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A THEORY OF THE COORDINATE REPRESENTATION OF SWITCHING FUNCTIONS

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Abstract

In this paper, a theoretical investigation on the coordinate representation of switching functions is developed and some new concepts concerning the classification of swithing functions are derived.

1. Switching Functions and Linear Functions

The switching algebra is a finite Boolean algebra generated from n independent variables $x_1, x_2, \ldots, x_n$. Its elements are called switching functions or simply functions sometimes, and are denoted by letters such as f, g, h etc. The 3 fundamental Boolean operations, *i.e.*, multiplication, addition and complementation and the relation of inclusion are denoted with $\cdot$ or mere juxtaposition of letters, $+$, $'$ and $\leqq$ respectively. The greatest and the least elements in regard to the relation $\leqq$ are denoted by 1 and 0 respectively. The functions of the special form $x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n}(x_1^{\alpha_1}+x_2^{\alpha_2}+\ldots+x_n^{\alpha_n})$ are called fundamental products (sums), where α's take the values 1 and 0 and x_i^1 and x_i^0 mean x_i and x_i' respectively. There are 2^n fundamental products in all and are numbered serially from 0 to 2^n-1 considering $\alpha_1\alpha_2\ldots\alpha_n$ as the binary expression of an integer. As is well known, every function can be uniquely expanded as a standard sum (product), *i.e.*, as a sum (product) of fundamental products (sums). The number of fundamental products appearing in the standard sum of a function f is called its order and is denoted by a symbol $0(f)$. If a function is the sum of fundamental product numbered $a, b, \ldots, k$, it may be represented conveniently by the so-called numerical symbol $(a, b, \ldots, k)$.

Now we introcduce another important Boolean operation which may be called linear addition as follows.

$$f\oplus g=f'g+fg'=(f+g)(f'+g'). \tag{1.1}$$

Concerning this operation, the following laws can be proved easily from the definition.

$$f\oplus g=g\oplus f \tag{1.2.1}$$

$$(f\oplus g)\oplus h=f\oplus(g\oplus h) \tag{1.2.2}$$

$$f(g\oplus h)=fg\oplus fh \tag{1.2.3}$$

$$(f \oplus g)' = f' \oplus g = f \oplus g' \tag{1.2.4}$$

$$f' \oplus g' = f \oplus g \tag{1.2.5}$$

It may be seen that $f \oplus g$ is reduced to fg' if and only if $f \geqq g$, and to $f + g$ if and only if $fg = 0$. Thus, in particular, we have

$$f \oplus f = 0, \quad f \oplus 0 = f, \quad f \oplus f' = 1, \quad f \oplus 1 = f' \tag{1.2.5}$$

Switching functions formed from n variables x_i and their complements x_i' by means of the operations $\oplus$ only will be defined as linear functions. By virtue of (1.2), any linear function f can be uniquely represented in the form of

$$f = \beta_0 \oplus \beta_1 x_1 \oplus \beta_2 x_2 \oplus \ldots \oplus \beta_n x_n \tag{1.3}$$

where β's take the values 1 and 0. The number of coefficients other than β_0 taking the value 1 will be called the length of the linear function f. Thus for instance, the length of x_1 is 1 and that of $x_1' \oplus x_2 \oplus x_3$ is 3. Linear functions with the coefficient $\beta_0 = 0$ ($\beta_0 = 1$) will be designated as odd (even). The reason is this: Any odd (even) linear function other than 0 (1) is the sum of every fundamental product $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ such that the number of variables x_i for which $\alpha_i = 1$ and $\beta_i = 1$ is odd (even). The above property of linear functions tells us that the order of any linear function except 1 and 0 is 2^{n-1}.

Evidently the totality of 2^{n+1} linear functions forms a commutative group with the operation $\oplus$. Its subgroup formed by 2^n odd linear functions only will be denoted by $\mathbf{L}_n$. Since $\mathbf{L}_n$ can be interpreted as a linear space with a two-element Boolean algebra $\{0, 1\}$ for its coefficient, and its subgroup as subspaces, the usual notions of linear spaces such as linear independence, base, dimension etc. will be naturally introduced into $\mathbf{L}_n$, but their familiarity will justify the omission of the description about the definitions and the consequences thereof.

2. The Coordinate Representation of Switching Functions

Let the fundamental products be put in correspondence with 2^n vertices neighboring to the origin of the unit cube of a 2^n dimensional Euclidean space. Extending this correspondence up to general switching functions in the obvious manner, there arises a one-to-one correspondence between switching functions and vertices of the unit cube.

