stress on the shear plane, as suggested by M. E. Merchant.[16] An approximate relation for shear strength τ is

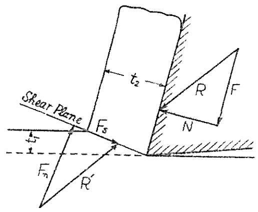

FIG. 5. Cutting mechanism.

$$\tau = \tau_0 + k\sigma$$

where τ_0 and k are constants for the metal being cut and σ is the compressive stress on the shear plane.

Then, if the undeformed chip thickness varied by the periodical motion of the workpiece, the frictional force could not change instantly according to variations of chip thickness. That is to say the frictional force lags behind the variations of chip thickness caused by the oscillation of the workpiece. In addition, the greater part of frictional force acts as the compressive force being applied to the shear plane, as shown in Fig. 5. Therefore, it can also be considered that the force required to shear the metal on the shear plane has the after-effect or lag.

When the machining of mild steel operates at high speed, the chip formation will be as shown in Fig. 5. At comparative low speeds, however, the sliding of the metal does not occur on a single plane, but in a zone extending from the cutting edge to the work surface. Accordingly, the part extending from the chip to the work surface is generally curved, which fact is closely connected with the formation of built up edge. In those cuttings which produce the continuous chip with built up edge, the presence of after-effect or lag in cutting force can be considered.

Further experiments[13][14][15] were carried out to examine the relation between the various cutting conditions and the properties of time lag of cutting force. Without exception, in 21 experimental records similar to Fig. 2 in which the frequency of oscillation of workpiece, amplitude of oscillation, cutting angle of tool, or the cutting speed was changed, the presence of time lag of cutting force was ascertained in machining of mild steel.

From the foregoing, it was proved experimentally that even in cuttings without chatter vibration, fluctuations in horizontal cutting force lag behind variations of undeformed chip thickness caused by the periodical motion of workpiece in horizontal direction, and then that this lag of cutting force does not appear as an effect of occurring chatter vibration but the presence of lag is an essential characteristic of machining metals.

Consequently, we can prove experimentally that the time lag of cutting force is a cause and not an effect of chatter vibration.

Then, in this paper, based on this experimental results a differential equation for chatter vibration is introduced, and theoretical analyses on chatter vibration were carried out. By these theoretical analyses, the cause of chatter vibration will be clarified theoretically and many characteristics of occurring chatter vibration will be explained collectively by the theoretical expressions.

Chapter II. Chatter Vibration due to Deflection of Main Spindle of Lathe—Linear Theory applied to the Analyses on Primary Chatter[15) 17)]

1. Differential equation for chatter vibration

We first consider our problem relating to the horizontal vibration of workpiece. Following is an analysis of chatter vibration occurring in orthogonal cutting operations. Our analytical results, however, include other cutting operations.

Let C in Fig. 6 denote the center of the workpiece before cutting. When a cut is operated on the workpiece, center C displaces to point O. If we denote the displacement by a and the undeformed chip thickness by d, it would follow that

$$F(d)=ka \tag{2.1}$$

where $F(d)$ is the horizontal cutting force corresponding to the chip thickness d, and ka is the spring force.

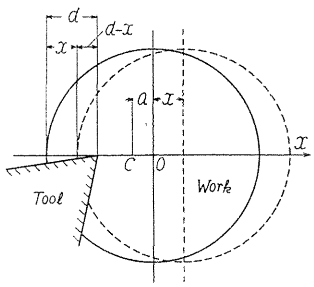

FIG. 6. Workpiece and cutting edge.

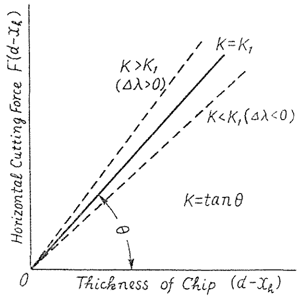

FIG. 7. Relation between thickness of chip and cutting force.

Taking 0 as the point of origin and indicating the displacement of workpiece by x, when the movement is in the direction away from the cutting edge, it is considered positive. Assuming, for the sake of simplicity, that the system has one degree of freedom, the expression for the chatter vibration becomes

$$m\ddot{x}+c\dot{x}+k(x+a)=F\{d-x(t-h)\} \tag{2.2}$$

where m, c, and k are well-known physical constants, $k(x+a)$ is the spring force, $F\{d-x(t-h)\}$ is the horizontal cutting force, and where the deflection of the tool is neglected. In the expression for the horizontal cutting force, $d-x$ is the undeformed chip thickness during vibration, and h indicates the time lag in horizontal cutting force existing behind the horizontal vibration of workpiece. Namely, based on our experimental results in Chapter I, the horizontal cutting force is represented as a function of undeformed chip thickness having the time lag h.

Setting

$$\left.\begin{aligned} &\frac{c}{m}=2n, \quad \frac{k}{m}=p^2, \quad x(t-h)=x_h \\ &\frac{1}{m}F(d-x_h)-\frac{k}{m}a=f(d-x_h) \end{aligned}\right\} \tag{2.3}$$

and

Equation (2.2) becomes

$$\ddot{x}+2n\dot{x}+p^2x-f(d-x_h)=0. \tag{2.4}$$

We will deal with our problem on the basis of Equation (2.4).

2. Chatter vibration occurring in the region $d-x\geqq 0$

We will now consider that the horizontal cutting force is proportional to the magnitude of thickness of chip, as shown in Fig. 7, so that

$$F(d-x_h)=K\cdot(d-x_h), \qquad d-x\geqq 0 \tag{2.5}$$

in which K is the coefficient of proportion depending upon the cutting conditions, and is expressed by

$$K=k_s b \tag{2.6}$$

where k_s is the specific horizontal cutting force of the metal being machined and b is the length of the cutting edge engaged in cutting.

On substituting Equations (2.1) and (2.5) into Equation (2.3), we obtain

$$f(d-x_h)=-\lambda x_h \tag{2.7}$$

where $\lambda=\dfrac{K}{m}$. Substituting this Equation in Equation (2.4), we have the following equation:

$$\ddot{x}+2n\dot{x}+p^2x+\lambda x_h=0. \tag{2.8}$$

We try to satisfy Equation (2.8) by a solution of the form

$$x=x_0e^{zt}, \qquad z=\alpha+j\omega \tag{2.9}$$

where α is a decrement (if $\alpha<0$) or an increment (if $\alpha>0$), ω is the frequency and $j=\sqrt{-1}$.

On substituting Equation (2.9) for (2.8) we obtain

$$z^2+2nz+p^2+\lambda e^{-zh}=0. \tag{2.10}$$

The real and imaginary parts of this expression yield

$$\left\{\begin{aligned} &p^2-\omega^2+\alpha^2+2n\alpha+\lambda e^{-h\alpha}\cos\omega h=0 && (2.11)\\ &2(\alpha+n)\omega-\lambda e^{-h\alpha}\sin\omega h=0, && (2.12)\end{aligned}\right.$$

For a harmonic vibration ($\alpha=0$), we have

$$\begin{cases} p^2-\omega^2+\lambda\cos\omega h=0 & (2.13)\\ 2n\omega-\lambda\sin\omega h=0. & (2.14)\end{cases}$$

The harmonic vibration can exist only when Equations (2.13) and (2.14) are fulfilled simultaneously. Combining these equations, we obtain the following expressions

$$\begin{cases} \omega_1=\omega(\lambda,\ n,\ h,\ p) & (2.15)\\ g(\lambda,\ n,\ h,\ p)=0. & (2.16)\end{cases}$$

Only when Equation (2.16) is satisfied, can a harmonic vibration with frequency ω_1 exist, and its amplitude should be decided from the initial conditions. If we consider that the parameters n, h, and p (not λ) are assigned, Equation (2.16) gives a certain value for parameter λ, which is the so-called harmonic value λ_1, *i.e.*,

$$\lambda_1=\frac{2n\omega_1}{\sin\omega_1 h} \tag{2.17}$$

and the frequency of the harmonic vibration is given by

$$\frac{\omega_1^2-p^2}{2n\omega_1}=\cot\omega_1 h. \tag{2.18}$$

The value of ω_1 can be obtained graphically as will be indicated later.

If the magnitude of λ deviates from a harmonic value expressed by Equation (2.17), the vibration ceases to be harmonic ($\alpha\neq 0$). We are now interested only in solution of Equation (2.8) not far from the harmonic solution ($\alpha=0$). In that neighborhood α is small. Let us consider here the small change $\Delta\lambda$ in the parameter λ from its harmonic value λ_1 for which $\alpha=0$, $z=j\omega_1$. For $\lambda_1+\Delta\lambda$, the corresponding value of z will be

$$z=j\omega_1+\Delta z,\qquad \Delta z=\Delta\alpha+j\Delta\omega.$$

Substituting this value into Equation (2.10), separating the real and the imaginary parts and carrying out calculations to the first order of small quantities $\Delta\lambda$, $\Delta\alpha$, and $\Delta\omega$, we have

$$\begin{cases} \dfrac{\Delta\alpha}{\Delta\lambda}=\dfrac{\lambda_1^2 h+2n(p^2+\omega_1^2)}{\lambda_1\{(2n-\lambda_1 h\cos\omega_1 h)^2+(2\omega_1+\lambda_1 h\sin\omega_1 h)^2\}} & (2.19)\\[2ex] \dfrac{\Delta\omega}{\Delta\lambda}=\dfrac{2\omega_1(2n^2+\omega_1^2-p^2)}{\lambda_1\{(2n-\lambda_1 h\cos\omega_1 h)^2+(2\omega_1+\lambda_1 h\sin\omega_1 h)^2\}}\,. & (2.20)\end{cases}$$

The right-hand side of Equation (2.19) is always positive; hence $\Delta\alpha\gtreqless 0$ for $\Delta\lambda\gtreqless 0$, but since $\alpha=0$ for $\Delta\lambda=0$, this means $\alpha\gtreqless 0$ for $\Delta\lambda\gtreqless 0$. Therefore, if $\Delta\lambda>0$ (that is, $\lambda>\lambda_1$) the motion occurs with increments, and if $\Delta\lambda<0$ ($\lambda<\lambda_1$) the vibration dies out, and only for $\Delta\lambda=0$ (that is, $\lambda=\lambda_1$) the harmonic vibration exists.

Now, let us examine the foregoing results relating to Fig. 7. In the figure, when the magnitude of K, which is the coefficient of proportion, is just equal to K_1, *i.e.*, $K=K_1=m\lambda_1$, there exists a harmonic vibration, and when $K>K_1$ (that is,

$\lambda > \lambda_1$) the motion occurs with increment ($\alpha > 0$). and when $K < K_1$ ($\lambda < \lambda_1$) the vibration dies out ($\alpha < 0$). Consequently, the whole area above $K = K_1$ line constitutes an unstable range.

For parameters n, h, and p, the same expressions as Equations (2.19) and (2.20) can easily be solved.

By the above, it is proved that the system given by Equation (2.8) are capable of self-excited vibrations due to the time lag h.

N. Minorsky,[18] in 1948, derived an equation for stabilization of ships. His equation includes the term of retarded velocity, and resembles our Equation (2.4) in form. He reached the conclusion that the system expressed by his equation is capable of self-excitation due to the retarded action.

We will now calculate the magnitude of λ_1 from Equation (2.17), where the harmonic value λ_1 is the function of n, h, and p. Although it depends strongly upon the structure of main spindle and main bearings of a lathe, the magnitude of n (damping coefficient of system) is generally $30 \sim 200$/sec. in practice, using the experimental results reported by S. Doi[12] and others.[4][5] Employing the same results,[12] the magnitude of h (time lag in horizontal cutting force existing behind horizontal vibration of workpiece) is generally between 0.0005 and 0.001 sec. The natural circular frequency p of the vibrational system is usually in a range from 200 to 1,200/sec.,[4][5][12] although these values are closely related to the rigidity of the main spindle of lathe and the mass of the chuck and that of the workpiece.

Figs. 8, 9 and 10 are obtained by numerical calculations using these values. Fig. 8 shows the relationship between the harmonic value λ_1 and the damping coefficient n of the system, where it is plotted for one value of p and several values of h. Here it is seen that as the magnitude of n increases, λ_1 increases almost linearly. The whole area above λ_1-curve constitutes the range of self-excitation for each vibrational system. We can see that when the magnitude of n is small, the self-excited vibration is likely to occur, which in fact is reasonable.

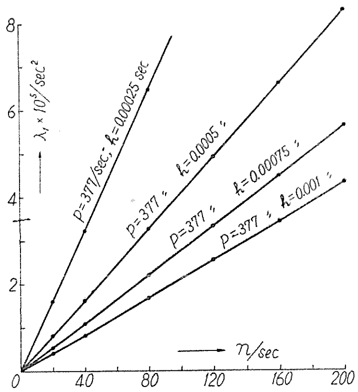

FIG. 8. Relation between damping coefficient n and harmonic value λ_1.

The solid lines in Fig. 9 where they are plotted for one value of h and several values of n, show the relationship between p and λ_1. In each curve, it is seen that even if the magnitude of p increases considerably, the harmonic value λ_1 increases very little, which means that the increase in rigidity of vibrational system does not increase the stability of the system. This theoretical result seems to be contrary to the actuality. The reason for seeming contradiction is that each curve represented by a solid line is plotted on the assumption that n is constant while p increases, but in practice the magnitude of n usually increases with increase of p, *i.e.*, the expression for the relationship between n and p is written

$$n = \frac{p \log v}{\pi}$$

where v is the damping ratio. If we here assume that the magnitude of n increases in proportion to p, the corresponding $p-\lambda_1$ curve would be represented by the broken line in Fig. 9. The area to the left of the broken line constitutes the unstable range. By this it can be seen that the increase in rigidity of the vibrational system results in increase of the stability of system, which is in good agreement with the observed facts.[19]

Fig. 10 shows $h-\lambda_1$ curve, where it can be seen that the magnitude of λ_1 decreases sharply with increase of h. Thus the chatter is most likely to occur under cutting conditions in which the time lag in horizontal cutting force is large. For example, the machining operation using a round-nosed tool seems to cause a large time lag in cutting force, because the lag in cutting force is presumed to be related to the degree of chip failure, as described in Chapter I. In such cutting operation, the chatter that is most likely to occur is the one experienced by the lathe turner.[19] Further, since the magnitude of time lag has a tendency to decrease with increase of cutting speed,[13] the chatter cannot occur when the cutting speed is higher than a certain value, which fact is in good agreement with the experimental results reported by S. Doi[12] and E. Salje.[5]

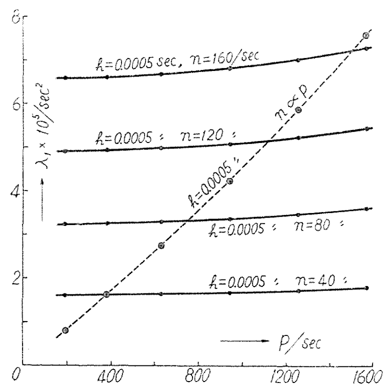

FIG. 9. Relation between natural frequency p and harmonic value λ_1.

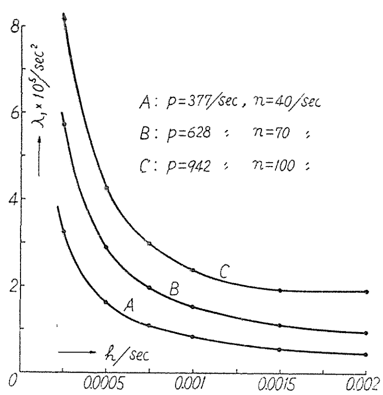

FIG. 10. Relation between time lag h and harmonic value λ_1.

The stability or instability of the vibrational system, according to the above description, is illustrated in Figs. 8, 9, and 10. In addition, the magnitude of λ is large in such cutting conditions as where the magnitude of cutting angle of tool, the length of cutting edge engaged in cutting, and the wear of the cutting edge are all large, and where the specific cutting force of the metal being machined is large. Accordingly, it can be said that the chatter is more likely to occur in these cutting conditions. Moreover, we have dealt with our problem with the proviso that the deflection of tool end during vibration be neglected. However, we can say that since the tool which is apt to deflect, *i.e.*, the tool lacking rigidity, causes the apparent decrease of λ, the chatter is not more likely to occur with decrease of tool rigidity. These theoretical findings are also in good agreement with the experimental results published by S. Doi[12][19] and E. Salje[5] and others.

By the above description, it is proved that the system expressed by Equation (2.8) is capable of self-excitation due to time lag h in horizontal cutting force, and it is found that many characteristics of occurring chatter vibration can be explained theoretically.

Hitherto we have dealt with our subject in the region where $d-x\geqq 0$. According to our theory, in this region it is evident that the stable vibration ($\alpha=0$) can hardly exist, because the harmonic vibration can exist only for the condition $\lambda=\lambda_1$ as described above. It is almost impossible to do the cutting operation under such a cutting condition where the magnitude of λ is just equal to λ_1. In most cases the magnitude of λ is either $\lambda<\lambda_1$ or $\lambda>\lambda_1$, and therefore, the vibration usually dies out ($\lambda<\lambda_1$, $\alpha<0$) or the amplitude increases indefinitely ($\lambda>\lambda_1$, $\alpha>0$).

3. Chatter vibration going beyond the region $d-x\geqq 0$

On the basis of our study outlined above, in region $d-x\geqq 0$ the amplitude increases indefinitely if $\alpha>0$, which seems to be contrary to the observed facts. If the amplitude increases, however, the motin deviates from region $d-x\geqq 0$ for a certain time during one cycle at which time the workpiece leaves the cutting edge. Hence in the region in which the workpiece leaves the cutting edge, where $d-x<0$, the cutting force is zero, and the motion is a free vibration with damping. The eqaution for such vibration of large amplitude can be written

$$\begin{cases} \ddot{x}+2n\dot{x}+p^2x+\lambda x_h=0 & \text{for} \quad d-x\geqq 0 \\ \ddot{x}+2n\dot{x}+p^2(x+a)=0 & \text{for} \quad d-x<0. \end{cases} \tag{2.21}$$

This equation can be solved graphically by Jacobsen's "phase-plane-delta" method[20] as follows: Writing Equation (2.21)

$$\ddot{x}+p^2(x+\delta)=0$$

in which

$$\begin{cases} \delta=\dfrac{1}{p^2}(2n\dot{x}+\lambda x_h) & \text{for} \quad d-x\geqq 0 \\ \delta=\dfrac{2n\dot{x}}{p^2}+a & \text{for} \quad d-x<0. \end{cases}$$

Using the expressions

$$x=x, \qquad \frac{\dot{x}}{p}=v, \qquad \ddot{x}=p^2v\frac{dv}{dx}$$

Equation (2.21) becomes

$$v\frac{dv}{dx}+x+\delta=0$$

which gives the relation

$$\frac{dx}{dv}=\frac{-v}{x+\delta}. \tag{2.22}$$

Taking the displacement x as ordinate and the velocity of motion as abscissa, Equation (2.22) gives a field direction graphically at any point. Therefore, we can approximate a solution curve through any given point in the following way: Fig. 11 shows that the slope of the line R-P will be $(x+\delta)/v$; consequently the equation of the normal to line R-P is identical with Equation (2.22). Moreover, in a step method δ remains constant for the step since an integration of Equation (2.22) gives the equation of a circle with its center at R.