Taking the center of the cube as the origin of a new coordinate system, say, P-system, and translating the axes to the new origin, the coordinates of the vertices will be turned into $(\delta_0/2, \delta_1/2, \ldots \delta_{2^n-1}/2)$ where δ's take the values 1 and -1. The Euclidean distance between a vertex and the center is $2^{n/2-1}$. Let the inner product of the radius vectors for two vertices f and g be denoted as usual by (f, g), then we have

$$(f, g) = (g, f) = \frac{1}{4}\{0(f' \oplus g) - 0(f \oplus g)\} \tag{2.1}$$

and, in particular,

$$(0, f) = \frac{1}{4}\{0(f') - 0(f)\}. \tag{2.2}$$

Hence we have

$$(f, g) = (0, f \oplus g) \tag{2.3}$$

and consequently, in general,

$$(f \oplus h, g) = (f, g \oplus h) = (0, f \oplus g \oplus h). \tag{2.4}$$

As the order of every linear function except 0 and 1 is 2^{n-1}, we obtain

$$(0, y) = 2^{n-2}\delta_{y0} \tag{2.5}$$

and, in general,

$$(y, z) = (0, y \oplus z) = 2^{n-2}\delta_{yz} \tag{2.6}$$

by virtue of (2.3), where y and z are arbitrary linear functions, and δ_{yz} is 1 if and only if $y = z$, -1 if and only if $y' = z$ and 0 otherwise. Thus the following theorem is established.

Theorem 2.1: 2^n lines connecting the center of the 2^n-cube and the vertices corresponding to linear functions are orthogonal to each other.

This theorem suggests the investigation of switching functions by means of their coordinate representations in regard to a new coordinate system, say, L-system with the origin at the center of the cube and the axes passing the vertices corresponding to linear functions. The positive sense of the each axis of L-system will be defined as that corresponding to the odd linear function passed by it. In connection with the above definition of the positive senses of the axes, we agree that the letters y, z and sometimes u with or without suffices will be used exclusively for odd linear functions.

Now the coordinate transformation from P-system to L-system is given by the following orthogonal matrix.

$$T = 2^{-n/2+1}\begin{pmatrix} \mathbf{r}_0 \\ \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_{2^n-1} \end{pmatrix}, \tag{2.7}$$

where $\mathbf{r}_i$, $i = 0, 1, \ldots, 2^n-1$ are the row vectors with the coordinates of odd linear functions in P-system as their components, and the columns of T correspond to fundamental products. Here it is agreed, once for all, that by the coordinates of a switching function f we mean $2^{n/2}$ times the coordinates of the vertex corresponding to f in L-system. Thus the coordinate f_y of f for y will be given by

$$f_y = 2(y, f). \tag{2.8}$$

Hence, from (2.1) and (2.2), we have

$$f_0 = 2^{n-1} - 0(f) \tag{2.9}$$

and, for $y \neq 0$,

$$f_y = 0(fy) - 0(fy'). \tag{2.10}$$

From the results just obtained, the following theorem is immediate.

Theorem 2.2: Coordinates of a switching function are integers and the sum of their squares is equal to $2^{2(n-1)}$. Except for the case of $n=1$, the parity of all the coordinates coincides with that of the order of the function.

For the purpose of illustration, the coordinates of linear functions and fundamental products will be examined. Since each linear function is located on an axis, it has only one non-zero coordinate with the absolute value 2^{n-1}. From (2.10) its coordinates will be given by

$$z_y^{\varepsilon} = 2^{n-1}\varepsilon\delta_{yz}, \tag{2.11}$$

where ε takes the values 1 and -1, and z^1 and z^{-1} mean z and z' respectively. This notation of linear functions will be used consistently hereafter. Fundamental produts are characterized by the property that their order is 1. Hence except the coordinate for 0, their every coordinate has the absolute value 1. Thus the coordinates of a fundamental products p will be given by

$$p_0 = 2^{n-1} - 1, \qquad p_y = \varepsilon_y \tag{2.12}$$

where we assume that $p \leqq y^{\varepsilon_y}$. Since $p \leqq y^{\varepsilon_y}$ and $p \leq z^{\varepsilon_z}$ imply $p \leqq (y \oplus z)^{-\varepsilon_y \varepsilon_z}$, it can be proved by a mathematical induction on the length of linear function that, for $p = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_n^{\varepsilon_n}$ and $y = x_1 \oplus x_2 \oplus \ldots \oplus x_k$, we have

$$p_y = (-1)^{k+1}\varepsilon_1\varepsilon_2 \ldots \varepsilon_k. \tag{2.13}$$

From the above result, it will be seen that the special fundamental product $q = x_1' x_2' \ldots x_n'$ has the coordinates

$$q_y = 2^{n-1}\delta_{y0} - 1. \tag{2.14}$$

3. Boolean Operation and Relations

Our next task is the coordinate representation of Boolean operations and relations.

(*1*) *Complementation*

Vertices corresponding to a function and its complement are located symmetrically in regard to the origin as is seen from their coordinates in P-system. Therefore we have

Theorem 3.1: For every function f, we have

$$f_y' = -f_y \tag{3.1.1}$$

and

$$f_{y'} = -f_y. \tag{3.1.2}$$

(*2*) *Linear Addition*

Before proceeding to general cases, we consider the linear addition of a function f and a linear function z^{ε}. It will be easily found from (2.4) and (2.8) that

$$(f \oplus z^{\varepsilon})_y = \varepsilon f_{y \oplus z}. \tag{3.2}$$

This shows that the linear addition of a linear function has a very simple effect on the coordinates of the augendum, *i.e.*, it causes only a permutation and sometimes an inversion of signs of coordinates. Incidentally it will be noted that, by taking $z^\xi=1$, (3.2) is reduced to (3.1.1).

Now let us turn to general cases. Applying (2.8), (2.4) and (3.2) and taking account of the agreement on the coordinates, the following theorem will be derived.

Theorem 3.2: For any two functions f and g, we have

$$(f\oplus g)_y=\frac{1}{2^{n-1}}\sum_{z\in \mathbf{L}_n} f_{y\oplus z}g_z=\frac{1}{2^{n-1}}\sum_{z\in \mathbf{L}_n} f_z g_{y\oplus z}. \tag{3.3}$$

(3) *Addition and Multiplication*

To begin with, the sum of two functions whose product is 0 will be examined. From (2.9) and (2.10), we have

$$(f+g)_0=f_0+g_0-2^{n-1}$$

and, for $y\neq 0$,

$$(f+g)_y=f_y+g_y.$$

Thus we have:

Theorem 3.3: $fg=0$ if and only if

$$(f+g)_y=f_y+g_y-2^{n-1}\delta_{y0}, \tag{3.4.1}$$

and $f\geqq g$ if and only if

$$(fg')_y=f_y-g_y+2^{n-1}\delta_{y0}, \tag{3.4.2}$$

Proof: The "only if" part of the first half is already shown. Its "if" part will be proved later by Theorem 3.4. The second half will be proved by (3.1.1) and the first half since $f\geqq g$ is equivalent to $f'g=0$.

Now the assumption that the product of the two functions is 0 will be removed. Since Theorem 3.3 implies that $f_y-2^{n-1}\delta_{y0}$ may be given as the sum of terms $p_y-2^{n-1}\delta_{y0}$ for each fundamental product p such that $p\leqq f$, and since $f+g=fg+(f\oplus g)$ and $fg(f\oplus g)=0$, we have $(f+g)_y+(fg)_y=f_y+g_y$ and $(f+g)_y-(fg)_y=(f\oplus g)_y-2^{n-1}\delta_{y0}$. Solving $(f+g)_y$ and $(fg)_y$ from the above simultaneous equations, we have:

Theorem 3.4: For any two functions f and g, we have

$$(f+g)_y=\frac{1}{2}\{-2^{n-1}\delta_{y0}+f_y+g_y+(f\oplus g)_y\}, \tag{3.5}$$

and

$$(fg)_y=\frac{1}{2}\{2^{n-1}\delta_{y0}+f_y+g_y-(f\oplus g)_y\}. \tag{3.6}$$

When we assume $(f+g)_y=f_y+g_y-2^{n-1}\delta_{y0}$, (3.5) will give $(f\oplus g)_y=(f+g)_y$

i.e. $f \oplus g = f + g$. This being equivalent to $fg = 0$, the "if" part of the first half of Theorem 3.3 is proved.

4. Some Useful Theorems

In this section, some useful theorems for the study of switching functions by means of the coordinate representation will be derived.

Theorem 4.1: For a switching function f, a necessary and sufficient condition for $f \leqq z^{\varepsilon}$ is

$$2^{n-1}\delta_{y0} - f_y = -\varepsilon(2^{n-1}\delta_{yz} - f_{y\oplus z}). \tag{4.1}$$

The set of all odd linear functions y such that $f \leqq y^{\varepsilon_y}$ forms a subgroup $\mathbf{L}_f$ of $\mathbf{L}_n$. Let its base be $\{y_1, y_2, \ldots y_k\}$ and assume that $f \leqq y_i^{\varepsilon_i}$, $i = 1, 2, \ldots, k$, then, for $y = y_1 \oplus y_2 \oplus \ldots \oplus y_h \in \mathbf{L}_f$ we have

$$f = (-1)^{h+1}\varepsilon_1\varepsilon_2 \ldots \varepsilon_h 0(f). \tag{4.2}$$

Proof: From (3.6) and (2.11), we have

$$(fz^{\varepsilon})_y = \frac{1}{2}(2^{n-1}\delta_{y0} + f_y + 2^{n-1}\varepsilon\delta_{yz} - \varepsilon f_{y\oplus z})$$