Let $\rho = \overline{RP}$, $ds = \overline{PQ}$, and $d\theta = \angle PRQ$ in Fig. 11, and the following relation will be obtained

$$d\theta = \frac{ds}{\rho} = \frac{dx}{v}.$$

Consequently, the explicit expression for time t is

$$dt = \frac{dx}{pv} = \frac{1}{p}d\theta \qquad \text{or} \qquad d\theta = pdt.$$

This expression can be integrated for the duration of the step, and gives

$$\theta - \theta_0 = p(t - t_0).$$

If we now use a step method by taking steps at intervals $\frac{h}{N}$ (N is any positive integer), the last expression becomes $\theta - \theta_0 = p\frac{h}{N}$.

Therefore, the graphical solution of Equation (2.22) can be obtained by a stepwise construction. This step method should be started from the point $x = x_0$, $v = 0$ at which the value of x_h can be considered nearly equal to x in good accuracy.

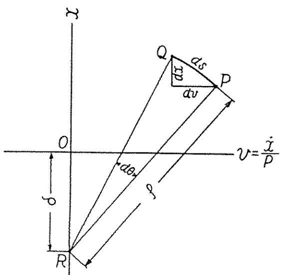

FIG. 11. Phase-plane-delta method.

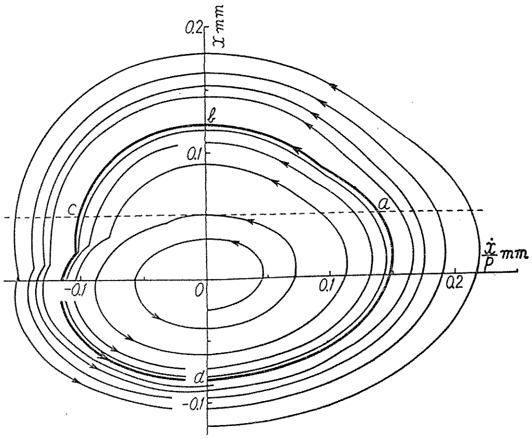

FIG. 12. Graphical solution of vibration.

Fig. 12 is an example of a graphical solution using the cutting conditions of Doi's experiments,[12] *viz.*, $d = 0.05$ mm, $p = 377$/sec. ($f_0 = 60$/sec.), $n = 40$/sec., $h = 0.0005$ sec., and $K = k_s b = 120 \times 6$ kg/mm (k_s is specific horizontal cutting force and b is thickness of flange or length of cutting edge engaged in cutting) and $\lambda = 2.45 \times 10^5/\text{sec.}^2$ The harmonic value of the parameter λ in this case is

$$\lambda_1 = 1.62 \times 10^5/\text{sec.}^2$$

so that $\alpha > 0$.

As shown in Fig. 12, curves starting near the origin spiral away from it, while those starting far from the origin spiral toward it, and there exist a limit cycle a-b-c-d-a. In the segment a-b-c of the limit cycle, in which $d - x < 0$, the workpiece leaves the cutting edge during vibration. Thus it is evident that the amplitude cannot increase indefinitely even if $\alpha > 0$, and that there exists a stable limit cycle with such an amplitude sufficient to cause the workpiece to leave the cutting edge for a certain period of one cycle during vibration. In practice, this type of chatter vibration is most common.[12] The amplitude of the limit cycle is about 0.1 mm which is in good agreement with the experimental results by S. Doi.[12]

4. On the vertical vibration

Previously we have dealt with our problem relating only to the horizontal vibration of workpiece, because, as S. Doi reported, chatter vibration is caused by the fluctuations in area of cut due to the horizontal vibration of workpiece. The reasons given by S. Doi[19] are as follows: (i) When the chatter is excited, the horizontal amplitude of workpiece is enlarged faster than that of vertical direction; on the other hand, when the chatter dies out the horizontal amplitude decreases faster than that of vertical direction. (ii) According to experiments on prevention of chatter, chatter can be effectively avoided by supressing both horizontal vibrations of workpiece and of cutting edge. (iii) Even when the natural frequency of the vibrational system in vertical direction differs from that in horizontal direction, the frequency of chatter vibration in both directions is the same as that related to the natural frequency of vibrational system in horizontal direction.

If the horizontal vibration of workpiece is initially started at its own natural frequency, the area of cut will fluctuate with that frequency, causing the vertical cutting force to do the same, and the vertical vibration will be set up. Therefore, the workpiece generally vibrates in a somewhat elliptical orbit. The differential equation of chatter vibration in vertical direction is then expressed by

$$m'\ddot{y} + c'\dot{y} + k'(y + a') = F'(d - x_{h'}) \tag{2.23}$$

where m', c', and k' are physical constants which are not always equivalent to those in horizontal direction, and where $F'(d-x_h)$ is the vertical cutting force which is represented as the function of the chip thickness having time lag h'. The magnitude of this time lag in vertical cutting force, according to our previous experimental results, is slightly smaller than that in horizontal cutting force.[13]

We will here assume that the vertical cutting force is proportional to the magnitude of chip thickness as well as in the previous treatment, so that

$$F'(d - x_{h'}) = K' \cdot (d - x_{h'}).$$

Using this expression, we have

$$\ddot{y} + 2n'\dot{y} + p'^2 y + \lambda' x_{h'} = 0 \tag{2.24}$$

where

$$\frac{c'}{m'} = 2n', \qquad \frac{k'}{m'} = p'^2, \qquad \frac{K'}{m'} = \lambda'.$$

We confine our attention to the harmonic vibration so that the horizontal vibration of workpiece can be expressed by $x = x_0 \sin \omega t$.

On substituting the last equation into Equation (2.24) we obtain

$$\ddot{y} + 2n'\dot{y} + p'^2 y = -\lambda' x_0 \sin \omega(t - h'). \tag{2.25}$$

A particular solution of Equation (2.25) will have the form

$$y = y_0 \sin(\omega t - \theta) \tag{2.26}$$

in which y_0 and θ are constants. Substituting this expression into Equation (2.25) we find that Equation (2.26) is satisfied when the constants y_0 and θ fulfill the following equations

$$\left\{\begin{aligned} y_0 &= \frac{\lambda' x_0}{\sqrt{(\omega^2 - p'^2)^2 + 4n'^2\omega^2}} && (2.27)\\ \tan(\omega h' + \theta) &= \frac{2n'\omega}{\omega^2 - p'^2}. && (2.28)\end{aligned}\right.$$

In Equations (2.27) and (2.28), the physical constants of the system in vertical direction, denoted by n' and p', are not always exactly equivalent to those in horizontal direction, because the condition of fitting the main spindle to the main bearings, the factor having the most important effect on the magnitude of n', and the rigidity of main spindle are not always equal in both directions. The difference between them, however, is generally small. We then represent n' and p' by $n' = n + \Delta n$, $p'^2 = p^2 + \Delta p^2$, where n and p are in horizontal direction.

Substituting these relations into Equations (2.27) and (2.28), and carrying out calculations to the first order of small quantities Δn, Δp^2, we obtain

$$\left\{\begin{aligned} y_0 &= \frac{\lambda'}{\lambda} x_0 \left(1 - \frac{\Delta n}{n}\sin^2\omega h + \frac{\Delta p^2}{\omega^2 - p^2}\cos^2\omega h\right) && (2.29)\\ \tan(\omega h' + \theta) &= \tan\omega h\left(1 + \frac{\Delta n}{n} + \frac{\Delta p^2}{\omega^2 - p^2}\right). && (2.30)\end{aligned}\right.$$

In Equations (2.29) and (2.30), the second and third terms in parentheses on the right-hand side are generally of small quantity. When $\Delta n = 0$ (that is, $n = n'$) and $\Delta p^2 = 0$ (that is, $p = p'$), these terms are vanished, and thus Equations (2.29) and (2.30) become

$$\left\{\begin{aligned} y_0 &= \frac{\lambda'}{\lambda} x_0 && (2.31)\\ \theta &= \omega(h - h'). && (2.32)\end{aligned}\right.$$

The vertical motion of the workpiece during chatter vibration is then expressed by

$$y = y_0 \sin(\omega t - \theta)$$

and the horizontal vibration of workpiece is, of course, written as

$$x = x_0 \sin\omega t.$$

Accordingly, it can be said that when the harmonic vibration of workpiece exists in horizontal direction, the harmonic vibration of the workpiece exists also in vertical direction with the amplitude y_0 indicated by Equation (2.29) and the phase difference θ denoted by Equation (2.30), and that the frequency of vertical vibration is the same as that of horizontal vibration. Therefore, the workpiece vibrates around an orbit in a somewhat elliptical form. This vibrating manner, however, varies according to the various characteristics of vibrational system and cutting conditions, as seen in Equations (2.29) and (2.30). This theoretical fact is in good agreement with the experimental results given by S. Doi[12] and others.[5)6)7]

5. Conclusions

We have introduced a differential equation for chatter vibration based on our previous experimental results that fluctuations in horizontal cutting force lag slightly behind variations of undeformed chip thickness caused by the horizontal vibration of workpiece. By this equation, it has been proved theoretically that the chatter vibration is a kind of self-excited vibration caused by this lag. Furthermore, expressions for conditions of self-excitation have been introduced by our equation. By these expressions we have been able to explain theoretically many characteristics of occurring chatter vibration.

Chapter III. Chatter Vibration due to Deflection of Main Spindle of Lathe—Non-linear Theory applied to the Analyses on Primary Chatter[21]

1. Introduction

In the previous chapter, it was proved theoretically that chatter is a kind of self-excited vibration caused by the time lag of cutting force, and that many characteristics of occurring chatter vibration can be explained almost entirely by our equation.

However, since the previous treatment was carried out on the assumption (adopted for simplicity) that the cutting force was in proportion to the thickness of chip, there were some theoretical facts which contradicted the actuality. These seemingly contradictory facts are, in the main, as follows: (i) According to the linear theory, a stable vibration can hardly exist in the region where $d-x \geqq 0$, In practice, chatter occurring in this region is, in general, undoubtedly unstable. In most cases its amplitude either decreases or increases, but even in this region the stable vibration can exist, to some extent, under certain cutting conditions. Fig. 13 is an example of experimental record showing such stable vibration. In this figure, curves (a) and (b) are the horizontal vibration of workpiece and the horizontal component of cutting force respectively, and straight line (c) represents the stationary position of cutting tool where the cutting force is zero. It is seen that the cutting force does not reach zero in every cycle and the workpiece does not leave the cutting edge at all during vibration. (ii) According to the linear theory, the amplitude of harmonic vibration ($\alpha=0$) must be decided from the initial conditions, which is contradictory to the actuality.

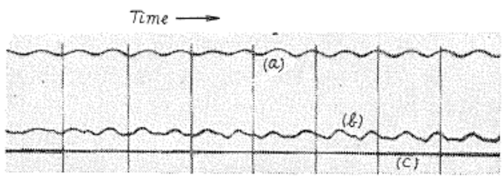

FIG. 13. Horizontal vibration of workpiece and horizontal cutting force.

In this chapter, therefore, the analyses are made on the non-linear theory.

2. Equation of chatter vibration

The analyses are made relating only to horizontal vibration of workpiece for the same reason as given in Chapter II. The equation for chatter is either

$$m\ddot{x} + c\dot{x} + k(x+a) = F(d - x_h) \tag{3.1}$$

or

$$\ddot{x} + 2n\dot{x} + p^2 x - f(d - x_h) = 0 \tag{3.2}$$

where

$$2n = \frac{c}{m}, \quad p^2 = \frac{k}{m}, \quad x_h = x(t-h), \quad f(d - x_h) = \frac{1}{m}\{F(d-x_h) - ka\}.$$

As previously mentioned, we based our analyses for reasons of simplicity on the assumption that cutting force was in proportion to thickness of chip as indicated by

$$F(d - x_h) = K \cdot (d - x_h), \qquad d - x \geqq 0.$$

In practice, however, the relationship between them is usually non-linear (see Fig. 14). The non-linearity is comparatively large particularly in horizontal cutting force.

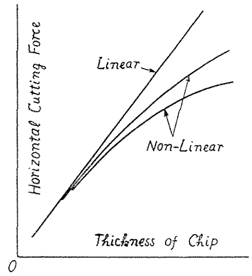

FIG. 14. Relation between thickness of chip and cutting force.

Expressions for the relationship between horizontal cutting force and thickness of chip have been represented in many ways.[22] Convenient for the following treatment, the characteristic is given here in the from

$$F(d - x_h) = K \cdot (d - x_h) - K_3 \cdot (d - x_h)^3, \; d - x \geqq 0 \tag{3.3}$$

where K and K_3 are coefficients which are closely connected with cutting conditions. Coefficient K_3 is generally small in comparison with K.

In the same sense as Equation (2.1) in Chapter II we have

$$Kd - K_3 d^3 = ka. \tag{3.4}$$

Consequently,

$$f(d - x_h) = -\lambda x_h + g(x_h) \tag{3.5}$$

where $\lambda = \dfrac{K}{m}, \quad \lambda_3 = \dfrac{K_3}{m}, \quad g(x_h) = \lambda_3 x_h^3 - 3\lambda_3 d x_h^2 + 3\lambda_3 d^2 x_h.$

On substituting the last expressions for Equation (3.2), the differential equation for chatter vibration becomes

$$\ddot{x} + 2n\dot{x} + p^2 x + \lambda x_h - g(x_h) = 0. \tag{3.6}$$

This equation includes the additional non-linear term $g(x_h)$ not included in the linear equation.

3. *Analyses*

We now take up the non-linear Equation (3.6). In general, the magnitude of coefficient K_3 is considerably smaller than that of K. Hence the term $g(x_h)$ is usally considered to be of small quantity, and consequently we can deal with Equation (3.6) as a quasi-linear difference-differential equation. We deal witht he quasi-linear equation by means of Kryloff-Bogoliuboff method[23] in the light of results of the linear treatment in Chapter II.

We assume the solution of Equation (3.6) can be made through form

$$x = A(t)\sin\gamma(t), \qquad \gamma(t) = \omega t + \beta(t) \tag{3.7}$$

where A is the amplitude and β the phase angle, both unknown functions of time t. It is necessary that $\dot{x}$ is of the form

$$\dot{x} = A\omega\cos\gamma \tag{3.8}$$

which imposes an additional condition

$$\dot{A}\sin\gamma + A\dot{\beta}\cos\gamma = 0 \tag{3.9}$$

Differentiating $\dot{x}$ in Equation (3.8) we obtain

$$\ddot{x} = \dot{A}\omega\cos\gamma - A\omega^2\sin\gamma - A\omega\dot{\beta}\sin\gamma. \tag{3.10}$$

Substituting (3.7), (3.8) and (3.10) into Equation (3.6) we have

$$(\dot{A}\omega + 2An\omega)\cos\gamma + (Ap^2 - A\omega^2 - A\omega\dot{\beta})\sin\gamma + \lambda A\sin(\gamma-\phi) - g\{A\sin(\gamma-\phi)\} = 0 \tag{3.11}$$

where $\phi = \omega h$.

Combining Equations (3.9) and (3.11) and carrying out calculations, we obtain the following two differential equations

$$\left\{\begin{aligned} &\dot{A} + \frac{A\xi}{\omega}\sin\gamma\cos\gamma + \frac{A\eta}{\omega}\cos^2\gamma - \frac{g}{\omega}\cos\gamma = 0 && (3.12)\\ &\dot{\beta} - \frac{\xi}{\omega}\sin^2\gamma - \frac{\eta}{\omega}\sin\gamma\cos\gamma + \frac{g}{\omega A}\sin\gamma = 0 && (3.13)\end{aligned}\right.$$

where $\qquad \xi = p^2 - \omega^2 + \lambda\cos\phi, \qquad \eta = 2n\omega - \lambda\sin\phi \qquad (3.14)$

and $\qquad g = g\{A\sin(\gamma-\phi)\}$.

Thus, instead of the single differential equation of the second order (3.6) for unknown x, we have two differential equations of the first order for two unknown A, β.

In the case of the linear system, it was proved that there can exist a stable vibration when the following relations are fulfilled simultaneously as by

$$p^2 - \omega^2 + \lambda\cos\phi = 0, \qquad 2n\omega - \lambda\sin\phi = 0$$

namely, when $\xi = 0$, $\eta = 0$ in Equation (3.14) are satisfied.

In the present case of quasi-linear system, we restrict the treatment to a small neighborhood around the condition in which the stable vibration occurs in linear system. Then, wa can assume the quantities ξ and η to be small. This means that during one period $T = \frac{2\pi}{\omega}$ of the trigonometric functions, the quantities A and β vary very little, and that therefore we can apply the standard procedure of averaging per period T which leads to the following equations

$$\left\{\begin{aligned} &\frac{A(t+T)-A(t)}{T}+\frac{A\xi}{2\pi\omega}\int_0^{2\pi}\sin\gamma\cos\gamma\,d\gamma+\frac{A\eta}{2\pi\omega}\int_0^{2\pi}\cos^2\gamma\,d\gamma\\ &\qquad\qquad\qquad\qquad\qquad\qquad-\frac{1}{2\pi\omega}\int_0^{2\pi}g\cos\gamma\,d\gamma=0\\ &\frac{\beta(t+T)-\beta(t)}{T}-\frac{\xi}{2\pi\omega}\int_0^{2\pi}\sin^2\gamma\,d\gamma-\frac{\eta}{2\pi\omega}\int_0^{2\pi}\sin\gamma\cos\gamma\,d\gamma\\ &\qquad\qquad\qquad\qquad\qquad\qquad+\frac{1}{2\pi\omega A}\int_0^{2\pi}g\sin\gamma\,d\gamma=0.\end{aligned}\right.$$

If we consider slow variations in amplitude A and phase β in the course of many periods T, we can replace $A(t+T)-A(t)$ by ΔA, etc., and can assume T of the previous problem to be ΔT of the present one.

Thus in the first approximation, we can write the first terms in the preceding equations $\frac{dA}{dt}$ and $\frac{d\beta}{dt}$, and can arrive at the following equations of the first approximation:

$$\left\{\begin{aligned} \frac{dA}{dt}&=-\frac{1}{2\omega}\left\{A(2n\omega-\lambda\sin\phi)-3\lambda_3d^2A\sin\phi+\frac{3}{4}\lambda_3A^3\sin\phi\right\} &&(3.15)\\ \frac{d\beta}{dt}&=\frac{1}{2\omega}\left\{(p^2-\omega^2+\lambda\cos\phi)-3\lambda_3d^2\cos\phi-\frac{3}{4}\lambda_3A^2\cos\phi\right\}. &&(3.16)\end{aligned}\right.$$

We now carry out analyses based on Equations (3.15) and (3.16) instead of on Equation (3.6).

(3.1) Conditions for self-excitation

First, we introduce an expression for conditions of self-excitation based on Equation (3.15). To excite the vibration beginning at the rest point, the following condition must be satisfied $\qquad \frac{dA}{dt}>0.$

In the present problem, this condition for excitation becomes

$$A(2n\omega-\lambda\sin\phi)+3\lambda_3d^2A\sin\phi+\frac{3}{4}\lambda_3A^3\sin\phi<0.$$

If the self-excitation proceeds from the rest point where the amplitude is $A\fallingdotseq 0$, the third term on the left-side of the preceding equation can be neglected. Therefore, the condition for self-excitation becomes

$$2n\omega-\lambda\sin\phi+3\lambda_3d^2\sin\phi<0. \qquad (3.17)$$

Now we carry out calculations relating to the last equation in the light of the results in linear treatment. In the linear theory, the condition for self-excitation was expressed by $\qquad \lambda\geqq\lambda_1$

in which the stable vibration occurs with frequency ω_1 when $\lambda=\lambda_1$. In our non-linear system, even if we restrict the treatment to a small neighborhood around $\lambda=\lambda_1$ and $\omega=\omega_1$, the critical value of λ will deviate a little from λ_1, and the frequency will also deviate.