Therefore (4.1) is equivalent to $(fz^{\varepsilon})_y = f_y$ *i.e.* to $f \leqq z^{\varepsilon}$. Strictly speaking, only

$$f_z = \varepsilon 0(f) \tag{4.3}$$

is sufficient for $f \leqq z^{\varepsilon}$, since (4.3) implies $(fz^{\varepsilon})_0 = f_0$ *i.e.* $0(fz^{\varepsilon}) = 0(f)$. Thus the first half is established. Next, let $\mathbf{L}_f$ be the set of all the odd linear functions y such that $f \leqq y^{\varepsilon_y}$. Clearly $0 \in \mathbf{L}_f$. As $f \leqq y^{\varepsilon_y}$ and $f \leqq z^{\varepsilon_z}$ imply $f \leqq (y \oplus z)^{-\varepsilon_y\varepsilon_z}$, $\mathbf{L}_f$ is a subgroup of $\mathbf{L}_n$. Using the fact that $f_{y\oplus z} = -\varepsilon_y\varepsilon_z 0(f)$ if $f_y = \varepsilon_y 0(f)$ and $f_z = \varepsilon_z 0(f)$, (4.2) will be proved by a mathematical induction on the length of linear functions. This completes the proof.

The above theorem indicates that to each switching function f there corresponds a subgroup $\mathbf{L}_f$ of $\mathbf{L}_n$. This group $\mathbf{L}_f$. will be defined as the group associated to the function f.

Now the coordinates of functions of the form $f = y_1^{\varepsilon_1} y_2^{\varepsilon_2} \ldots y_k^{\varepsilon_k}$ will be calculated for the later uses, where $\{y_1, y_2, \ldots, y_k, \ldots y_n\}$ is a bases of $\mathbf{L}_n$. The results are as follows.

$$f_0 = 2^{n-1} - 2^{n-k}, \tag{4.4.1}$$

and for $y \in \mathbf{L}_f$, $y \neq 0$, say, $y = y_1 \oplus y_2 \oplus \ldots \oplus y_h$

$$f_y = (-1)^{h+1}\varepsilon_1\varepsilon_2 \ldots \varepsilon_h 2^{n-k}, \tag{4.4.2}$$

and for $y \notin \mathbf{L}_f$

$$f_y = 0 \tag{4.4.3}$$

where the group associated to f is determined by the base $\{y_1, y_2, \ldots y_k\}$.

In order to verify the above results, it has only to prove (4.4.1), because, in the first place, (4.4.2) is a direct consequence of (4.4.1) and (4.2), and, in the second place, from Theorem 2.2, (4.4.2), we have

$$2^{2(n-1)} = \sum_{y \in \mathbf{L}_n} f_y^2 = \sum_{y \in \mathbf{L}_f} f_y^2 + \sum_{y \notin \mathbf{L}_f} f_y^2 = 2^{2(n-1)} + \sum_{y \notin \mathbf{L}_f} f_y^2$$

i.e. (4.4.3). The proof of (4.4.1) will be carried out by a mathematical induction on k as follows.

First, as is readily seen from (2.11), (4.4.1) and (4.4.3) are valid for $k=1$. Let them be assumed to be true for $k=l$. Putting $f=y_1^{\varepsilon_1}y_2^{\varepsilon_2}\ldots y_l^{\varepsilon_l}$, $g=y_{l+1}^{\varepsilon_{l+1}}$ and $h=fg$, we obtain

$$h_0 = \frac{1}{2}(2^{n-1} + f_0 - \varepsilon_{l+1} f_{y_{l+1}})$$

by (3.6), (3.2) and (2.11), which, in turn gives

$$h_0 = \frac{1}{2}(2^{n-1} + 2^{n-1} - 2^{n-k}) = 2^{n-1} - 2^{n-l-1}$$

by virtue of the assumption of (4.4.1) and (4.4.3) for $k=l$. This completes the proof. (4.4.1) indicates that the order of the functions of the form $y_1^{\varepsilon_1}y_2^{\varepsilon_2}\ldots y_k^{\varepsilon_k}$ is 2^{n-k}, and, in particular, the order of the functions of the form $y_1^{\varepsilon_1}y_2^{\varepsilon_2}\ldots y_n^{\varepsilon_n}$ is 1, *i.e.* they are fundamental products. Furthermore, since there are 2^n different functions of the form $y_1^{\varepsilon_1}y_2^{\varepsilon_2}\ldots y_n^{\varepsilon_n}$, it can be concluded that every fundamental product can be expressed in this form. This remarkable property of linear functions will enable to introduce new concepts on the classification of switching functions in a later place.

Now we proceed to the next theorem.

Theorem 4.2: Let $\{y_1, y_2, \ldots, y_k\}$ be a base of an arbifrary subgroup $\mathbf{L}$ of $\mathbf{L}_n$, then, for every function f, we have

$$\sum_{z \in \mathbf{L}} \delta_z f_{y \oplus z} \equiv 2^{n-1} (\text{mod. } 2^k) \tag{4.5.1}$$

and

$$\left| \sum_{z \in \mathbf{L}} \delta_z f_{y \oplus z} \right| \leqq 2^{n-1}, \tag{4.5.2}$$

where $\delta_z = \pm 1$ and $\delta_{u \oplus z} = \delta_u \delta_z$ for any u and z such that $u, z \in \mathbf{L}$.