Setting $\lambda=\lambda_1+\Delta\lambda$, $\omega=\omega_1+\Delta\omega$, $\phi=\phi_1+\Delta\phi$ ($\phi_1=\omega_1 h$, $\Delta\phi=h\Delta\omega$), and substituting these expressions into Equation (3.17), and carrying out calculations by neglecting the higher order of small quantities $\Delta\lambda$, $\Delta\omega$ and $\Delta\phi$ we obtain

$$\left\{1-\left(\frac{2n}{\sin\phi_1}-\lambda_1 h\cot\phi_1\right)\frac{\Delta\omega}{\Delta\lambda}\right\}\Delta\lambda>3\lambda_3 d^2.$$

Putting the expression $\frac{\Delta\omega}{\Delta\lambda}$ (obtained in linear treatment, Equation (2.20) in Chapter II) into the last equation, the condition for self-excitation becomes

$$\Delta\lambda>\lambda_3 d^2 R \qquad \text{or} \qquad \lambda>\lambda_1+\lambda_3 d^2 R \tag{3.18}$$

where
$$R=\frac{3n\{4(n^2+\omega_1^2)+4nh(p^2+\omega_1^2)+\lambda_1^2 h^2\}}{(1+nh)\{h\lambda_1^2+2n(p^2+\omega_1^2)\}} \tag{8.19}$$

which is a dimentionless quantity and is always positive, as seen in this expression.

Consequently, vibration can start from the rest point only when $\lambda>\lambda_1+\lambda_3 d^2 R$. The term $\lambda_3 d^2 R$ is an additional one owing to the no-linear term $g(x_h)$ in Equation (3.6) and is not included in the linear equation. Of course, if the non-linear term be vanished, *i.e.*, if $\lambda_3=0$, then the additional term $\lambda_3 d^2 R=0$, and thus the expression for excitation coincides completely with that in linear treatment.

(3.2) Stationary amplitude and frequency

Stationary amplitude A_s is determined by putting $\frac{dA}{dt}=0$ into Equation (3.15), resulting in

$$2n\omega-\lambda\sin\phi+3\lambda_3 d^2\sin\phi+\frac{3}{4}\lambda_3 A^2\sin\phi=0. \tag{3.20}$$

Setting $A=A_s$, $\lambda=\lambda_1+\Delta\lambda$, $\omega=\omega_1+\Delta\omega$, $\phi=\phi_1+\Delta\phi$, and putting these relations into Equation (3.20), and carrying out calculations to the first order of small quantities $\Delta\lambda$, $\Delta\omega$, and $\Delta\phi$,

we obtain
$$A_s^2=4\left(\frac{\Delta\lambda}{\lambda_3 R}-d^2\right) \qquad \text{or} \qquad A_s^2=4\left(\frac{\lambda-\lambda_1}{\lambda_3 R}-d^2\right) \tag{3.21}$$

in which
$$R=\frac{3n\{4(n^2+\omega_1^2)+4nh(p^2+\omega_1^2)+\lambda_1^2 h^2\}}{(1+nh)\{h\lambda_1^2+2n(p^2+\omega_1^2)\}}. \tag{3.22}$$

In Equation (3.21), stationary amplitude A_s is represented as a function of parameters λ, n, h, p, λ_3 and d, etc., which are closely connected with the cutting conditions or with the characteristics of vibrational system. The non-linear theoretical finding that the stationary amplitude is a function of the various cutting conditions and is related to characteristics of vibrational system is considered to be a reasonable result, while in linear theory, the stationary amplitude must be decided only from the initial conditions, as described in Chapter II.

Next, in order to ascertain whether the stationary amplitude is stable or not, it is necessary to differentiate the right-hand side of Equation (3.15) with respect to A, *viz.*:

$$\psi(A) = -A(2n\omega - \lambda \sin\phi) + 3\lambda_3 d^2 A \sin\phi + \frac{3}{4}\lambda_3 A^3 \sin\phi.$$

If $\frac{d\psi}{dA} < 0$ the stationary amplitude is stable, and if $\frac{d\psi}{dA} > 0$ it is unstable.

Hence we have $$\frac{d\psi}{dA} \fallingdotseq -\frac{3}{2}\lambda_3 A_s^2 \sin\phi_1.$$

Since $\sin\phi_1 > 0$, we obtain $\frac{d\psi}{dA} < 0$.

Therefore, in the present case the stationary amplitude is stable.

The frequency of stationary vibration (ω_s) can be calculated from Equation (3.16). In the stationary state, it is clear that

$$\frac{d\gamma}{dt} = \omega + \frac{d\beta}{dt} = \text{const.} = \omega_s.$$

Setting $$\omega_s = \omega_1 + \Delta\omega_s$$

where ω_1 is the frequency of stationary vibration in the linear theory and $\Delta\omega_s$ is a correction of frequency which can be expressed by

$$\Delta\omega_s = \Delta\omega + \delta\omega.$$

Here $\Delta\omega$ may be given by Equation (2.20) in Chapter II, *viz.*,

$$\Delta\omega = \frac{2\,\omega_1(2n^2 + \omega_1^2 - p^2)}{\lambda_1\{(2n - \lambda_1 h \cos\phi_1)^2 + (2\omega_1 + \lambda_1 h \sin\phi_1)^2\}}\,\Delta\lambda \tag{3.23}$$

where $\Delta\lambda > 0$. This $\Delta\omega$ (component of correction) is, therefore, the frequency shift due to the increase of λ in the linear theory. The other component, $\delta\omega$, is the nonlinear frequency correction which can be calculated from Equation (3.16), *viz.*:

$$\delta\omega = \frac{1}{2\omega}\left\{(p^2 - \omega^2 + \lambda\cos\phi) - 3\lambda_3 d^2\cos\phi - \frac{3}{4}\lambda_3 A^2 \cos\phi\right\}.$$

Setting $\lambda = \lambda_1 + \Delta\lambda$, $\omega = \omega_1 + \Delta\omega$, $\phi = \phi_1 + \Delta\phi$, $A = A_s$, and placing these relations into the preceding equation and carrying out calculations, we have

$$\Delta\omega_s = \frac{h\lambda_1^2(p^2 - \omega_1^2) - 4n^3(p^2 + \omega_1^2) + 2n\lambda_1^2(1 - nh)}{2\lambda_1 n\omega_1\{4(n^2 + \omega_1^2) + 4nh(p^2 + \omega_1^2) + \lambda_1^2 h^2\}}\,\Delta\lambda. \tag{3.24}$$

Therefore, the stationary frequency can be expressed by

$$\omega_s = \omega_1 + \Delta\omega_s. \tag{3.25}$$

(3.3) Condition causing stationary vibration in region $d - x \geqq 0$

Hitherto our analyses have been made in region $d - x \geqq 0$ in which the vibrating workpiece does not, during any cycle, leave the cutting edge at all. We will here examine the condition in which the stationary vibration occurs in this region.

We begin by taking Equation (3.21) and calculating the condition for $A_s^2 > 0$, we now have

$$\Delta\lambda > \lambda_3 d^2 R \qquad \text{or} \qquad \lambda > \lambda_1 + \lambda_3 d^2 R \tag{3.26}$$

which coincides with Equation (3.18) previously introduced, expressing the condition for self-excitation.

Because of the condition that the workpiece while vibrating does not leave the cutting edge at all during any cycle is given by $A_S^2 < d^2$, it follows that

$$\frac{5}{4}\lambda_3 d^2 R > \Delta\lambda \qquad \text{or} \qquad \lambda_1 + \frac{5}{4}\lambda_3 d^2 R > \lambda.$$

Consequently, the condition in which the stationary vibration occurs in the region $d - x \geqq 0$ can be expressed by

$$\frac{5}{4}\lambda_3 d^2 R > \Delta\lambda > \lambda_3 d^2 R \qquad \text{or} \qquad \lambda_1 + \frac{5}{4}\lambda_3 d^2 R > \lambda > \lambda_1 + \lambda_3 d^2 R. \tag{3.27}$$

Hence it can be said that when the magnitude of λ is in the range expressed by Equation (3.27), there can exist a stationary vibration during which the workpiece leaves the cutting edge at no time. Amplitude and frequency of this stationary vibration are calculated by Equations (3.21) and (3.25) respectively.

Furthermore, it is clear that when $\lambda > \lambda_1 + \frac{5}{4}\lambda_3 d^2 R$ the vibration occurs with an amplitude sufficiently large to cause the workpiece to leave the cutting edge for a certain time during one cycle, and that when $\lambda < \lambda_1 + \lambda_3 d^2 R$ the vibration dies out.

So far we have made analyses regarding parameter λ only. The same treatment can be given as regards the other parameters n, h, and p, and expressions for conditions of self-excitation can also be obtained for them.

4. *Numerical calculations and considerations*

We begin by calculating the conditions for self-excitation. For numerical values of parameters n, h, and p, etc., we make use of Doi's experimental data.[12] For an example, by employing values $n = 40$/sec., $h = 0.0005$ sec., $p = 377$/sec. ($f_0 = 60$/sec.), magnitudes of harmonic value λ_1 and frequency ω_1 are obtained, giving

$$\lambda_1 = 1.62 \times 10^5/\text{sec.}^2, \qquad \omega_1 = 546/\text{sec.}$$

Calculating the magnitude of R from Equation (3.22) by using the preceding values, we obtain $R = 3.02$, which is a dimentionless quantity.

The magnitude of $\lambda_3 \left(\lambda_3 = \frac{K_3}{m}\right)$, which represents the non-linearity of horizontal cutting force, depends on various cutting conditions, *i.e.*, on the properties of metal being machined, on the cutting angle of tool engaged in cutting, on the nature of the vibrational system and other conditions. According to results of experiment[24] in which mild steel was machined, the magnitude of λ_3 is about

$$\lambda_3 = 0.23 \sim 0.92 \times 10^5/\text{sec.}^2\text{mm}^2$$

where the cutting angle of tool varies from 80° to 60°. The magnitude of λ_3 is large when the machining is done with a tool having small cutting angle, which is a matter of common knowledge.

Fig. 15 shows results of calculations from Equation (3.27) when the preceding values are used. Curves A and B correspond to the condition in which $\lambda_3=0.92\times10^5$, A indicating the curve of $\lambda=\lambda_1+\frac{5}{4}\lambda_3 d^2R$ and B representing that of $\lambda=\lambda_1+\lambda_3 d^2R$. Curves C and D correspond to the state in which $\lambda_3=0.23\times10^5$, straight line E standing for the case where $\lambda_3=0$.

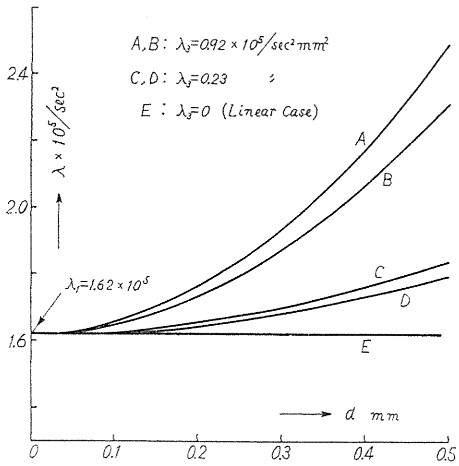

FIG. 15. Relation between feed d and λ.

As for non-linear case: In curves A, B corresponding to condition of large λ_3, the vibration does not occur when the magnitude of λ is below curve B; when the magnitude of λ is in the range between curves A and B, there can occur the stationary vibration during which the workpiece does not leave the cutting edge at all at any period of each cycle. When the magnitude of λ is above curve A, vibration occurs with a sufficiently large amplitude to cause the workpiece to leave the cutting edge for a certain time during one cycle. These facts can be said to also relate to curves C and D corresponding to the condition for small λ_3.

In the linear case represented by straight line E, on the other hand, only when $\lambda=\lambda_1$ can there exist a stationary vibration during which the workpiece does not leave the cutting edge at all. Moreover, according to the previous analyses, the amplitude of this stationary vibration is decided by the initial conditions. In practical machining operations, it is almost impossible to operate a cut where the condition $\lambda=\lambda_1$ is just satisfied. This means that theoretically for linear case there can hardly exist a stationary vibration during which the workpiece does not leave the cutting edge at all, which in fact contradicts the actuality. Furthermore, it is contrary to actuality to say that the stationary amplitude must be decided only by the initial conditions.

In the linear case, moreover, the critical value of λ (*i.e.*. λ_1) is independent of the magnitude of the feed d, as seen by the straight line E in Fig. 15. It can consequently be said that in the linear case the comparative difficulty for chatter to occur has no connection with the magnitude of feed. It is a fact,[19] however, that the chatter is more likely to occur in cuttings with a thin chip, *i.e.*, with a small feed, than in cuttings with a thick chip, a large feed.

In the present treatment of the non-linear case, the magnitude of λ have a certain width where there occur stationary vibrations during which the workpiece does not leave the cutting edge at all (see Fig. 15). It can also be noted that this range has a large width when the magnitude of λ_3 and the feed are large. Consequently, the stationary vibration is more likely to occur in such cutting condition. It is additionally clear that the width of the range in which the stationary vibration occurs is relatively large when the magnitude of λ_1 is small in contrast to λ_3, as seen in Equation (3.27). We have shown in Chapter II Fig. 8 that the magnitude of λ_1 is small when the damping coefficient of the vibrational system is small. For example, when the workpiece being cut consists of a long bar held by both centers, the corresponding value of λ_1 is small because the damping coefficient in this case is very small. In such a case the stationary vibration is most likely. This theoretical fact is in good agreement with the experimental results reported by S. Doi.[19]

By Equation (3.21), the amplitude of this stationary vibration is decided as being a function of parameters n, h, p, λ_3, and d, etc. which are closely connected to cutting conditions and characteristics of vibrational system. This theoretical fact is considered to be a proper conclusion.

Moreover, since the whole area above curve B or D in Fig. 15 constitutes the range of self-excitation for each vibrational system, it is reasonable that when the magnitude of feed is large, chatter is not likely to occur. This conclusion explains the nature of chatter vibration occurring in practice, although this nature may be caused partially by the following fact: The workpiece is likely to leave the cutting edge during vibration when the magnitude of feed is small. In the cutting conditions such as where vibrating workpiece leaves the cutting edge for a certain time during one cycle, the magnitude of time lag in cutting force is larger than that in cuttings where the workpiece does not leave the cuttng edge at all, as found in other experiments by the author[14] and S .Doi.

From the foregoing, it was ascertained that the characteristics of occurring chatter vibration which could not be expressed by the linear theory could be well explained by the non-linear theory.

Furthermore, Equation (3.18) which we introduced through the non-linear theory, is a general expression for the conditions of self-excitation and includes the linear theory. Consequently, it can be said that this Equation explains many characteristics of occurring chatter vibration as well as explaining the linear treatment.

5. *Conclusions*

An analytical research on chatter vibration was made on the basis of the non-linear theory through which we have introduced the general expression for the conditions of self-excitation, including the linear state. Theoretically we proved that some facts which could not be expressed by the linear theory were verified by the non-linear theory.

It is believed that chatter vibration is a kind of self-excited vibration caused by the time lag in cutting force existing behind the horizontal vibration of workpiece.

Chapter IV. Chatter Vibration due to Deflection of Main Spindle of Lathe—On the Frequency of Primary Chatter[25]

1. *Introduction*

In the present chapter, the frequencies of chatter vibration are calculated based on the differential equation introduced in Chapters II and III. Some research works relating to the frequency of chatter have been made by S. Doi,[12] R. S. Hahn,[4] E. Salje[5] and others. According to their results, it was reported that, in general, the frequency of chatter vibration is almost always slightly higher than the natural frequency of the vibrational system, although sometimes it may be lower. However, it seems to the author that a comprehensive study on the frequency of chatter vibration has not yet been reported.

Calculations on the frequency of chatter vibration are presented in this chapter and are discussed with the aid of experimental results, which are here examined in comparison with those of other investigators.

2. Frequency of chatter vibration introduced by the linear theory

As stated in Chapter II, the differential equation for chatter vibration is expressed by

$$\ddot{x}+2n\dot{x}+p^2x-f(d-x_h)=0. \tag{4.1}$$

Assuming that the cutting force is in proportion to the thickness of chip, Equation (4.1) becomes

$$\ddot{x}+2n\dot{x}+p^2x+\lambda x_h=0. \tag{4.2}$$

We have proved that the vibrational system given by linear Equation (4.2) is capable of causing stationary vibration in the region $d-x \geqq 0$ where the workpiece does not leave the cutting edge at all when the following conditions are fulfilled simultaneously:

$$\begin{cases} p^2-\omega^2+\lambda\cos\phi=0 & (4.3)\\ 2n\omega-\lambda\sin\phi=0 & (4.4)\end{cases}$$

where $\phi=\omega h$.

Accordingly, only when the magnitude of λ has a certain value expressed by

$$\lambda_1=\frac{2n\omega_1}{\sin\phi_1} \tag{4.5}$$

can stationary vibration arise, and its frequency ω_1 be represented by

$$\frac{\omega_1^2-p^2}{2n\omega_1}=\cot\phi_1 \tag{4.6}$$

where $\phi_1=\omega_1 h$.

Moreover, it has been verified that when $\lambda<\lambda_1$ the vibration dies out and when $\lambda>\lambda_1$ there is a stable vibration with a sufficiently large amplitude to cause the workpiece to leave the cutting edge for a certain time during one cycle. The equation of such vibration of large amplitude can be written by

$$\begin{cases} \ddot{x}+2n\dot{x}+p^2x+\lambda x_h=0, & d-x\geqq 0\\ \ddot{x}+2n\dot{x}+p^2(x+a)=0, & d-x<0.\end{cases} \tag{4.7}$$

(2.1) Frequency of stationary vibration which occurs in the region $d-x\geqq 0$, *i.e.*, frequency ω_1 $(\omega_1=2\pi f_1)$

For reasons which will appear later, frequency ω_1 of the stationary vibration which occus in the region $d-x\geqq 0$ where the workpiece does not leave the cutting edge at all, has a fundamental meaning when we deal with frequencies of chatter vibration.

Equation (4.6) can be written

$$\frac{1}{2nh}\phi - \frac{p^2 h}{2n}\frac{1}{\phi} = \cot\phi.$$

Setting $\quad y_1 = \frac{1}{2nh}\phi, \quad y_2 = \frac{p^2 h}{2n}\frac{1}{\phi}, \quad$ and $\quad y_3 = \cot\phi$

the preceding equation becomes

$$y_1 - y_2 = y_3 \tag{4.8}$$

This equation can be solved by substracting the ordinates of a hyperbola $y_2 = \frac{p^2 h}{2n}\frac{1}{\phi}$ from those of straight line $y_1 = \frac{1}{2nh}\phi$ which passes through the origin (see Fig. 16). The roots of Equation (4.8) can be obtained by finding the abscissas of the points of intersection of two curves $y_1 - y_2$ and y_3. The roots are indicated by $\phi_1, \phi_2, \ldots,$ as seen in Fig. 16. They correspond to the different modes of vibration. The most important of these modes is generally the first one, and we confine our attention only to this first mode.

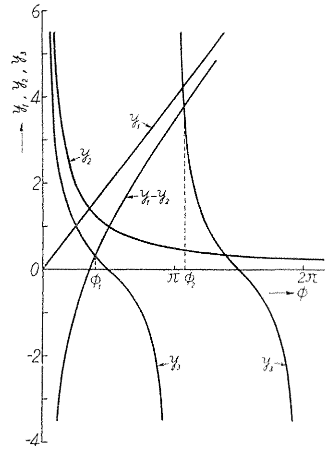

FIG. 16. Graphical calculation of frequency.

The frequency $\omega_1 \left(\omega_1 = \frac{\phi_1}{h}\right)$ is thus obtained by graphical method. According to Equation (4.5), since $\sin\phi_1 > 0$, it is clear that ϕ_1 may be either in the first quadrant $\left(\frac{\pi}{2} > \phi_1 > 0\right)$ or in the second quadrant $\left(\pi > \phi_1 > \frac{\pi}{2}\right)$. When ϕ_1 is in the first quadrant, $\cos\phi_1 > 0$. Hence we have $\omega_1 > p$ from Equation (4.6), *viz.*, $\phi_1 > \phi_0$, where $\phi_0 = ph$. If $\phi_1 = \frac{\pi}{2}$, $\cos\phi_1 = 0$. Hence $\omega_1 = p$ or $\phi_1 = \phi_0 = \frac{\pi}{2}$. When ϕ_1 is in the second quadrant, $\cos\phi_1 < 0$, hence $\omega_1 < p$ or $\phi_1 < \phi_0$. The relationship between ϕ_1 and ϕ_0 is illustrated in Fig. 17.

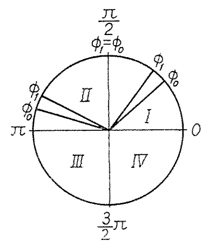

FIG. 17. Relation between ϕ_1 and ϕ_0

From the above description, it can be said that the decision as to whether frequency ω_1 is large or small in comparison with the natural frequency p, rests firmly on in which quadrant ϕ_1 is, whether it is in the first quadrant, or the second. Because ϕ_1 and ϕ_0 are always in the same quadrant, as seen in Fig. 17, it can be said that if ϕ_0 is in the first quadrant, *i.e.*, $\frac{\pi}{2} > \phi_0 > 0$, frequency ω_1 is larger than the natural frequency, p, *viz.*, $\omega_1>p$, and further that when $\phi_0=\frac{\pi}{2}$, $\omega_1=p$. Moreover, if ϕ_0 is second quadrant, i.e., $\pi>\phi_0>\frac{\pi}{2}$, frequency ω_1 is smaller than the natural frequency, *viz.*, $\omega_1<p$.

In practice, the natural frequency p of the vibrational system is commonly in the approximate range 200~1,200/sec.,[12) 4) 5)] while these values are closely connected with rigidity of the main spindle of lathe, mass of the chuck, and mass of workpiece. At the same time, the magnitude of h is in the approximate range 0.0005 ~ 0.001 sec.,[11)] in general, and is, of course, related closely to the various cutting conditions. Consequently, it can be considered that, in practice, ϕ_0 is commonly in the first quadrant, which gives us $\omega_1>p$.

(2.2) Frequency of chatter vibration going beyond the region $d-x \geqq 0$, *i.e.*, frequency ω_c ($\omega_c=2\pi f_c$)

In the previous section, we examined frequency ω_1 of stationary vibration which occurs in the condition where $\lambda=\lambda_1$. We here calculate the frequency ω_c of chatter vibration which occurs in the condition where $\lambda>\lambda_1$. In this latter case, there exists a stationary vibration having such a large amplitude as to cause the workpiece to leave the cutting edge for a certain time during one cycle. In practice, this type of chatter appears most often.[12)] In the region in which the workpiece leaves the cutting edge, cutting force is zero, and thus the motion of the workpiece is a free vibration. Therefore, in comparison to ω_1, the magnitude of frequency ω_c of such vibration can be considered to approach that of the natural frequency p.

The total energy of the vibrational system is expressed by the sum of two components, *viz.*, potential energy P and kinetic energy K as represented by

$$E=P+K=\frac{1}{2}m\dot{x}^2+\frac{1}{2}k(x+a)^2.$$

Differentiating both sides of this equation by t, we obtain

$$\frac{dE}{dt}=m\dot{x}\{\ddot{x}+p^2(x+a)\}.$$

Using this expression, Equation (4.7) becomes

$$\left\{\begin{aligned} &\frac{dE}{dt}=m\dot{x}(-2n\dot{x}-\lambda x_h+p^2a), \qquad d-x\geqq 0\\ &\frac{dE}{dt}=m\dot{x}(-2n\dot{x}), \qquad d-x<0. \end{aligned}\right. \tag{4.9}$$

Since the system expressed by Equation (4.9) is considered to be an asymmetric one, we assume that the stationary solution of Equation (4.9) is the form

$$x=b+A\sin\omega_c t. \tag{4.10}$$

Hence

$$x_h=b+A\sin(\omega_c t-\phi_c) \tag{4.11}$$

where $\phi_c=\omega_c h$,

Fig. 18 illustrates the form of the stationary solution. The solid line indicates curve x and the broken line represents curve x_h. The period 1–2–3 indicates one cycle of vibration and in the period 1–2 in which $d-x_h<0$ where the cutting force is zero, the motion is a free vibration.

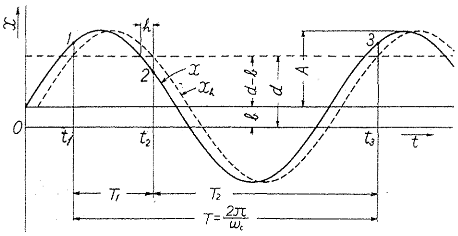

FIG. 18. Stationary vibration of workpiece.

So far as we deal with stationary vibration, it is considered that total energy E of the system must not increase or decrease per one cycle. Therefore, it follows that

$$\oint dE = 0.$$

Referring to Fig. 18, this expression becomes

$$\oint dE = \int_{t_1}^{t_2} dE + \int_{t_2}^{t_3} dE = 0. \tag{4.12}$$

Substituting Equations (4.10) and (4.11) into Equation (4.9), using the last expression and carrying out calculations, we obtain

$$n\omega_c^2 A(t_3-t_1) - \frac{n\omega_c A}{2}(\sin 2\omega_c t_3 - \sin 2\omega_c t_1) + \lambda(d-b)(\sin \omega_c t_3 - \sin \omega_c t_2)$$
$$+ \frac{\lambda A \cos\phi_c}{4}(\cos 2\omega_c t_3 - \cos 2\omega_c t_2) + \frac{\lambda A \sin\phi_c}{4}(\sin 2\omega_c t_3 - \sin 2\omega_c t_2)$$
$$+ \frac{\lambda\omega_c A \sin\phi_c}{2}(t_3-t_2) = 0.$$

By putting the following relations into this equation

$$\left\{\begin{aligned} & t_3 - t_1 = \frac{2\pi}{\omega_c} \\ & t_3 - t_2 = \frac{1}{\omega_c}(\pi + 2\sin^{-1}\gamma), \qquad \text{where } \gamma = \frac{d-b}{A} \\ & \sin 2\omega_c t_3 - \sin 2\omega_c t_1 = 0 \\ & \sin \omega_c t_3 - \sin \omega_c t_2 = 2\sqrt{1-\gamma^2}\sin\phi_c \\ & \cos 2\omega_c t_3 - \cos 2\omega_c t_2 = -4\gamma\sqrt{1-\gamma^2}\sin 2\phi_c \\ & \sin 2\omega_c t_3 - \sin 2\omega_c t_2 = 4\gamma\sqrt{1-\gamma^2}\cos 2\phi_c \end{aligned}\right.$$

we obtain

$$\gamma\sqrt{1-\gamma^2} + \sin^{-1}\gamma = \frac{2\pi n\phi_c}{\lambda h \sin\phi_c} - \frac{\pi}{2}. \tag{4.13}$$

Since it is certain that

$$\frac{\phi_c}{\sin\phi_c} \fallingdotseq \frac{\phi_1}{\sin\phi_1}$$

Equation (4.13) can be written

$$\gamma\sqrt{1-\gamma^2} + \sin^{-1}\gamma = \frac{2\pi n\phi_1}{\lambda h \sin\phi_1} - \frac{\pi}{2}. \tag{4.14}$$

As seen in Fig. 18, it is clear that γ is closely connected with the time rate of free vibration period per one cycle, *viz.*, when $\gamma = 1$, the vibrating workpiece does not leave the cutting edge at all during each cycle, and when $\gamma = 0$, the workpiece leaves the cutting edge just for the half period of one cycle, and when $\gamma = -1$, the vibrating workpiece is completely away from the cutting edge, which in fact is impossible in practice.

In Equation (4.14), when $\lambda = \lambda_1$, we have $\gamma = 1$. This means that vibrating workpiece does not leave the cutting edge at all, which fact is in good agreement with our theory. When $\lambda = \infty$, it follows that $\gamma = -1$, which is a reasonable result.

As described above, the magnitude of γ which is closely connected with the time rate of free vibration period per one cycle can be obtained as a function of λ by Equation (4.14).

Now, let T_1 denote the time during which the vibrating workpiece is away from the cutting edge (see Fig. 18); its motion is then free vibration, and let T_2 denote the time during which the workpiece is not away from the cutting edge and only at this time is the cutting operated. Setting

$$\varepsilon = \frac{T_1}{T_1 + T_2}, \qquad 1 - \varepsilon = \frac{T_2}{T_1 + T_2}$$

and referring to Fig. 18, we obtain

$$\varepsilon = \frac{1}{2} - \frac{1}{\pi}\sin^{-1}\gamma. \tag{4.15}$$

Accordingly, the time rate ε of free vibration period per one cycle is obtained as a function of γ which is related to λ. Then, if the magnitude of λ be given, the corresponding value of ε can be obtained by Equations (4.14) and (4.15).

Fig. 19 shows the relationship between ε or $(1-\varepsilon)$ and $\frac{\lambda}{\lambda_1}$ calculated by Equations (4.14) and (4.15). As seen in Fig. 19, when $\frac{\lambda}{\lambda_1} = 1$ $(\lambda = \lambda_1)$, $\varepsilon = 0$ and $(1-\varepsilon) = 1$, which means that the vibrating workpiece does not leave the cutting edge at all during each cycle and there is no free vibration period. As $\frac{\lambda}{\lambda_1}$ increases from 1, ε increases sharply at the start and then gradually increases.

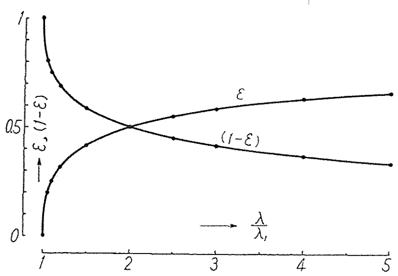

FIG. 19. Relation between λ and ε or $(1-\varepsilon)$.

Frequency ω_c of stationary vibration during which workpiece leaves the cutting edge for a certain period during one cycle can now be written as a function of natural frequency p, frequency ω_λ in cutting operation, and ε as follows:

$$\omega_c = \varepsilon p + (1-\varepsilon)\,\omega_\lambda \tag{4.16}$$

where ω_λ is also a function of λ and when $\lambda = \lambda_1$ it must coincide with ω_1, *viz.*,

$\omega_\lambda = \omega_1$ when $\lambda = \lambda_1$. When magnitude of λ is near λ_1, using expression $\frac{\Delta\omega}{\Delta\lambda}$ obtained in Chapter II, frequency ω_λ can be written approximately as

$$\omega_\lambda = \omega_1 + \frac{2\omega_1(2n^2 + \omega_1^2 - p^2)}{\lambda_1\{(2n - \lambda_1 h \cos\omega_1 h)^2 + (2\omega_1 + \lambda_1 h \sin\omega_1 h)^2\}}(\lambda - \lambda_1) \qquad (4.17)$$

When the magnitude of λ is considerably large as compared with λ_1, although the calculations are very troublesome, the expression for ω_λ can be obtained from Equations (2.11) and (2.12) in Chapter II as follows:

$$\omega_\lambda = \frac{-\mu(\rho+\nu)(\omega_1^2-p^2)+\sqrt{\{\mu(\rho+\nu)(\omega_1^2-p^2)+2\rho\omega_1^2\}^2+4\rho\nu\omega_1^2(2n^2+\omega_1^2-p^2)+4\nu^2n^2\omega_1^2}}{2\rho\omega_1^2} \qquad (4.18)$$

where $\qquad \nu = \frac{\lambda - \lambda_1}{\lambda_1}, \quad \rho = 1 + nh(1+\nu), \quad \mu = \frac{2}{\frac{\pi}{\phi_1} - 2}.$

In Equations (4.17) and (4.18), it is clear that if $\lambda = \lambda_1$, it follows that $\omega_\lambda = \omega_1$, which is a reasonable result.

According to the above, we can obtain the expression for frequency ω_c of the stationary vibration which occurs in the conditions where $\lambda > \lambda_1$ as a function of λ. Referring to Equation (4.16), it can be said that frequency ω_c depends not only upon characteristics of the vibrational system, *i.e.*, p, n, and etc., but also on various cutting conditions, for example λ, h, and others.

3. Frequency of chatter vibration introduced by the non-linear theory

In Section 2 of this chapter, the frequency of chatter vibration was introduced on the basis of the linear theory. We will here examine it on the non-linear theory.

As described in Chapter III, the non-linear equation for chatter vibration can be written

$$\ddot{x} + 2n\dot{x} + p^2 x + \lambda x_h - g(x_h) = 0 \qquad (4.19)$$

where $\qquad g(x_h) = \lambda_3 x_h^3 - 3\lambda_3 d x_h^2 + 3\lambda_3 d^2 x_h$

in which λ_3 indicates the degree of non-linearity of horizontal cutting force.

By carrying out analyses on the basis of non-linear Equation (4.19), it has been proved that

(i) when $\lambda < \lambda_1 + \lambda_3 d^2 R$, vibration does not occur;

(ii) when $\lambda > \lambda_1 + \frac{5}{4}\lambda_3 d^2 R$, vibration occurs with a sufficiently large amplitude to cause the workpiece to leave the cutting edge for a certain time during one cycle;

(iii) when $\lambda_1 + \frac{5}{4}\lambda_3 d^2 R > \lambda > \lambda_1 + \lambda_3 d^2 R$, there can exist a stationary vibration during which the workpiece does not leave cutting edge at all.

The frequency of type (iii) vibration has been introduced as

$$\begin{cases} \omega_s = \omega_1 + \Delta\omega_s, \qquad \Delta\omega_s = -Q(\lambda - \lambda_1) & (4.20) \\ -Q = \dfrac{h\lambda_1^2(p^2 - \omega_1^2) - 4n^3(p^2 + \omega_1^2) + 2n\lambda_1^2(1 - nh)}{2\lambda_1 n\omega_1\{4(n^2 + \omega_1^2) + 4nh(p^2 + \omega_1^2) + \lambda_1^2 h^2\}} & (4.21) \end{cases}$$

where ω_1 is the frequency of stationary vibration in the linear theory and $\Delta\omega_s$ is the correction of frequency in the non-linear theory.

We now examine the magnitude of $\Delta\omega_s$. As seen in Equation (4.20), the magnitude of $\Delta\omega_s$ depends upon that of Q, hence we carry out calculations relating to Q from Equation (4.21). Fig. 20 is an example of calculations using the following numerical values: $p = 377$/sec. and $h = 0.0005$ sec. which are held constant, and damping coefficient n which is changed variously. These numerical values are frequently used in Chapters II and III. At this calculation stage of Q, the magnitudes of λ_1 and ω_1 for each value of n are preliminarily computed by Equations (2.17) and (2.18) in Chapter II.

As seen in Fig. 20, the magnitude of Q is about $(1 \sim 3) \times 10^{-5}$ sec., which is a very small quantity even though it depends on various conditions. Now, the magnitude of $(\lambda - \lambda_1)$ in Equation (4.20), although it is widely changed under various cutting conditions, is generally considered to be of the quantity $(0 \sim 2) \times 10^5/\text{sec.}^2$, as seen in Fig. 15 in Chapter III. Hence the magnitude of $\Delta\omega_s$ is about $-(0 \sim 6)$/sec., namely, it is at most about 1% of ω_1, which can be considered a negligible quantity. Consequently, although the frequency ω_s of type (iii) vibration in the non-linear theory is slightly smaller in magnitude in comparison with frequency ω_1 in the linear theory, we can write approximately

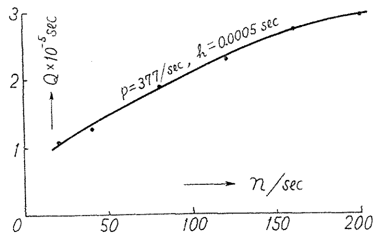

FIG. 20. Relation between n and Q.

$$\omega_s \doteqdot \omega_1. \tag{4.22}$$

Next, the frequency of type (ii) vibration must be considered. We can recognize that the effect of non-linearity of cutting force on the frequency of type (ii) vibration is far less than that on the frequency of type (iii) vibration, because during type (ii) vibration, the workpiece is away from the cutting edge for a certain time during one cycle. At the time while the the workpiece is away from the cutting edge, the motion is a free vibration with natural frequency p. On the other hand, in type (iii) vibration the workpiece vibrates with frequency ω_s during whole period of one cycle.

Consequently, so far as our study relates to frequencies of chatter vibration, we need not calculate frequencies by non-linear theory, but it is justifiable to examine them by linear theory.

4. *Numerical calculations and experimental results*

Here, we carry out numerical calculations of frequencies and these theoretical results will be discussed in comparison with experimental records.

In the first place, frequency ω_1 must be examined because it is basic to all other frequencies. As seen in Equation (4.6), ω_1 is a function of parameters n, h, and p, etc., which are closely related with characteristics of the vibrational system and various cutting conditions. The magnitude of h, according to experimental results of S. Doi,[12] is generally about 0.0005 ~ 0.001 sec. The magnitude of n is closely connected with that of p, and both are decided by the state of vibrational system. We will then obtain them experimentally as regard to various vibrational systems.

A series of tests are made, in which natural frequency p of lathe spindle system is varied by clamping various masses to the chuck. Figs. 21 and 22 show experimental recordings of free vibration for each vibrational system made by means of optical method.[11] Fig. 21 shows recordings for lathe *A*, and Fig. 22 is those for lathe *B*. Calculating magnitudes of n and p from these recordings, we obtain the following table.

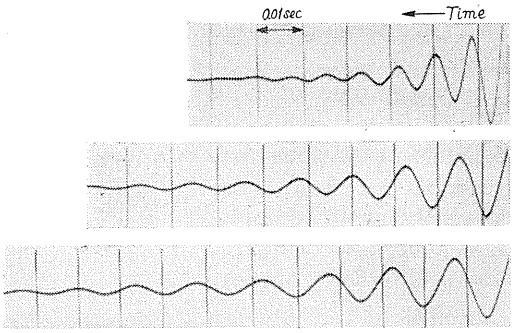

FIG. 21. Free vibrations for various vibrational systems (Lathe *A* systems).