Proof: Putting $g=y_1^{\varepsilon_1}y_2^{\varepsilon_2}\ldots y_k^{\varepsilon_k}$ in (3.3) and using (4.4), we have

$$(f \oplus g)_y = f_y - \frac{1}{2^{k-1}} \sum_{z \in \mathbf{L}} \delta_z f_{y \oplus z}$$

where $\delta_0 = 1$, $\delta_{y_i} = -\varepsilon_i$, $i=1, 2, \ldots, k$ and $\delta_{u \oplus z} = \delta_u \delta_z$ for any u, $z \in \mathbf{L}$, because, for instance, for $z = y_1 \oplus y_2 \oplus \ldots \oplus y_h$ we have from (4.4.2)

$$-\delta_z = (-1)^{h+1} \varepsilon_1 \varepsilon_2 \ldots \varepsilon_h = -(-\varepsilon_1)(-\varepsilon_2)\ldots(-\varepsilon_h) = -\delta_{y_1}\delta_{y_2}\ldots\delta_{y_h}$$

i.e.

$$\delta_z = \delta_{y_1}\delta_{y_2}\ .\ .\ .\ \delta_{y_h}.$$

Inserting the above expression into (3.6), we have

$$(fg)_y = \frac{1}{2}\,(2^{n-1}\delta_{y0} + g_y + \frac{1}{2^{k-1}}\sum_{z\in\mathbf{L}} \delta_z f_{y\oplus z}).$$

First, we assume that $y \in \mathbf{L}$, then the above expression will be rewritten as

$$(fg)_y = \frac{1}{2}\,(2^{n-1}\delta_{y0} - 2^{n-k}\delta_y + \frac{1}{2^{k-1}}\sum_{z\in\mathbf{L}} \delta_z f_{y\oplus z})$$

i.e.

$$\sum_{z\in\mathbf{L}} \delta_z f_{y\oplus z} = 2^{n-1}\delta_y - 2^k\{2^{n-1}\delta_{y0} - (fg)_y\}.$$

Here $fg \leqq y^{-\delta_y}$ holds, because $g = y_1^{\varepsilon_1}y_2^{\varepsilon_2}\ .\ .\ .\ y_k^{\varepsilon_k} \leqq y^{-\delta_y}$. Hence, by (2.10), we have

$$2^{n-1}\delta_{y0} - (fg)_y = \delta_y 0(fg).$$

Therefore

$$\sum_{z\in\mathbf{L}} \delta_z f_{y\oplus z} = \delta_y\{2^{n-1} - 2^k 0(fg)\}. \tag{4.6}$$

Since $0 \leqq 0(fg) \leqq 2^{n-k}$ by (4.4.1), (4.5.1) and (4.5.2) are verified.

Next, assume that $y \notin \mathbf{L}$, then we have

$$(fg)_y = \frac{1}{2^k}\sum_{z\in\mathbf{L}} \delta_z f_{y\oplus z},$$

and consequently

$$\sum_{z\in\mathbf{L}} \delta_z f_{y\oplus z} = 2^k(fg)_y = 2^k\{0(fgy) - 0(fgy')\}. \tag{4.7}$$

Since $0 \leqq 0(fgy) \leqq 2^{n-k-1}$ and $0 \leqq 0(fgy') \leqq 2^{n-k-1}$ by (4.4.1) and since $k \leqq n-1$ because of the fact that there exists y such that $y \notin \mathbf{L}$, (4.5.1) and (4.5.2) are verified.

Corollary 4.2.1: For every switching function of 3 or more than 3 variables, either $f_y \equiv f_0$ (mod. 4) for every y or else the number of coordinates such that $f_y \equiv f_0$ (mod. 4) and that of coordinates such that $f_y \equiv f_0 + 2$ (mod. 4) are equal to each other.

Proof: Taking $\mathbf{L} = \{0, y, z, y \oplus z\}$ and $\delta_y = \delta_z = 1$ in (4.5.1), we have

$$f_0 + f_y + f_z + f_{y\oplus z} \equiv 0 \pmod{4} \tag{4.8}$$

because $2^{n-1} \equiv 0$ (mod. 4) for $n \geqq 3$. This can be satisfied if $f_y \equiv f_0$ (mod. 4) for every y. On the other hand, if $f_y \equiv f_0 + 2$ (mod. 4) for a certain y, then (4.8) will be rewritten as

$$f_z + f_{y\oplus z} = 2f_0 + 2 \pmod{4}.$$

Hence, if $f_z \equiv f_0$ (mod. 4), we have $f_{y \oplus z} \equiv f_0 + 2$ (mod. 4) and if $f_z \equiv f_0 + 2$ (mod. 4), we have $f_{y \oplus z} \equiv f_0$ (mod. 4). Since $\mathbf{L}_n$ can be partitioned into 2^{n-1} rest classes $\{z, y \oplus z\}$ by its subgroup $\{0, y\}$, the latter half of the corollary is proved.