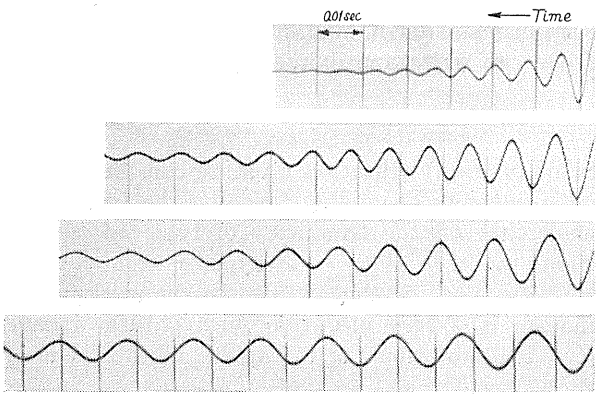

FIG. 22. Free vibrations for various vibrational systems (Lathe *B* systems).

TABLE 1

No. of System	Clamping Mass (kg)	p/sec.	n/sec.
A-1	0	754	150
A-2	3.67	504	72
A-3	7.27	403	54
B-1	0	880	100
B-2	3.67	666	45
B-3	7.27	540	34
B-4	23.76	352	12

Lathe A used in this experiments is of small size having a swing of 300 mm with diameter of main spindle 46 mm. Lathe B is of a medium size having a swing of 400 mm with diameter of main spindle 52 mm. The two main bearings of lathe A are common plane bearings, while those of lathe B are taper roller bearings. It can be seen in Figs. 21 and 22 that amplitude of free vibrations for lathe A systems decrease more rapidly than those for lathe B systems. Hence magnitudes of n for lathe A systems are considerably larger than those for lathe B systems, as seen in Table 1.

Setting these numerical values into Equation (4.8), and carrying out calculations relating to frequency ω_1 by graphical method, we obtain Fig. 23. Curves A, B in the figure are those of vibrational systems of lathe A and lathe B respectively, and curve C indicates $\omega_1 = p$ line drawn for reference. Since ϕ_0 is in the first quadrant in any system treated here, it is seen in the figure that both curves A, B are above line C, *viz.*, $\omega_1 > p$. The magnitudes of ω_1 for lathe A systems, however, are considerably larger than those for lathe B systems due to the difference of damping coefficients in the two systems.

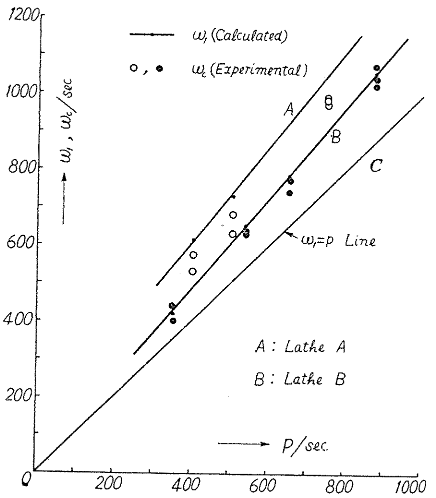

FIG. 23. Relation between natural frequency p and frequency of chatter ω_1 or ω_c.

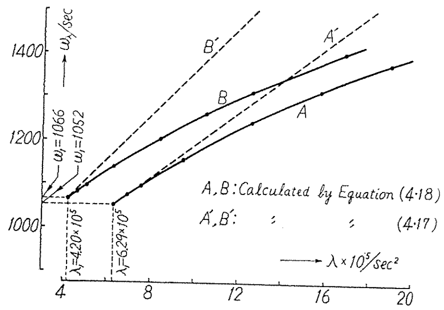

FIG. 24. Relation between λ and ω_λ.

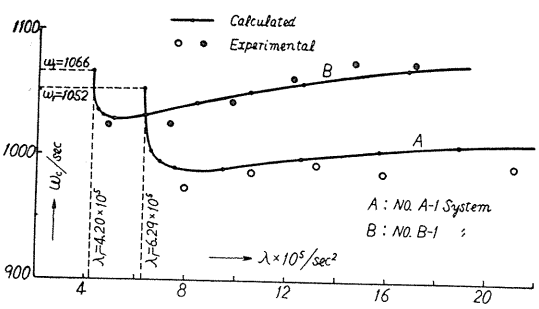

FIG. 25. Relation between λ and ω_c.

We next examine frequency ω_c of stationary vibration during which the workpiece is away from the cutting edge for a certain period of one cycle by using Equation (4.16) as, for example, relating to systems No. A-1 and No. B-1. In order to obtain ω_c, calculations for ω_λ must be preliminarily done. Fig. 24 shows results of calculations for ω_λ by using Equations (4.17) and (4.18). In the figure, curves A, A' are for the system A-1, and curves B, B' are for the system B-1. Curves A, B indicated by solid lines are obtained from Equation (4.18), and straight lines A' B', represented by broken lines are calculated by Equation (4.17) which gives an approximate value for ω_λ. The magnitudes of λ_1 and ω_1 for both systems are as follows:

$$\lambda_1 = 6.29 \times 10^5/\text{sec.}^2, \qquad \omega_1 = 1{,}052/\text{sec.} \qquad \text{for No. A-1 system}$$

$$\lambda_1 = 4.20 \times 10^5/\text{sec.}^2, \qquad \omega_1 = 1{,}066/\text{sec.} \qquad \text{for No. B-1 system.}$$

It is seen in Fig. 24 that the magnitude of ω_λ increases with increase of λ which fact seems to be reasonable, and further that the approximate results obtained from Equation (4.17) (broken lines A', B') first coincide approximately with those calculated from Equation (4.18) (solid lines A, B), but later the difference between them increases with increase of λ. In the following calculations for ω_c, the magnitude of ω_λ obtained from Equation (4.18) is used.

The magnitudes of frequency ω_c for both systems (No. A-1 and No. B-1) can be now calculated from Equation (4.16) by using the numerical values for ε, $(1-\varepsilon)$ and ω_λ which are shown in Figs. 19 and 24. Fig. 25 shows the results of calculations for frequency ω_c, and curves A, B are those for No. A-1, No. B-1 systems respectively. In both curves, when $\lambda = \lambda_1$, of course $\omega_c = \omega_1$. In curve A which corresponds to the system having a large magnitude of n in contrast to p, it is seen that ω_c is alway smaller than ω_1. In case of a system having small magnitude of n (curve B), ω_c becomes slightly larger than ω_1 as the magnitude of λ increases, *i.e.*, in the neighborhood around $\lambda = 4\lambda_1$, although ω_c is smaller than ω_1 at the begining of curve B. (Of course, it is evident that curve B must converge toward $\omega_c = p$ line in $\lambda = \infty$, because it must follow that $\varepsilon = 1$ in $\lambda = \infty$). In curve B, this tendency of ω_c to become larger than ω_1 is remarkable when the magnitude of n is relatively small in contrast to p. However, the damping coefficient n of the vibrational system expressed by curve B is very small in magnitude. Furthermore, chatter vibrations occurring in such conditions as where $\lambda = 4\lambda_1$ are so severe that it is difficult to do the cutting operation. Accordingly, it can be said that the magnitude of ω_c is generally smaller than ω_1, unless the cutting condition is not so severe. In this meaning, it is justifiable to say that frequency ω_c of stationary vibration during which workpiece leaves the cutting edge for a certain period during one cycle is generally smaller than ω_1. Consequently, ω_c is usually considered to be within the range between straight line C and curves A or B in Fig. 23, although this range is variously changed by the characteristics of vibrational system. This theoretical finding can also be applied in the system in which ϕ_0 is in the second quadrant, although in this system ω_1-curve (which corresponds to curves A or B in Fig. 23) is below the straight line C.

In order to examine the foregoing theoretical calculations for frequencies of chatter vibration, the following experiments are now carried out on each vibrational system shown in Table 1. To experiment with the cutting condition where the vibratory motion is not affected in any way by the undulatory surfaces produced by the preceding revolution of workpiece, tests are made by cutting a square thread which is preliminarily prepared on the workpiece of mild steel, as shown in Fig. 26. The tool employed in this experiment is made from a 16 mm square rod of high speed steel with cutting angle 80° and clearance angle 10°. The tool is fitted into the tool post in the way shown in Fig. 26, and fed in longitudinal direction corresponding to the pitch of square thread of workpiece.

At the time of experiment, the magnitude of λ is changed by varying the width of square thread b. (As previously described, $\lambda = \frac{K}{m}$, $K = k_s b$, where k_s = specific horizontal cutting force.)

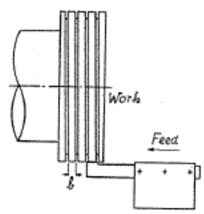

FIG. 26. Cutting manner.

Fig. 27 shows an example of experimental recordings of the horizontal vibration of workpiece during chatter by means of an optical method.[11] Fig. 27 is for No. A-1 system, and records (1), (2) and (3) correspond to conditions $b = 3$, 4, and 5 mm respectively, where cutting speed and depth of cut are held constant. The magnitude of λ_1 for this system (No. A-1) is $\lambda_1 = 6.29 \times 10^5/\text{sec.}^2$, which magnitude corresponds to width of square thread $b = 2.4$ mm.

From these experimental recordings, frequencies of chatter vibration were obtained and are plotted in Fig. 25. Marks ○ and ● in curves A, B are for systems No. A-1 and No. B-1 respectively. It is seen in the figure that in tendency experimental results are in good agreement with results of theoretical calculations.

The same experiments are made on all systems shown in Table 1. Fig. 28 shows an example of experimental recordings. Here, (1), (2) and (3) correspond to the systems No. A-1, No. A-2 and No. A-3 respectively, where all cutting conditions are held constant. These results are plotted in Fig. 23 by marks ○, ●. Mark ○ represents for lathe A systems, and mark ● for lathe B systems. It is seen in the figure that in both systems the experimental values of frequency are almost in the range between straight line C and curves A or B. In lathe B systems, however, it is seen that experimental values deviate slightly from this range, which fact was discussed previously.

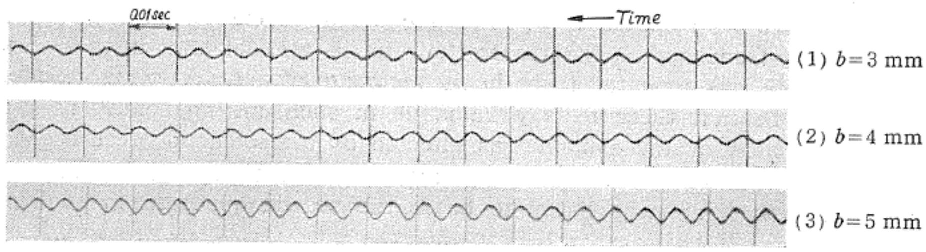

FIG. 27. Horizontal vibrations of workpiece.

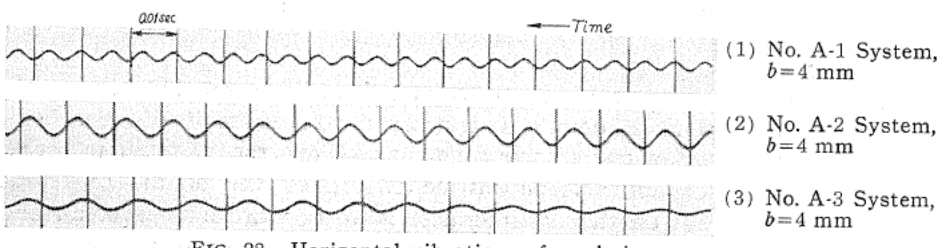

FIG. 28. Horizontal vibrations of workpiece.

From the foregoing, it is ascertained that frequencies of chatter vibration which are calculated based on the differential equation which we introduced in Chapters II and III are in good agreement with experimental results. We will now examine these results, comparing them with those of other investigators.

R. S. Hahn[41] introduced the following empirical formula relating to frequency of chatter vibration

$$\omega_c = \omega_q \sqrt{1 + \frac{k_w}{k_q}}$$

where ω_q = natural circular frequency which corresponds to p in this paper,
ω_c = circular frequency of chatter vibration,
k_w = rate of increase of thrust force with chip thickness *i.e.*, a workpiece constant which seems to coincide with λ in this paper,
k_q = spring constant of vibrational system at tool point which corresponds to k in this paper.

It is seen in his formula that frequency of chatter ω_c is larger than natural frequency ω_q of the system, and that ω_c is expressed as a function of k_w (*i.e.*, λ in this paper). These facts are considered to be in accord with our results. It is also seen in Hahn's formula that frequency of chatter ω_c is positively decided only by k_w, k_q and ω_q, and further that the magnitude of ω_c always increases with increase of k_w. The frequency of chatter, however, depends upon various cutting conditions and characteristics of vibrational systems, as seen in our Equations (4.16) and (4.18). Moreover, the magnitude of ω_c does not always increase with increase of k_w, but the relationship between ω_c and k_w seems to vary in many ways depending upon characteristics of vibrational system and on cutting conditions, as, for example, seen in Fig. 25 in this paper.

The experimental results reported by the author[26] together with S. Doi also show that in general the frequency of chatter is slightly larger than natural frequency of the system, which fact is in good agreement with our results.

According to other experimental results reported by S. Doi[19] and E. Salje,[5] however, there are some cases in which frequency of chatter is slightly smaller than natural frequency. The reason for this is, we feel, that in these test conditions ϕ_0 ($\phi_0 = ph$) appeared to be in the second quadrant. A still more important reason may be that these tests were made in such cutting conditions that the vibratory motion of the system was affected by the so-called "feed-back" effect which caused an apparent increase of magnitude of time lag. It is therefore considered that ϕ_0 is more likely to be in the second quadrant in such cutting conditions.

As for the so-called feed-back effect, we will take that up later in Chapter V and will then go into frequency of chatter vibration again.

5. *Conclusions*

Frequencies of chatter vibration are examined based on the differential equation given in Chapters II and III. It is ascertained that results of theoretical calculations for frequency of chatter vibration can well express the relationship between frequency of chatter and various cutting conditions as well as the relationship between frequency of chatter and characteristics of vibrational system, and then that theoretical calculations are in good agreement with experimental results.

Summing up, it has been clarified that by our differential equation for chatter vibration, frequencies of chatter are well expressed, in addition to the characteristics of occurring chatter vibration having been well expressed.

Chapter V. Chatter Vibration due to Deflection of Main Spindle of Lathe—Regenerative Chatter

1. Introduction

In the foregoing chapters, we deal with so-called primary chatter which occurs in cutting conditions where there is no interaction between the vibratory motion of the system and the undulation previously produced on a work surface. We proved that primary chatter is a kind of self-excited vibration caused by a lag in horizontal cutting force existing behind the horizontal vibration of workpiece. In addition, it was clarified that this lag of cutting force is an essential characteristic of machining metals.

The magnitude of this lag, however, is greatly changed by the various cutting conditions, and is considered to be closely related with the cutting mechanism. Accordingly, we can guess that sometimes the magnitude of time lag is very small and further that under certain circumstances, for example in cutting of brittle metals such as cast iron, it appears to be zero. It can be considered that in a cutting having small time lag the primary chatter is not likely to occur, and especially that in a cutting having no time lag it cannot occur. However, it is a fact that, even in a cutting of brittle metals such as cast iron, so-called regenerative chatter occurs frequently in such cutting conditions as where the vibratory motion of the system is subject to the effect of undulatory surface produced during the preceding revolution of workpiece, *i.e.*, in cuttings where the workpiece is subject to so-called "feed-back" effect. It is needless to say that regenerative chatter can also be caused in a cutting having time lag of cutting force.

To find the reason for causing so-called regenerative chatter, we deal in this chapter with regenerative chatter experimentally and theoretically.

2. On the time lag of cutting force in a cutting of cast iron

We first examine the time lag of cutting force in cuttings with cast iron by using the same method as employed in Chapter I (see Fig. 1), where the workpiece is oscillated forcibly in horizontal direction and the thickness of chip is thus changed periodically.

Fig. 29 is an example of experimental results obtained in cutting conditions where the cutting speed is 1.0 m/min., cutting angle of tool 90°, thickness of flange 5 mm, and the frequency of oscillation of workpiece is 1.5/sec. with an amplitude

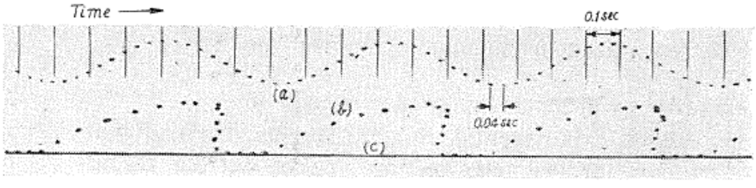

FIG. 29. Experimental record.

0.22 mm. Curves (a) and (b) represent the horizontal oscillation of workpiece and the fluctuations of cutting force respectively. Straight line (c) indicates the stationary position of cutting tool where the cutting force is zero. Measuring the position of each dot in curves (a), (b) by a sensitive comparator, we obtain Fig. 30.

In Fig. 30, curves (a), (b) and (c) represent the horizontal oscillation of workpice, the fluctuations in horizontal cutting force and those in vertical cutting force, respectively. Each dot on the same vertical line is the same instant. It is seen in the figure that point A, at which the workpiece moves nearest to the cutting edge and the thickness of chip becomes maximum, coincides almost completely with points B and C at which the cutting force reaches the maximum value, and then that the time lag of cutting force in both directions does not exist (for reference see Fig. 3 which is a result of cutting with mild steel).

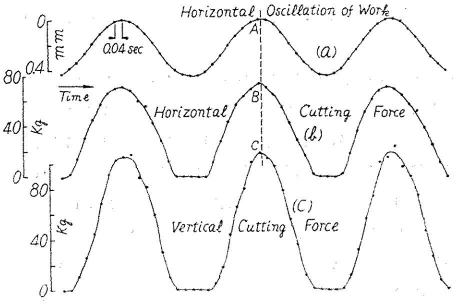

FIG. 30. Horizontal oscillation of workpiece and fluctuation of cutting force.

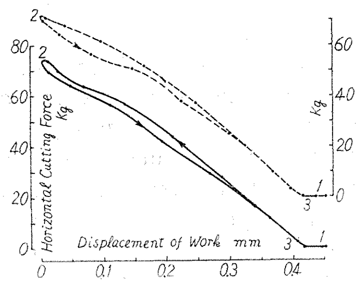

FIG. 31. Force-displacement curve.

It is a fact that, in a cutting of brittle metals such as cast iron, the continuous chip which appears in cuttings with steel is not produced, but the discontinuous or segmental chip is made, *i.e.*, the cutting mechanism of cast iron differs completely from that of steel. Accordingly, in cuttings with cast iron the action of frictional force along the tool face is considered to be small in comparison with cutting of steel, and then the after-effect or lag of cutting force can hardly be considered.

Now, Fig. 31 shows the relationship between the horizontal displacement of workpiece and the horizontal cutting force obtained from Fig. 30. The solid line shows a previous cycle and the broken line, the following cycle. It is seen in the figure that the approaching stroke 1–2 of oscillating workpiece to the cutting edge almost coincides with the recess stroke 2–3 because the time lag of cutting force hardly exist (for reference see Fig. 4 which corresponds to a cutting of mild steel).

It can be noted that in many experimental records similar to Fig. 29 in which the cutting conditions are changed in many ways, the time lag of cutting force can hardly be recognized in cuttings with cast iron.

3. *Experiments on regenerative chatter vibration*

From the above experimental results, it was clarified that, in cuttings with cast iron, there can hardly exist the time lag of cutting force. Accordingly, in practical machinings of cast iron, the so-called primary chatter cannot occur unless there is any other cause to excite the vibration above and beyond the time lag of cutting force.

To verify this, the following experiments are now carried out by cutting a square thread which is preliminarily prepared on the workpiece of cast iron in the same way as shown in Fig. 26. In this cutting operation, therefore, the workpiece is not subject to so-called feed-back effect in any way. Fig. 32 shows an example of experimental records measuring the horizontal vibration of workpiece during cutting operation by using the optical method.[11] For reference, Fig. 33 shows the same record obtained by cutting mild steel. By these figures, it is justifiable to say that so-called primary chatter can hardly occur in cuttings with cast iron, while in cutting of mild steel it can occur because of the existence of time lag of cutting force. Fig. 34 shows the experimental record obtained by cutting the same square thread of cast iron, only an interruption is made purposely in one part of square thread so as to cause a large disturbance during cutting. It is seen in Fig. 34 that even though the workpiece is suddenly disturbed to a great degree at point A at which the interruption is just cut, the transient vibration dies out immediately and chatter vibration can hardly be observed.

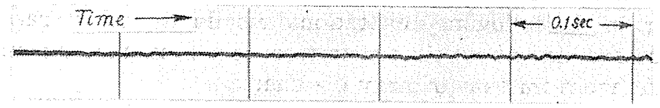

FIG. 32. Experimental record (cutting of cast iron).

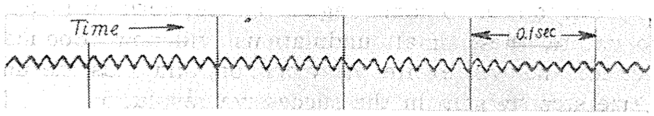

FIG. 33. Experimental record (cutting of mild steel).

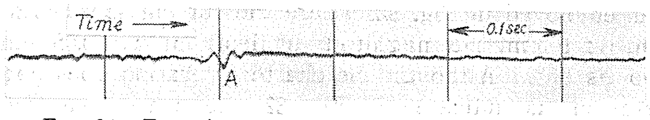

FIG. 34. Experimental record (cutting of cast iron).

From the foregoing, it is ascertained that so-called primary chatter can hardly occur in cuttings having no time lag of cutting force, and thus it is made more clear that the fundamental cause of primary chatter is no more than time lag of cutting force.

In cuttings with cast iron, however, chatter vibration occurs frequently in such cutting conditions as where the vibratory motion of workpiece is subject to the effect of undulatory surface produced during the preceding revolution of workpiece, *i.e.*, in cuttings where the vibratory motion of the system is subject to so-called "feed-back" effect. Fig. 35 shows an example of recordings of chatter vibration occurring in such cutting conditions, where a flange of cast iron is machined with side lathe tool which is fed in a traverse direction by the cross spindle of lathe, as shown in Fig. 36. In this cutting operation, therefore, the workpiece is subject to feed-back effect.

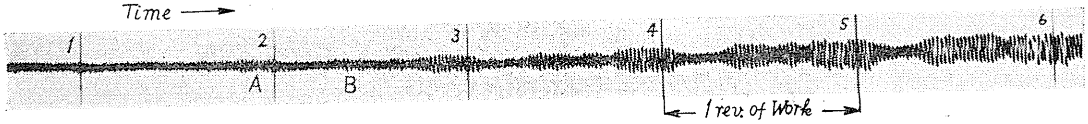

FIG. 35. Horizontal vibration of workpiece.

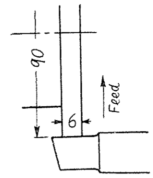

FIG. 36. Cutting manner.

Fig. 35 shows the horizontal vibration of workpiece obtained by using the optical method,[11] where the cutting speed is 20 m/min., the thickness of flange 6 mm, the magnitude of feed 0.04 mm, and the cutting angle of tool is 90°. In the figure, vertical lines indicate the revolution marks of workpiece. It is seen in the figure that chatter vibration occurs even in a cutting of cast iron. Furthermore, it can be seen that, at two parts indicated by *A* and *B*, small transient vibrations of workpiece are generated immediately after the begining of cutting, and that in the successive revolution the amplitudes of these transient vibrations are not only increased just at the points corresponding to *A* and *B* but the vibrating regions are extended around the circumference. The vibration of workpiece is thus heightened step by step in the following revolutions and finally the vibration of large amplitude extends around the whole circumference. This exciting manner of vibration is quite different from that of primary chatter.

It can be noted that, even though it is very small, if some sort of transient horizontal vibration of workpiece be generated (see Fig. 35 *A*, *B*), small undulations could be produced to some region on the work surface. As the workpiece continues to rotate, these small undulations will now produce fluctuating forces on the workpiece and will excite the vibration, and thus the undulations will also increase in size step by step in the successive revolution.

The following experiments are now made to ascertain how the undulations will grow in each successive cutting. A square thread of cast iron is machined in the same way as employed in Fig. 32, where the cutting speed is 30 m/min., the width of square thread 6 mm, the magnitude of depth of cut 0.05 mm, and the cutting angle of tool is 90°. Although the chatter vibration does not occur in the first cutting operation, as indicated in Fig. 32, the chatter can be caused when the cutting is operated repeatedly on the same square thread with the same cutting conditions. The chatter is heightened step by step just in the same way as in Fig. 35, and the small undulations initially produced on the work surface become large in each successive cutting operation. Fig. 37 shows the profiles of undulation in every cuttings measuring by a sensitive dial indicator. It is seen in the figure that the undulations increase in size with the following cutting operations, and further that the phase of undulation always lags behind that of the preceding undulation. The magnitude of phase lag, however, is not always equal, *i.e.*, first it is about 90° and decreases gradually with every cuttings, as seen in Fig. 37.

Now, in Fig. 37 *A*-*B*-*C* indicates one cycle of vibration and *A*-*B*, *B*-*C* represent the approaching stroke of workpiece toward the cutting edge, the recess stroke respectively. It is seen that because of the presence of phase lag of undulation, the area being cut in approaching stroke is less than that being cut in recess stroke, and therefore that the action of cutting force in recess stroke is larger than that in approaching stroke.

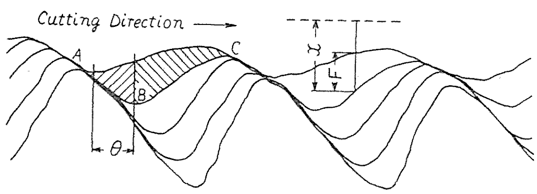

FIG. 37. Undulations on work surface.

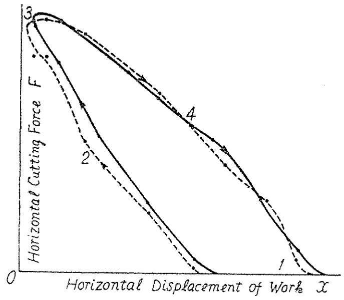

FIG. 38. Force-displacement curve.

Fig. 38 shows the relationship between the horizontal displacement of workpiece x and the horizontal cutting force F, both are obtained from Fig. 37. In Fig. 38, the period 1-2-3 corresponds to the approaching stroke of workpiece to the cutting edge and 3-4-1 to the recess stroke. Solid line indicates the previous cycle and broken one the following cycle. It is seen in the figure that the cutting force does not reach a maximum value at the time where the workpiece moves horizontally nearest to the cutting edge, and slightly after that time, it becomes maximum and subsequently sustains itself at a comparatively high value during the recess stroke as if there is a time lag of cutting force. Thus, because of the presence of phase lag, the energy corresponding to the closed area is furnished to the vibrational system for exciting or maintaining vibration. It is needless to say that if there was a time lag of cutting force, much more energy would be furnished to the system.

It is well-known that when the chatter vibration occurs in practical maciining operations, orderly patterns are produced on a work surface. Fig. 39 shows the chatter patterns produced in a common outer turning of cast iron by a roughing tool, where the workpiece is subject to feed-back effect because the length of cutting edge engaged in cutting is larger than the magnitude of feed. It is seen in Fig. 39 that the phase of undulation of this pattern lags regularly behind that of previous passage in the same way as in Fig. 37 and orderly spiral patterns are thus produced. When the direction of feed is reversed, a pattern of counter-spirals is produced.

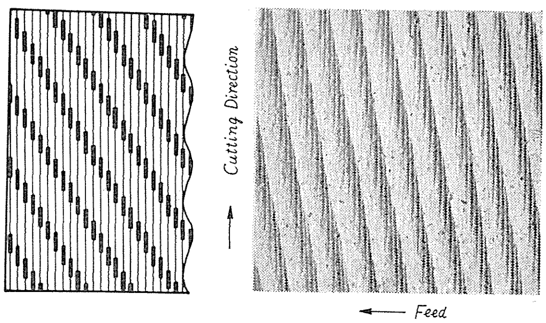

Fig. 39. Chatter patterns

4. Theoretical analyses on regenerative chatter vibration

In the previous section, the reason for causing regenerative chatter was clarified experimentally. Basing on these experimental results, we here deal with regenerative chatter theoretically.

(4.1) Differential equation for regenerative chatter vibration

We analyze regenerative chatter relating only to the horizontal vibration of workpiece for the same reason as described in Chapter II.

It is clear that in regenerative chatter the instantaneous forces acting on the workpiece depend not only upon the present state of motion of workpiece but also upon the motion of the workpiece in the revolution immediately preceding. If we assume a system with one degree of freedom, the expression for regenerative chatter becomes

$$m\ddot{x}+c\dot{x}+k(x+a)=F\{d-x(t-h_0)-x\} \tag{5.1}$$

where the same symbols as in Chapter II are used, and where F is the horizontal cutting force, which is represented as a function of chip thickness $d-x(t-h_0)-x$. The symbol h_0 represents the time lag corresponding to the phase difference θ between the undulatory work surface produced in the preceding revolution and the present motion of workpiece, and is expressed by

$$h_0=\frac{\theta}{\omega} \tag{5.2}$$

where ω is frequency of vibration of workpiece.

If there be a time lag of cutting force, as seen in a cutting of mild steel, the expression for regenerative chatter would be

$$m\ddot{x}+c\dot{x}+k(x+a)=F\{d-x(t-h_0-h)-x(t-h)\} \tag{5.3}$$

where h is the time lag of cutting force.

In order to deal with generalities, we now carry out analyses based on Equation (5.3). We here assume, as we did in previous chapters, that horizontal cutting force is in proportion to magnitude of chip thickness, so that

$$\left\{\begin{array}{l} F\{d-x(t-h_0-h)-x(t-h)\}=K\cdot\{d-x(t-h_0-h)-x(t-h)\} \\ \qquad d-x(t-h_0-h)-x(t-h)\geqq 0 \end{array}\right. \tag{5.4}$$

in which K is the proportion coefficient depending upon cutting conditions.

By substituting Equation (5.4) into Equation (5.3), and using the relation $ka=Kd$, we obtain the following equation:

$$\ddot{x}+2n\dot{x}+p^2x+\lambda x_h+\lambda x_H=0 \tag{5.5}$$

where $2n=\frac{c}{m}$, $p^2=\frac{k}{m}$, $\lambda=\frac{K}{m}$, $x_h=x(t-h)$, $x_H=x(t-H)$, $H=h_0+h$.

We now deal with out problem on the basis of Equation (5.5).

(4.2) Analyses and considerations

Even though Equation (5.5) includes the additional term λx_H which is not included in linear Equation (2.8) of primary chatter, it is still linear and we can apply the same method as that used in Chapter II.

We will try to satisfy Equation (5.5) by a solution of the form

$$x=x_0e^{zt},\qquad z=\alpha+j\omega,\qquad j=\sqrt{-1}. \tag{5.6}$$

On substituting (5.6) for (5.5) we obtain

$$z^2+2nz+p^2+\lambda(e^{-zh}+e^{-zH})=0. \tag{5.7}$$

The real and imaginary parts of this expression yield

$$\begin{cases} p^2 - \omega^2 + \alpha^2 + 2n\alpha + \lambda(e^{-h\alpha}\cos\omega h + e^{-H\alpha}\cos\omega H) = 0 & (5.8) \\ 2(\alpha + n)\omega - \lambda(e^{-h\alpha}\sin\omega h + e^{-H\alpha}\sin\omega H) = 0. & (5.9) \end{cases}$$

For a harmonic vibration ($\alpha = 0$), we have

$$\begin{cases} p^2 - \omega^2 + 2\lambda\cos\frac{\theta}{2}\cos\left(\frac{\theta}{2} + \phi\right) = 0 & (5.10) \\ n\omega - \lambda\cos\frac{\theta}{2}\sin\left(\frac{\theta}{2} + \phi\right) = 0 & (5.11) \end{cases}$$

where $\phi = \omega h,\ \theta = \omega h_0.$

When we assume the time lag of cutting force to be zero, Equations (5.10) and (5.11) become

$$\begin{cases} p^2 - \omega^2 + \lambda(1 + \cos\theta) = 0 & (5.12) \\ 2n\omega - \lambda\sin\theta = 0. & (5.13) \end{cases}$$

Combining Equations (5.10) and (5.11), expressions for harmonic value λ_{1r} and frequency of harmonic vibration ω_{1r} become

$$\begin{cases} \lambda_{1r} = \dfrac{n\omega_{1r}}{\cos\frac{\theta}{2}\sin\left(\frac{\theta}{2} + \phi_{1r}\right)} & (5.14) \\ \dfrac{\omega_{1r}^2 - p^2}{2n\omega_{1r}} = \cot\left(\frac{\theta}{2} + \phi_{1r}\right) & (5.15) \end{cases}$$

wher $\phi_{1r} = \omega_{1r} h.$

When $h = 0$, the corresponding expressions become

$$\begin{cases} \lambda_{1r} = \dfrac{2n\omega_{1r}}{\sin\theta} & (5.16) \\ \omega_{1r} = n\cot\dfrac{\theta}{2} + \sqrt{n^2\cot^2\dfrac{\theta}{2} + p^2} & (5.17) \end{cases}$$

Only when λ is equal to λ_{1r} expressed by Equation (5.14) or (5.16), can the harmonic vibration exist with frequency ω_{1r} indicated by Equation (5.15) or (5.17).

Let us now consider the small change $\Delta\lambda$ in the parameter λ from its harmonic value λ_{1r} for which $\alpha = 0,\ z = j\omega_{1r}$. For $\lambda_{1r} + \Delta\lambda$, the corresponding value of z will be

$$z = j\omega_{1r} + \Delta z, \qquad \Delta z = \Delta\alpha + j\Delta\omega.$$

Substituting this value into Equation (5.7), and carrying out calculations to the first order of small quantities $\Delta\lambda$, $\Delta\alpha$, and $\Delta\omega$, we then have the following equation:

$$\frac{\Delta\alpha}{\Delta\lambda} = \frac{2n(\omega_{1r}^2 + p^2) + \lambda_{1r}^2(H + h)(1 + \cos\theta)}{\lambda_{1r}\{(2n - \lambda_{1r}h\cos\phi_{1r} - \lambda_{1r}H\cos(\theta + \phi_{1r}))^2 + (2\omega_{1r} + \lambda_{1r}h\sin\phi_{1r} + \lambda_{1r}H\sin(\theta + \phi_{1r}))^2\}} \tag{5.18}$$

When $h=0$, the corresponding expression for $\frac{\Delta\alpha}{\Delta\lambda}$ become

$$\frac{\Delta\alpha}{\Delta\lambda}=\frac{2n(\omega_{1r}^2+p^2)+\lambda_{1r}^2 h_0(1+\cos\theta)}{\lambda_{1r}\{(2n-\lambda_{1r}h_0\cos\theta)^2+(2\omega_{1r}+\lambda_{1r}h_0\sin\theta)^2\}}\,. \tag{5.19}$$

The right-hand side of Equations (5.18) and (5.19) is always positive; hence $\alpha\gtreqless 0$ for $\Delta\lambda\gtreqless 0$. Therefore, it can be said that the system given by Equation (5.3) is capable of causing chatter vibration due to both time lag h and phase lag θ. In a system having no time lag of cutting force, however, chatter vibration can be caused by only phase lag θ. The condition for causing vibrations is then expressed by

$$\lambda \geqq \lambda_{1\lambda} \tag{5.20}$$

We now calculate the magnitude of λ_{1r} from Equations (5.14) and (5.16). Fig. 40 shows the relationship between λ_{1r} and θ obtained by numerical calculations. This figure is for the system $p=377$/sec., $n=40$/sec., which is frequently employed in Chapters II~IV. Curve A shows $\lambda_{1r}-\theta$ relationship obtained from Equation (5.16) and is for the condition having no time lag of cutting force. On the other hand, curves B, C and D are obtained from Equation (5.14), where the time lag of cutting force h are 0.0005, 0.001 and 0.0015 sec. respectively. The whole area above each curve constitutes the unstable region ($\lambda\geqq\lambda_{1r}$) for each vibrational system.

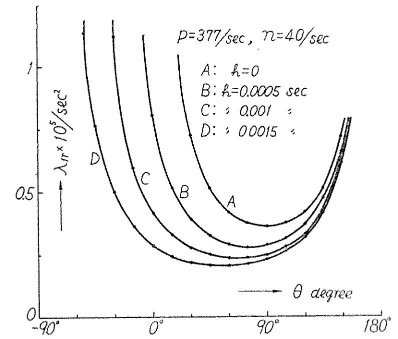

FIG. 40. Relation between phase lag θ and harmonic value λ_{1r}.

It is seen in curve A that the magnitude of λ_{1r} is infinity when $\theta=0^\circ$ and $\theta=180^\circ$, and that when $\theta=90^\circ$ the magnitude of λ_{1r} is minimum. Therefore, chatter cannot occur when $360^\circ>\theta>180^\circ$. If chatter vibration could occur, it could be said theoretically that the magnitude of phase lag must be in the range $180^\circ>\theta>0^\circ$. In curve B, C or D, this range extends a little to the left. At the same time, the point at which the magnitude of λ_{1r} is minimum shifts to the left, and this tendency to shift to the left is large when h is large. In the machining of mild steel, the magnitude of h is generally considered to be in the range $h=0.0005\sim0.001$ sec., as described in Chapter II. When $h=0.0005\sim0.001$ sec., the magnitude of θ at which λ_{1r} is minimum is about $60^\circ\sim75^\circ$, as seen in the figure. Therefore, it can be considered that chatter is most likely to occur when $\theta=60^\circ\sim75^\circ$.

The author[26] and S. Doi reported that chatter vibration (regenerative) has a natural tendency to maintain a certain magnitude of phase lag, and that the magnitude of this phase lag is such that it can feed energy into the vibrational system at the maximum rate, and further that, in their experiments made by cutting mild steel, its magnitude is usually about 60°. This experimental result seems to be in good agreement with our theoretical finding.

Next, we compare the harmonic value λ_{1r} in regenerative chatter with λ_1 in primary chatter. Fig. 41 is obtained by numerical calculations using the same values as those used in Chapter II, where we assumed that the magnitude of n

(damping coefficient of vibrational system) increases in proportion to p (natural frequency of system).