Corollary 4.2.2: The sum of the absolute values of any two coordinates of a function never exceeds 2^{n-1}.

The proof is obvious from (4.5.2).

Now we proceed to the essential problem of characterizing the coordinates of switching functions.

Theorem 4.3; A necessary and sufficient condition so that an ordered 2^n-tuple $\{f_y\}$, $y \in \mathbf{L}_n$ may represent a switching function is that

$$\sum_{y \in \mathbf{L}_n} \delta_y f_y = \pm 2^{n-1} \tag{4.9}$$

for every possible combination of values of $\delta_y = \pm 1$ such that $\delta_{y \oplus z} = \delta_y \cdot \delta_z$.

Proof: The necessity is obvious by Theorem 4.2. The sufficiency is the principal part of the theorem and will be proved as follows. Evidently a sufficient condition for a vector $\mathbf{f} = \{f_y\}$ to represent a switching function is that the each component of the vector $T^*\mathbf{f}$ is $\pm 2^{n/2-1}$, where T is the transformation matrix given by (2.7) and T^* is its transpose. Since the column vector of T corresponding to a fundamental product p may be given by

$$\{2^{-n/2}(p_y - 2^{n-1}\delta_{y0})\},$$

the above mentioned condition will be rewritten as

$$2^{-n/2} \sum_{y \in \mathbf{L}_n} (p_y - 2^{n-1}\delta_{y0}) f_y = \pm 2^{n/2-1}.$$

By means of the fact that $p_0 = 2^{n-1} - 1$ and $p_y = -\delta_y$ and $p_z = -\delta_z$ imply $p_{y \oplus z} = -\delta_y \delta_z$, the condition will be reduced to (4.9).

Here it may be noted that (4.9) provides a means for the identification of the switching function f represented by the $\{f_y\}$, since $x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_n^{\varepsilon_n} \leqq f$ if and only if (4.9) gives the value -2^{n-1} for $\delta_{x_i} = -\varepsilon_i$, $i = 1, 2, \ldots, n$. From a practical view point, however, the above theorem is rather tedious, because 2^n trials, in general, will be needed to test whether a given 2^n-tuple represents a switching function or not. Accordingly some deformations of the theorem such that the quantity of effort necessary for the test may be reduced to a practically manageable extent, will be desirable. One of the solutions of the problem is given by the following theorem reducing the number of trials down to 2^{n-1}.

Theorem 4.4: A necessary and sufficient condition so that an ordered 2^n-tuple $\{f_y\}$, $y \in \mathbf{L}_n$ may represent a switching function is that either

$$\sum_{y \in \mathbf{L}_{n-1}} \delta_y f_y = \pm 2^{n-1} \text{ and } \sum_{y \in \mathbf{L}_{n-1}} \delta_y f_{y \oplus y_n} = 0, \tag{4.10.1}$$

or

$$\sum_{y \in \mathbf{L}_{n-1}} \delta_y f_y = 0 \text{ and } \sum_{y \in \mathbf{L}_{n-1}} \delta_y f_{y \oplus y_n} = \pm 2^{n-1} \tag{4.10.2}$$

for every possible combination of values of $\delta_y = \pm 1$ such that $\delta_{y\oplus z} = \delta_y \delta_z$, where $\{y_1, y_2, \ldots, y_n\}$ is a base of $\mathbf{L}_n$ and $\mathbf{L}_{n-1}$ is the subgroup of $\mathbf{L}_n$ determined by the base $\{y_1, y_2, \ldots, y_{n-1}\}$.

Proof: Since $\mathbf{L}_n$ can be partitioned into two rest classes $\mathbf{L}_{n-1}$ and $y_n \oplus \mathbf{L}_{n-1}$ the condition (4.9) is equivalent to

$$\sum_{y\in\mathbf{L}_n} \delta_y f_y = \sum_{y\in\mathbf{L}_{n-1}} \delta_y f_y + \sum_{y\in y_n\oplus\mathbf{L}_{n-1}} \delta_y f_y = \sum_{y\in\mathbf{L}_{n-1}} \delta_y f_y + \delta_{y_n} \sum_{y\in\mathbf{L}_{n-1}} \delta_y f_{y\oplus y_{n-1}} = \pm 2^{n-1},$$

whence the sufficiency is immediate.

The necessity is also proved easily by Theorem 4.2 because it tells us that

$$\sum_{y\in\mathbf{L}_{n-1}} \delta_y f_y = \pm 2^{n-1} \text{ or } 0,$$

and

$$\sum_{y\in\mathbf{L}_{n-1}} \delta_y f_{y\oplus y_{n-1}} = \pm 2^{n-1} \text{ or } 0.$$

Here it will be observed that the identification of the switching function f represented by the coordinates $\{f_y\}$ may be performed capitalizing on the fact that $y_1^{\varepsilon_1} y_2^{\varepsilon_2} \ldots y_n^{\varepsilon_n} \leqq f$ if and only if $\sum_{y\in\mathbf{L}_{n-1}} \delta_y f_y = -2^{n-1}$ or $\sum_{y\in\mathbf{L}_{n-1}} \delta_y f_y = 0$ and $\sum_{y\in\mathbf{L}_{n-1}} \delta_y f_{y\oplus y_n} = 2^{n-1}\varepsilon_n$ for $\delta_{y_i} = -\varepsilon_i$, $i = 1, 2, \ldots, n-1$.