In Fig. 41, curve A represents the harmonic value λ_{1r} calculated by Equation (5.14), where $h=0.0005$ sec., $\theta=60°$. Curve B is obtained from Equation (5.16), where $h=0$, $\theta=60°$. For reference, curve C is the same one as shown by broken line in Fig. 9, where it is for the primary chatter. It is seen in all curves that the magnitude of λ_{1r} or λ_1 increases with increase of p, but that the magnitudes of λ_{1r} of curves A, B are considerably smaller than the magnitude of curve C, and further that the increasing manner of curves A, B is different from mode of curve C. The whole area above each curve constitutes the unstable region. It can therefore be said that in curve A, chatter is far more likely to occur than in curve C, and that even in curve B in which the time lag $h=0$, chatter is more likely to occur than in curve C. In other words, in cuttings where the work surface produced during previous revolution of workpiece has an effect on the present motion of the vibrational system (in cuttings, being subject to so-called feed-back effect), chatter occurs remarkably in comparison with the cuttings in which the motion of vibratory system is not subject to the feed-back effect. Furthermore, chatter vibration can be caused by so-called feed-back effect even in cutting where there is no time lag of cutting force, for example, in cutting of brittle metals such as cast iron. These theoretical results are in good agreement with the practical experiences.

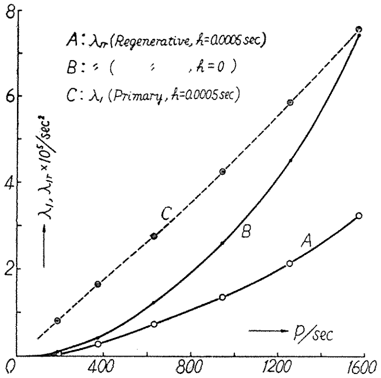

FIG. 41. Relation between natural frequency p and harmonic value λ_1 or λ_{1r}.

(4.3) Frequency of regenerative chatter vibration

The frequency of regenerative chatter vibration can be obtained graphically from Equation (5.15) by the same method as that used in Chapter IV, as shown in Fig. 42. The frequency for the case of $h=0$ is given immediately by Equation (5.17).

Equation (5.15) can be written

$$y_1 - y_2 = y_3$$

where
$$y_1 = \frac{1}{2nh}\phi, \quad y_2 = \frac{p^2 h}{2n}\frac{1}{\phi}, \quad y_3 = \cot\left(\frac{\theta}{2}+\phi\right). \tag{5.21}$$

In Fig. 42, ϕ_{1r} can be obtained by finding the abscissa of the point of intersection of two curves $y_1 - y_2$ and y_3. Since curve $y_3 = \cot\left(\frac{\theta}{2}+\phi\right)$ shifts to the origin in the amount of $\frac{\theta}{2}$ in comparison with curve $y_3 = \cot\phi$ for primary chatter, it is seen that the frequency of regenerative chatter ω_{1r} must be smaller than that of primary chatter ω_1. The difference between them, however, is influenced by the magnitude of phase lag θ and by the other conditions.

We now calculate both frequencies ω_{1r} and ω_1 by the above method, and com-

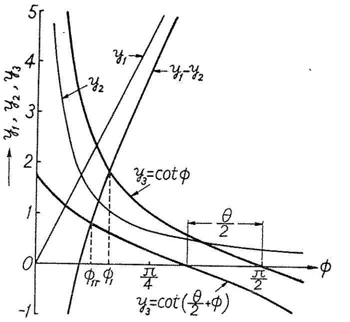

FIG. 42. Graphical calculation of frequency.

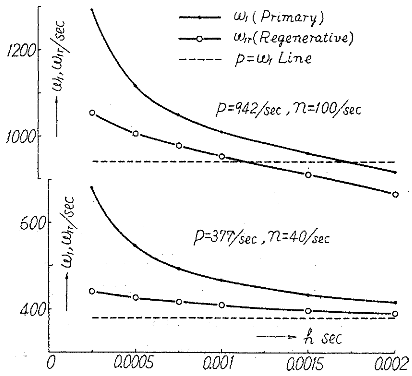

FIG. 43. Relation between time lag h and frequency of chatter ω_1 or ω_{1r}.

pare them. Fig. 43 shows the results of calculations for two vibrational systems, one is $p=942$/sec., $n=100$/sec., and the other is $p=377$/sec., $n=40$/sec. In the figure, ω_{1r} and ω_1 are plotted for h. Curves indicated by the mark ○ are for regenerative chatter in which $\theta=60°$ and those indicated by the mark ● are for primary chatter; for reference, the broken lines are $\omega_{1r}=p$. It is seen in the figure that frequency ω_{1r} is always considerably smaller than ω_1 for each system, and that in the system $p=942$, $n=100$, ω_{1r} becomes smaller than the natural frequency p when h is large. This tendency to become smaller than p is considered greater when $\theta>60°$.

5. *Conclusions*

The exciting manner of so-called regenerative chatter vibration has been found experimentally. Namely, small undulations initially produced on a work surface by the occurrence of transient vibratoin of the workpiece become more and more large and herewith the undulations extend the whole work surface because the phase of undulation always lags behind that of the preceding undulation. It has been proved that owing to the existence of this phase lag, some amount of energy can be furnished to the system for exciting or maintaining vibration.

Then, a differential equation for regenerative chatter vibration was introduced based on the experimental results. By this equation, it has been proved theoretically that regenerative chatter is a kind of self-excited vibration caused by the phase lag of present motion of the vibrational system existing behind the undulatory work surface produced during the preceding revolution of workpiece, although it is subject to some influences like a forced vibration. Moreover, the expression for conditions occurring regenerative chatter vibration has been introduced on the basis of our differental equation. By this expression, many characteristics of occurring regenerative chatter vibration were examined comparing with those of primary chatter.

Chapter VI. Chatter Vibration due to Flexible Cutting Tool having One Degree of Freedom

1. Introduction

Hitherto we have dealt with chatter vibration caused by deflection of the main spindle of a lathe and the workpiece. We now examine the chatter vibration caused by flexible cutting tool. Generally speaking, the frequency of this latter vibration is considerably higher than frequency of the former. Chatter caused by flexible cutting tool is usually accompanied by shrill sound and produces fine pitched undulations on the work surface by which we can distinguish it.

The first comprehensive treatise on chatter caused by flexible cutting tool was by R. N. Arnold.[1] Further papers on this subject were separately published by A. J. Chisholm,[2] R. S. Hahn[4] and E. Salje.[5]

R. N. Arnold and A. J. Chisholm reported that the chatter of cutting tool is a self-excited vibration caused by the falling characteristic of cutting force, which depends upon the cutting speed, in the same way that the self-excited frictional vibration is caused by the dry friction, which depends on the rubbing speed. We agree that the chatter of the cutting tool may in some cases be caused by the falling characteristic of cutting force. The author, however, feels that the fundamental cause of chatter vibration does not depend on this characteristic of cutting force, as suggested by R. S. Hahn and E. Salje.

To verify this, the author[3] together with S. Doi carried out experiments on the self-excited frictional vibration of a flexible rubbing tool similar to those on the chatter vibration of flexible cutting tool, and the character and cause of chatter vibration were examined and compared. As a result, it was ascertained that the character of chatter caused by the flexible cutting tool resembles that of frictional vibration (flexible rubbing tool), but that it was not caused by the falling characteristic of cutting force. We feel that the fundamental cause of the chatter vibration of a flexible cutting tool may be considered to be the same as that of the deflection of the main spindle of lathe or of the workpiece. In other words, chatter due to a flexible cutting tool is a kind of self-excited vibration caused by lag in fluctuation of horizontal cutting force existing behind the horizontal vibration of cutting edge.

In an ordinary cutting operation, it is possible for the cutting edge to move in both directions, *i.e.*, horizontally and vertically (see Fig. 44). The cause of chatter can be said to depend on the horizontal vibration of the cutting edge by which the area of cut fluctuates. It is the fluctuations in cutting area that produce the time lag in cutting force.

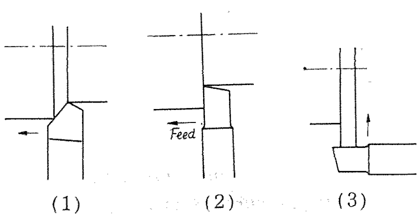

FIG. 44. Cutting manner (several degrees of freedom).

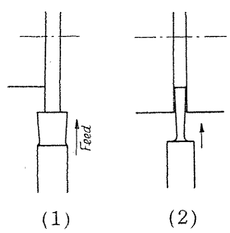

FIG. 45. Cutting manner (one degree of freedom).

The chatter vibration, however, occurs frequently and is accompanied by undulations on the work surface even in the cutting operation in which such tools as shown in Fig. 45 are used, for example, even in the cutting operation using cutting-off tool in which the cutting edge is capable of moving only in a vertical direction, *i.e.*, the system having one degree of freedom.

To find the answer to the question "What can be the reason for these undulations?," we deal in this paper with the problem relating to chatter vibration due to the flexible cutting tool having one degree of freedom as a reason for undulations. We first make clear the way in which chatter vibration causes undulations on the work surface and then we elucidate the fundamental cause of this chatter.

2. Experiments and considerations

Tests are made under the orthogonal cutting condition as shown in Fig. 46. A flange machined preliminarily from a mild steel bar is held on the lathe by the chuck and machined with a tool 13.4 mm square and 130 mm long. The tool is always placed normal to the periphery of flange and is fed by the cross spindle of lathe in an axial direction. To ensure a complete elastic deflection of the tool, a special tool post with no upper slide is employed and the upper and lower surfaces of the tool are ground to secure rigid clamping of the tool. Vertical vibrations of the tool end are measured by means of vertical deflection angle of the tool.[22] Fig. 47 gives an example of records of experiments measuring vertical vibrations of cutting edge where the cutting conditions are as follows: Cutting angle of tool 80°, clearance angle of tool 10°, thickness of flange 1 mm, feed 0.01 mm, diameter of flange 65 mm, and cutting speed 51 m/min. Fig. 48 is a photograph of undulations produced on the work surface by the occurrence of chatter shown in Fig. 47.

In cutting operations such as shown in Fig. 46, it is impossible for the cutting edge to move in horizontal direction; it can vibrate only in vertical direction. Notwithstanding the fact that it is impossible for the cutting edge to move in horizontal direction, so-called "chatter marks" are usually produced on the work surface as shown in Fig. 48. The wave length of this chatter mark corresponds exactly

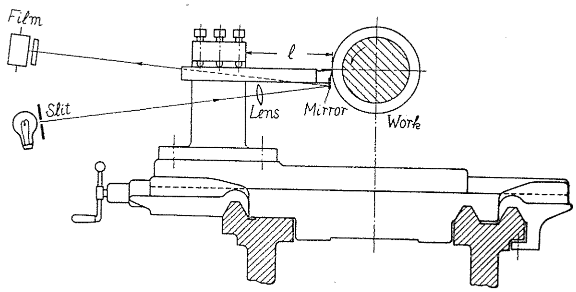

FIG. 46. Method of experiment.

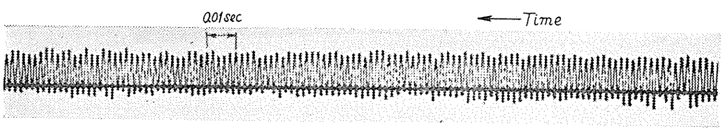

FIG. 47. Vertical vibration of cutting tool.

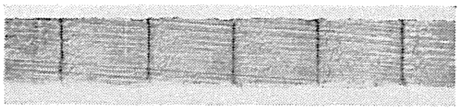

FIG. 48. Chatter mark.

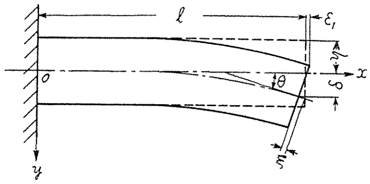

FIG. 49. Analysis of fluctuation of chip thickness during vibration.

to the frequency of the vertical vibration of cutting edge. It is not possible for the workpiece and the tool post to vibrate at such high frequency. Therefore, the chatter mark must be the result of the vertical vibration of cutting edge.

We will now examine the causes producing the chatter mark. When the cutting edge moves downward to displacement δ, it follows that the cutting edge is displaced slightly to displacement ε_1 as shown in Fig. 49, and the fluctuations in area of cut could be produced. Of course, the magnitude of ε_1 is considerably small, and depends not only on the position of the cutting edge with respect to the neutral axis of the tool shank and the tool support design, but also on the modes of vibration of the system.

We will try to calculate ε_1 assuming that the tool is an elastic body with one end built in and the other end free. The expression for the deflection of elastic bar to its normal mode of vibration is[28)]

$$y = A(\cos k_1 x - \cosh k_1 x) + B(\sin k_1 x - \sinh k_1 x) \tag{6.1}$$

where A and B are the arbitrary constants decided by boundary conditions, and k_1 is a well-known physical constant. In this case, it follows that $A = \frac{\delta}{2}$, $B = 0.3671\,\delta$ and $k_1 l = 1.875$, in which δ denotes the vertical displacement of cutting edge and l the length of tool. Setting

$$L = \int_0^l \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \tag{6.2}$$

the magnitude of ξ (see Fig. 49) is expressed by

$$\xi = L - l. \tag{6.3}$$

Substituting Equation (6.1) into (6.2) and carrying out calculations by means of Simpson's first rule, we obtain the following expression for L

$$L \fallingdotseq l + 0.1733\, k_1^2 l \delta^2. \tag{6.4}$$

Accordingly, it follows that
$$\xi = \frac{0.6093\,\delta^2}{l}. \tag{6.5}$$

The deflection angle (denoted by θ) at the free end of neutral axis is expressed by

$$\tan\theta = \left(\frac{dy}{dx}\right)_{x=l} = \frac{1.3764\,\delta}{l}. \tag{6.6}$$

Consequently, because of the vertical motion of the cutting edge, the magnitude of horizontal displacement ε_1 becomes

$$\varepsilon_1 = \xi - \eta \tan \theta. \tag{6.7}$$

Substituting Equations (6.5) and (6.6) into Equation (6.7), we obtain

$$\varepsilon_1 = \frac{\delta}{l}(0.6093\,\delta - 1.3764\,\eta) \tag{6.8}$$

where η denotes the height of the cutting edge with respect to the neutral axis of the tool shank, as shown in Fig. 49.

Further, the fluctuations in area of cut are also considered to be caused as follows: As shown in Fig. 50, when the cutting edge is set up lower or higher than the center of workpiece and the cutting edge vibrates with certain amplitude in vertical direction, the fluctuations of chip thickness produced will be in the magnitude of ε_2 owing to the curvature of the workpiece. Of course, the magnitude of ε_2 depends not only on the radius of curvature of the workpiece and the amplitude of vibration, but also on the setting height of the cutting edge with respect to the center of workpiece. As the expression for ε_2, we can lead with

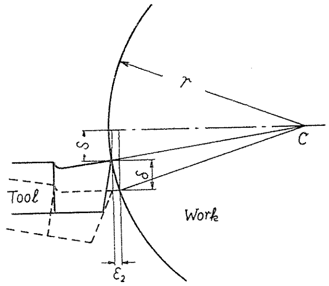

FIG. 50. Analysis of fluctuation of chip thickness during vibration.

$$\varepsilon_2 = \sqrt{r^2 - s^2} - \sqrt{r^2 - (s+\delta)^2} \fallingdotseq \frac{\delta^2 + 2s\delta}{2r} \tag{6.9}$$

where r is radius of curvature of workpiece, and s is setting height of cutting edge with respect to center of workpiece and is positive when set below the center of workpiece.

The expression for total fluctuations in area of cut then becomes

$$\varepsilon = \varepsilon_1 + \varepsilon_2 = \frac{\delta}{l}(0.6093\,\delta - 1.3764\,\eta) + \frac{\delta^2 + 2s\delta}{2r}. \tag{6.10}$$

Consequently, even when the cutting edge is able to move only in vertical direction, the fluctuations in area of cut can be produced by the vertical vibration of cutting edge.

Now, the wave form of the chatter mark can be calculated by Equation (6.10). Fig. 51 shows an example of calculations assuming that the cutting edge vibrates harmonically in vertical direction as $\delta = A \sin \omega t$.
In the figure, (1), (2) and (3) correspond to the conditions $s = 4.28$, 2.92 and 1.35 mm, respectively, where l, η, r, A and ω are held constant: $l = 130$ mm, $\eta = 4.5$ mm, $2r = 65$ mm, $A = 0.27$ mm and $\omega = 2\pi f = 6.28 \times 504 = 3{,}165/\text{sec}$. These numerical values of s, l, etc. are of the test conditions of experiments which will appear later. It is seen by these figures that in each profile there is a protruding part denoted by P in one pitch which is produced by the most downward displacement of cutting edge, and that the wave forms are to some extent unique. It is also seen that the

height of waves denoted by m increases as the magnitude of s increases. The wave form of (3) in Fig. 51 differs considerably from those of (1) and (2), in as much as there is a second small prominence at point Q in the same pitch, the reason for which is as follows: The magnitude of ε which corresponds to total fluctuations of chip thickness is the sum of ε_1 and ε_2 as seen in Equation (6.10). In general, ε_1 is a negative value, because the second term of right-hand side in Equation (6.8) is commonly larger than the first term, while ε_2 is a positive value when $s>0$. The absolute value of ε_2 is, in general, considerably larger than that of ε_1 especially when s is large. On the other hand, if the magnitude of s is small, the absolute value of ε_1 becomes larger than that of ε_2 at some points in one cycle during vibration. Thus in the neighborhood of point Q in Fig. 51-(3), the absolute value of ε_1 exceeds that of ε_2, and another prominence is formed. In Fig. 47-(1), (2), since at all points of a cycle during vibration, ε_2 is always larger than ε_1, there is no prominence other than P.

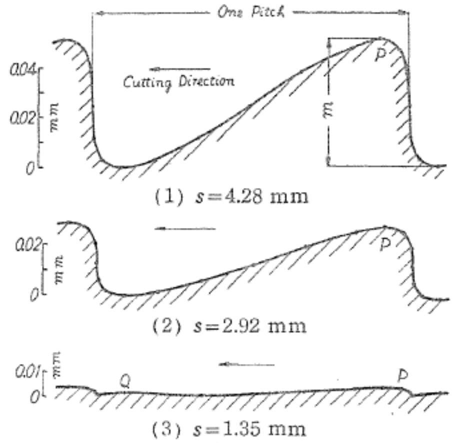

FIG. 51. Profiles of chatter mark obtained by theoretical calculations.

Next, to ascertain the foregoing considerations, the following experiments are carried out. The profile of the chatter mark (shown in Fig. 48) is recorded optically by means of a tracer method.[29] Fig. 52 shows an example of the experimental recordings. In the figure, (1), (2) and (3) are the results for the conditions $s=4.28$, 2.92 and 1.35 mm respectively, where l, η, r, A and ω are held constant: $l=130$ mm, $\eta=4.5$ mm, $2r=65$ mm, $A=0.27$ mm and $\omega=2\pi f=3{,}165$/sec. It is seen by these figures that the wave forms of these experimental recordings are to some extent unique and that they resemble closely those of Fig. 51 which are calculated theoretically by Equation (6.10).