5. Symmetric Transformations of Switching Functions

We now proceed to the consideration of the important transformations of switching functions caused by permutations and complementations of the independent variables $x_1, x_2, \ldots, x_n$. By a permutation and (or) a complementation of independent variables, a permutation of fundamental products will be induced, and this, in turn, induces a permutation of switching functions in a following natural way. That is: Let the permutation of fundamental products in question be denoted by a symbol σ, then any switching function f will be transformed into a function denoted by $\sigma(f)$ which is the sum of all the fundamental products $\sigma(p)$ such that $p \leqq f$. This transformation will play a fundamental role in the theory of synthesis of switching circuits and will be called as a symmetric transformation. As is well known, the totality of symmetric transformations forms a finite group $\mathbf{O}_n$ which may be called hyperoctahedral group, because it is isomorphic to the group of symmetries of n-dimensional hyperoctahedron. Since every symmetric transformation can be uniquely expressed as the product of a permutation operator π and a complementation operator γ in the form of $\gamma\pi$ (or $\pi\gamma$), and since there are $n!$ permutation operators and 2^n complementation operators, the order of $\mathbf{O}_n$ is $2^n n!$.

Now it will be observed that any symmetric transformation possesses the Boolean invariancy, *i.e.*, it preserves the orders of switching functions and all the Boolean operations and relation, *e.g.*,

$$0(\sigma(f)) = 0(f),$$

$$\sigma(f') = \{\sigma(f)\}',$$

$$\sigma(f+g) = \sigma(f) + \sigma(g),$$

$$\sigma(fg)=\sigma(f)\cdot\sigma(g),$$
$$\sigma(f\oplus g)=\sigma(f)\oplus\sigma(g),$$

and $\sigma(f)\geqq\sigma(g)$ if and only if $f\geqq g$.

By virtue of the Boolean invariancy of symmetric transformations, the following theorem is proved easily.

Theorem 5.1: Let f be a function and σ be a symmetric transformation, then we have

$$\sigma(f)_y=f_{\sigma^{-1}(y)}. \tag{5.1}$$

The theorem shows that the effects of a symmetric transformation on a function can be represented by a permutation and (or) changes of signs of its coordinates. Precisely, the effect of a permutation operator can be represented by a permutation of coordinates corresponding to linear functions of the same lengths, and that of a complementation operator by changes of signs of coordinates.

Here we shall introduce the concepts of "type" and "genus" which are of primary importance for the classification of switching functions. Their definitions are as follows.

"Two functions are of the same type or congruent if and only if they can be transformed from the one to the other by a symmetric transformation."

"Two functions are of the same genus if and only if they can be transformed from the one to the other by a symmetric transformation and (or) a complementation."

Thus the concept of genus is broader than that of type. In general a genus consists of two types, the one of which is the complement of the other. But there exist some genera consisting of only one type. Clearly every function belonging to such a genus (type) is congruent to its own complement and consequently is of the order 2^{n-1}. Functions (types, genera) with the above mentioned property will be called as self-complementary. Indeed, for $n\leqq 3$, every function (type, genus) of the order 2^{n-1} is self-complementary, but for $n\geqq 4$, there exist functions (types, genera) of the order 2^{n-1} which are not self-complementary.

Of course two functions f and g are of the same type (genus) if and only if there exists a symmetric transformation σ such that $f_{\sigma^{-1}(y)}=g_y(\varepsilon f_{\sigma^{-1}(y)}=g_y)$. But, for functions of odd orders, the above condition can be replaced with a weaker one. That is: Let f and g be functions of an odd order, then they are of the same genus if and only if there exists a permutation operator π such that $|f_{\pi^{-1}(y)}|=|g_y|$. As is readily seen, the essential part of the proof of the above fact is contained in the following theorem.

Theorem 5.2: Let f and g be functions of an odd order such that the absolute values of their corresponding coordinates are equal, then they are of the same genus.

Proof. Assume, first, that $n\leqq 2$, then every function of an odd order is either a fundamental product or its complement *i.e.* a fundamental sum. Therefore the theorem is valid because every fundamental product (sum) is of the one type.