For reference, Fig. 53 shows the profile of chatter marks produced by the chatter vibration occurring in such cutting operation as shown in Fig. 44-(3), *i.e.*, the system having two degrees of freedom. In this case, the cutting edge vibrates in both directions, horizontally and vertically. Therefore, the fluctuations of chip

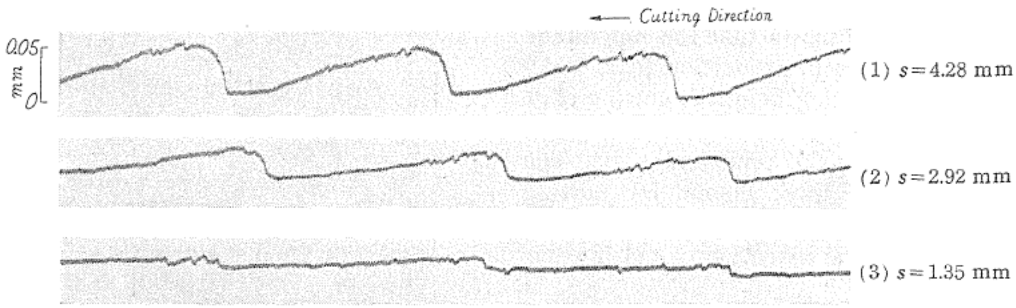

FIG. 52. Experimental records of profile of chatter mark.

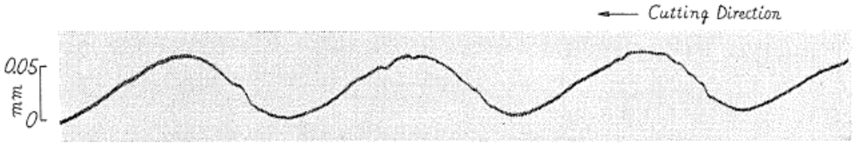

FIG. 53. Experimental record of profile of chatter mark (2 degrees of freedom).

thickness are large compared with those of Fig. 52 and thus the height of wave is considerably large. Moreover, the wave form in Fig. 53 is in sharp contrast to those in Fig. 52. Accordingly, the unique form of chatter marks shown in Fig. 52 is considered to be a special feature of chatter vibration due to the flexible cutting tool having one degree of freedom.

Further investigations were made relating to the height of wave denoted by m in Fig. 51. The magnitude of m is closely connected with various conditions as seen in Equation (6.10). Figs. 54, 55, and 56 show the relationships between m and various conditions. The curves indicated by broken lines in these figures give the results of theoretical calculations by using Equation (6.10), and the solid lines indicate the experimental results obtained by measuring the height of chatter marks using the sensitive dial indicator.

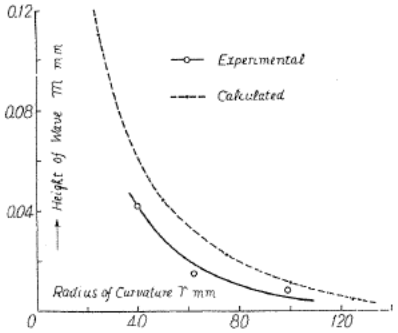

FIG. 54. Relation between radius of curvature of workpiece r and height of wave m.

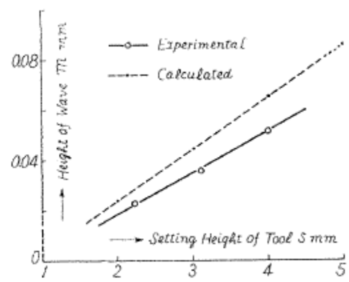

FIG. 55. Relation between setting height of tool s and height of wave m.

In Fig. 54, it is seen that the magnitude of m decreases with increase of diameter of workpiece, which is reasonable. Fig. 55 shows that the magnitude of m increases in proportion to s, s denoting the setting height of cutting edge with respect to the center of workpiece. Fig. 56 shows the relationship between the height of wave m and the cutting speed. It is well-known that the amplitude of vertical vibration of cutting tool is in proportion to the cutting speed as expressed by[1)2)3)]

$$A = \frac{V}{2\pi f}$$

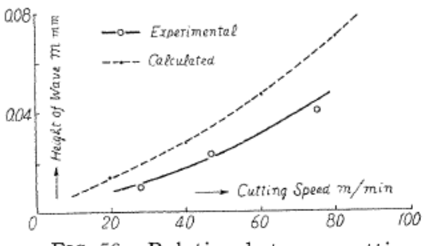

FIG. 56. Relation between cutting speed V and height of wave m.

where A is the amplitude of chatter vibration in vertical direction, and V denotes the cutting speed. Therefore, it can be said that Fig. 56 shows the relationship between m and A. It is seen in the figure that the magnitude of m increases with increase of cutting speed. Then it can be said that the magnitude of m increases with increase of the amplitude of vibration.

In Figs. 54 ~ 56, the experimental results have the same tendency as the theoretical calculations; but the former are slightly lower than the latter in magnitude. One reason for this is that the profile of chatter marks has a protruding part denoted by P in one pitch, as seen in Fig. 51. In cutting operation this part may be partially broken away together with the chip. As another reason, we can mention the effect of measuring pressure of dial indicator on the chatter marks.

From the above description, it has been ascertained that the horizontal displacement relative to the workpiece can be caused by the vertical movement of cutting edge even in the cutting operation as shown in Fig. 45 where the cutting edge of tool is able to move only in vertical direction, and thus it has also been clarified that the fluctuations in area of cut can be caused by the vertical movement of cutting edge during vibration even in the system having one degree of freedom. When the fluctuations in area of cut are caused by the vertical vibration of cutting edge, the phase of variation in vertical cutting force lags behind that of fluctuation in area of cut, as described in Chapter I of this paper. Of course, the magnitude of lag in phase in vertical direction is generally small compared with that in horizontal direction. Accordingly, the energy which must be available for maintaining the vertical vibration of cutting edge will be furnished to the vibrational system by this lag in vertical cutting force in the same way that the chatter vibration due to the deflection of main spindle of a lathe and of a workpiece is caused by the time lag in horizontal cutting force.

In Fig. 57, let line O-O denote the center of vertical vibration of cutting edge, and assume that the fluctuations in area of cut are caused by the vertical vibration of cutting edge as shown in the figure. At point 1 where the cutting edge reaches just to the center line vibrating downward, the cutting force corresponding to the area of cut denoted by a-a will act on the system, because the time lag in vertical cutting force exists. On the contrary, at point 2 where the cutting edge reaches just to the center vibrating upward, the vertical cutting force corresponding to the area of cut denoted by b-b will act on the system. It can therefore be said that during the period when the cutting edge moves downward, the action of the vertical cutting force of large magnitude is in the same direction as the movement of cutting edge, while in the period during which the cutting edge moves upward, the cutting force of comparatively small magnitude acts against the movement of cutting edge. Hence, the vibrational system gains the energy necessary to maintain vibration. However, the magnitude of fluctuation in area of cut and the manner of its fluctuation during one cycle of vibration are closely connected with the various cutting conditions, as described previously. Thus, there is a considerable increase or decrease in the amount of available

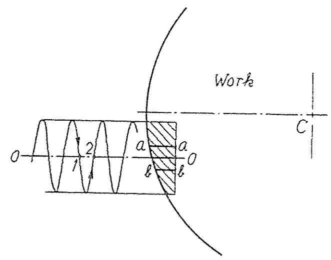

FIG. 57. Vertical displacement of cutting edge and fluctuation of area of cut.

energy according to the cutting conditions.

We will now examine the amounts of available energy versus various cutting conditions by means of numerical calculations using Equation (6.10). Fig. 58 shows an example of calculations assuming that the vertical cutting force can be represented as the linear function of area of cut having constant time lag. In the figure, (1), (2) and (3) are the results of calculation corresponding to the conditions $s=4.28$, 2.92 and 1.35 mm respectively, where all conditions except s are held constant the same as conditions in Fig. 51. In our calculations, the numerical values used for the magnitude of time lag in vertical cutting force (h) and that of the specific vertical cutting force (k_s) are as follows:

$$h \fallingdotseq 0.0002 \text{ sec.},\text{[30]} \qquad k_s = 300 \text{ kg/mm}^2.\text{[31]}$$

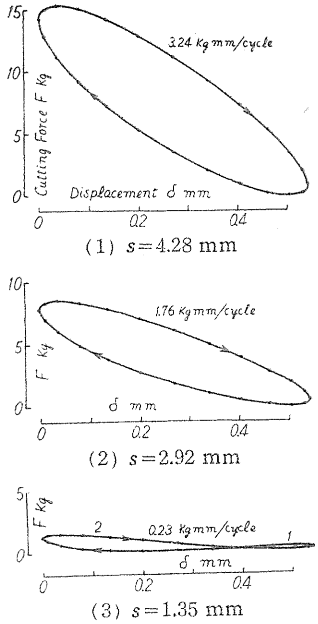

FIG. 58. Force-displacement curves.

In Fig. 58, the cutting force is plotted in the ordinate and the displacement of cutting edge is plotted in the abscissa, and in each figure the displacement to the right in direction corresponds to the downward movement of the cutting edge. The arrows indicate the rotational direction in each figure. It is seen that each curve in Figs. 58-(1) and (2) forms an ellipse, but that the curve in Fig. 58-(3) is quite different somewhat bottle-shaped, the reason for which is the same as described previously relating to Fig. 51-(3). The closed areas within these forms correspond to the amount of energy available for maintaining the vibration. In Fig. 58-(3), the available energy is expressed by the remaining area 2 after removal of area 1. Consequently, it is found by these figures that the chatter vibration is apt to occur in such conditions where s is large.

To verify the foregoing considerations, the following experiments were carried out. Fig. 59 shows the experimental recordings measuring the vertical vibration of cutting edge. In the figure, (1), (2), (3) and (4) are the recordings for conditions $s=4.28$, 3.51, 2.92, and 1.35 mm respectively. where all conditions except s are held constant the same as conditions in Fig. 47. Fig. 60 is the record measuring the free vibration of tool end. The frequencies of these chatter vibrations do not in any way depend on the cutting conditions; they are almost the same as natural frequency of the system. It is seen in Fig. 59 that the chatter vibration is stationary when it occurs in the condition where s is large; and that it becomes unstable gradually with decrease of s, and finally it almost dies out. It is evident that these facts sustain the foregoing considerations. However, the occurrence of chatter vibration depends not only upon the setting height of tool s, but also on various conditions such as the diameter of workpiece, the length of tool shank, and other conditions. Further, although the foregoing experiments were carried out by cutting a thin flange 1 mm in width, we can assume that in a cutting operation with a thicker flange, the chatter would be more likely to occur even when s is small.

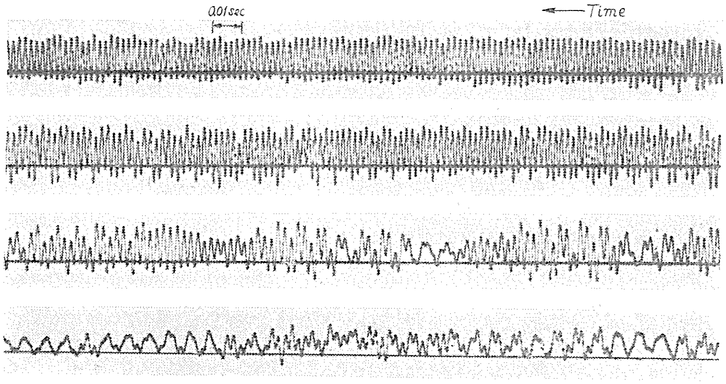

FIG. 59. Vertical vibrations of cutting tool.

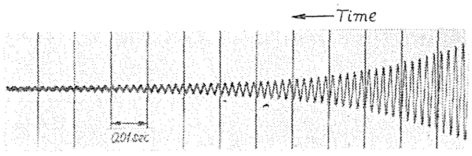

FIG. 60. Free vibration of cutting tool.

Now, if we accept the theory that the chatter vibration due to flexible cutting tool having one degree of freedom is caused by the falling characteristic of cutting force depending upon the cutting speed, the relative difficulty of occurring chatter, as described above, cannot be explained by this theory.

From the above description, it is found that the horizontal displacement relative to the workpiece can be caused by the vertical movement of cutting edge even in the cutting operation where it is impossible for the cutting edge to move in horizontal direction. In other words, it has been clarified that the fluctuations in area of cut can be caused by the vertical movement of cutting edge during vibration even in the system having one degree of freedom in vertical direction. When the fluctuations in area of cut are caused by the vertical vibration of cutting edge, the phase of variation in vertical cutting force lags behind that of the fluctuation in area of cut. The energy available for maintaining the vertical vibration of cutting edge is furnished by this lag in vertical cutting force. Consequently, it has been ascertained experimentally that the chatter vibration due to the flexible cutting tool having one degree of freedom must be a self-excited vibration caused by the lag in vertical cutting force existing behind the vertical vibration of cutting edge.

3. Theoretical analyses

We will now introduce a differential equation expressing the chatter vibration caused by the flexible cutting tool having one degree of freedom based upon the foregoing considerations.

In Fig. 61, let O denote the position of cutting edge before cutting, and s_i denote the setting height of cutting edge with respect to the center of workpiece indicated by C. When a cut is operated, the cutting edge displaces to the point

O_1. If we denote this displacement by s_d, and the undeformed chip thickness by d, it follows that

$$F(d) = k \cdot s_d \qquad (6.11)$$

where $F(d)$ is the vertical cutting force corresponding to the chip thickness d, and $k \cdot s_d$ is the spring force of tool. Taking O_1 as the point of origin and indicating the vertical movement of cutting edge by y, when the movement is in the downward direction, it is considered to be positive. The expression for the chatter vibration then becomes

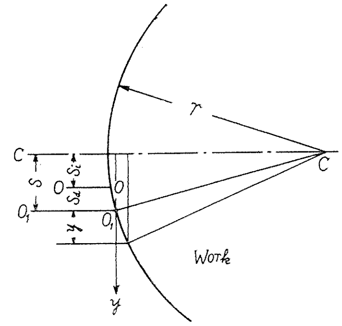

FIG. 61. Analysis of vibration.

$$m\ddot{y} + c\dot{y} + k(y + s_d) = F(d, y). \qquad (6.12)$$

For the sake of simplicity, we will assume that the vertical cutting force denoted by $F(d, y)$ is in proportion to the magnitude of area of cut, and that the vertical cutting force has a constant time lag h. Then the vertical cutting force during vibration must be expressed by

$$F(d, y) = K \cdot \left\{ d - (\alpha y_h^2 - \beta y_h) - \frac{y_h^2 + 2 s y_h}{2r} \right\} \qquad (6.13)$$

where $\quad y_h = y(t-h), \quad s = s_i + s_d, \quad \alpha = \dfrac{0.6093}{l}, \quad \beta = \dfrac{1}{l}(1.3764\eta - 2 \times 0.6093\, s_d).$

In Equation (6.13), the second and third terms within the braces of the right-hand side correspond to ε_1 and ε_2 defined previously. Using Equation (6.11), and substituting Equation (6.13) into (6.12), we obtain

$$\ddot{y} + 2n\dot{y} + p^2 y + u y_h + v y_h^2 = 0 \qquad (6.14)$$

where $\quad 2n = \dfrac{c}{m}, \quad p^2 = \dfrac{k}{m}, \quad u = \lambda\left(\dfrac{s}{r} - \beta\right), \quad v = \lambda\left(\dfrac{1}{2r} + \alpha\right), \quad \lambda = \dfrac{K}{m}.$

In Equation (6.14), the non-linear term $v y_h^2$ is generally negligibly small compared with the linear term $u y_h$. Therefore, we will first deal with the problem relating to the linear equation which neglects the non-linear term. The differential equation of chatter vibration is then expressed by

$$\ddot{y} + 2n\dot{y} + p^2 y + u y_h = 0. \qquad (6.15)$$

This equation is just the same in form as Equation (2.8) in Chapter II of this paper. Consequently, it can be said that, by using the same treatment as in Chapter II, the system given by Equation (6.15) is capable of self-excitation because of the time lag h, and that the expression for the condition of self-excitation is of the form

$$u \gtreqless u_1 \quad \text{or} \quad \lambda \gtreqless \lambda_1 \qquad (6.16)$$

where u_1 or λ_1 is the so-called harmonic value which is represented by

$$u_1 = \frac{2n\omega_1}{\sin \omega_1 h} \quad \text{or} \quad \lambda_1 = \frac{2n\omega_1}{\left(\frac{s}{r} - \beta\right) \sin \omega_1 h}. \tag{6.17}$$

In Equation (6.17), it is found that the magnitude of λ_1 is small when s is large, and when r and n are small, λ_1 is also small. Therefore, it can be said that the chatter vibration is likely to occur in such conditions where s is large, and r and n are small. This is in good agreement with the foregoing considerations and also with our experimental results.

Hitherto we have dealt with the problem relating to the linear Equation (6.15). However, in as much as we have restricted our discussions to the problem of the conditions of self-excitation, it is reasonable to consider that the analysis should be made on the basis of linear Equation (6.15) neglecting the term vy_h^2 instead of on the faultless Equation (6.14). The reason for this is that if the self-excitation develops from a rest point where the initial movement is $y \fallingdotseq 0$, the term of higher order can be neglected.

Based on the above description, it has also been ascertained by theoretical analysis that the self-excited vibration can be caused by the time lag h in vertical cutting force existing behind the vertical vibration of cutting edge even in the system having one degree of freedom, as shown in Fig. 48 in which the cutting edge is capable of moving only in vertical direction.

4. Conclusions

It has been found that the horizontal displacement of cutting edge relative to the workpiece can be caused by the vertical movement of cutting edge even in the cutting operation in which such tools as shown in Fig. 45 are used; for example, in the cutting operation using the cutting-off tool in which the cutting edge is capable of moving only in a vertical direction, *i.e.*, the system having one degree of freedom. In other words, it has been ascertained that the fluctuations in area of cut can be caused by the vertical movement of cutting edge during vibration even in the system having one degree of freedom in vertical direction. When the fluctuations in area of cut are caused by the vertical vibration of cutting edge, the phase of variation in vertical cutting force lags behind that of fluctuation in area of cut. The energy available for maintaining the vertical vibration of cutting edge is furnished by this lag in vertical cutting force. Consequently, it has been proved that the chatter vibration due to the flexible cutting tool having one degree of freedom must be a self-excited vibration caused by the lag in vertical cutting force existing behind the vertical vibration of cutting edge.

Acknowledgement

In concluding this paper, the author expresses his appreciation to Prof. Dr. S. Doi for his guidance and valuable suggestion, to Prof. Dr. Y. Kasuga and Prof. Dr. T. Yamamoto for their encouragements and valuable advices. He also thanks Mr. T. Yamada for his earnest assistance throughout this work. This research is indebted to the Laboratory of Machine Tools, Department of Mechanical Engineering, Nagoya University for laboratory use.

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29) Measuring magnification used was about 150 times.
30) Although the magnitude of time lag in vertical cutting force seems to depend on various cutting conditions, we here used this value tentatively referring to our experimental results in Chapter I of this paper and those reported by S. Doi (see Reference 11). The following considerations hold true without regard to its magnitude.
31) Referring to the experimental data reported by the author together with S. Doi (see Reference 3), we here adopted this value. Our qualitative discussions are not affected in any way by magnitude of k_s.