Next, assume that $n\geqq 3$. To begin with, we shall show that any function of an odd order can be transformed into a certain standard form by a suitable complementation operator. Let f be transformed into $f^* = \gamma(f)$ such that $f^*_{x_i} \equiv f^*_0$ (mod. 4), $i = 1, 2, \ldots, n$ by a complementation operator γ. Now from Theorem 4.2, we have

$$f^*_0 + f^*_{x_i} + f^*_{x_j} + f^*_{x_i \oplus x_j} \equiv 0 \ (\text{mod. } 4),$$

which means that $f^*_y \equiv f^*_0$ (mod. 4) for every y of the length 2. Again from the same theorem, we have

$$f^*_0 + f^*_{x_k} + f^*_{x_i \oplus x_j} + f^*_{x_i \oplus x_j \oplus x_k} \equiv 0 \ (\text{mod. } 4),$$

which means that $f^*_y \equiv f^*_0$ (mod. 4) for every y of the length 3. Continuing in this way, we find that $f^*_y \equiv f^*_0$ (mod. 4) for every y. This function f^* may be called the standard form of f. Now, let g^* be the standard form of g, then clearly $f^*_y = g^*_y$ or $f^*_y = -g^*_y$, according as $f_0 = g_0$ or $f_0 = -g_0$. This completes the proof.

In connection with the above theorem, there arises a natural question: Is the similar proposition true as well for the case of functions of even orders? Unfortunately, the answer is "no", because a counter example is found in the case of $n=4$, as will be reported shortly.

Now let us turn to the descriptions of various symmetry structures of switching functions and their recognitions by the coordinate representation.

For a given function f, the set of all the symmetric transformations σ such that $\sigma(f) = f$ forms a group, which may be called as the symmetry group of f. A function whose symmetry group contains all the permutation operators is defined to be strictly symmetric and a function which is congruent to a strictly symmetric function to be symmetric. With these definitions, the following theorem is obvious by Theorem 5.1.

Theorem 5.3: A function can be symmetric only if its coordinates corresponding to linear functions of the same lengths have the same absolute values. Especially, a function is strictly symmetric if and only if its coordinates corresponding to linear functions of the same length have the same values.

Corollary 5.3: A function of an odd order is symmetric if its coordinates corresponding to linear functions of the same lengths have the same absolute values.

Proof. The proof is immediate, since, for $n \leqq 2$, every function of an odd order is symmetric and, for $n \geqq 3$, the standard form of a function satisfying the condition of the corollary is strictly symmetric.

A function which is invariant under the complementation of a certain single variable is said to be independent to it. The following theorem is evident from Theorem 5.1.

Theorem 5.4: A function is independent to a variable if and only if all the coordinates corresponding to linear functions containing the variable are 0.

A function which is transformed into its complement (itself) by the simultaneous complementations of all the variables is said to be self-dual (anti-self-dual). These properties are interesting in the sense that a self-dual (anti-self-dual) function is transformed into itself (its complement) by the interchange of multiplications and additions, provided that it is expressed by multiplications, additions and complementations only. The following theorem is also obvious from Theorem 5.1.

Theorem 5.5: A function is self-dual (anti-self-dual) if and only if its coordinates corresponding to inear functions of even (odd) lengths are all 0.

6. Reducible and Irreducible Switching Functions

A switching function is said to be reducible when its associated group contains at least one linear function other than 0. Let f be a reducible function and let $\{y_1, y_2, \ldots, y_k\}$ be a base of the associated group $\mathbf{L}_f$. When we assume that $f \leqq y_i^{\varepsilon_i}$, $i=1, 2, \ldots, k$, we have

$$f_y = 2^{n-1}\delta_{y0} - 0(f)\delta_y \tag{6.1}$$

for $y \in \mathbf{L}_f$ from (4.2), where $\delta_{y_i} = -\varepsilon_i$, $i=1, 2, \ldots, k$ and $\delta_{y \oplus z} = \delta_y \delta_z$ for any y, $z \in \mathbf{L}_f$.

Now let any $n-k$ variables which are independent to the base $\{y_1, y_2, \ldots, y_k\}$ be $x_1^*, x_2^*, \ldots, x_{n-k}^*$ and let the group determined by the base $\{x_1^*, x_2^*, \ldots, x_{n-k}^*\}$ be $\mathbf{L}^*$, then, since $\mathbf{L}_n$ is the direct sum of $\mathbf{L}_f$ and $\mathbf{L}^*$, any odd linear function y will be expressed uniquely in the form of

$$y = z \oplus y^*, \qquad z \in \mathbf{L}_f, \qquad y^* \in \mathbf{L}^*. \tag{6.2}$$

Using (6.2) in (4.1), we have

$$2^{n-1}\delta_{y0} - f_y = \delta_{yz}(2^{n-1}\delta_{y^*0} - f_{y^*}). \tag{6.3}$$

Therefore the knowledge of f_{y^*} for $y^* \in \mathbf{L}^*$ suffices to determine all the coordinates f_y for $y \notin \mathbf{L}_f$. When use is made of (6.2) and (6.3) in (4.9), we have

$$\sum_{y \in \mathbf{L}_n} \delta'_y f_y = 2^{n-1} - \sum_{z \in \mathbf{L}_f} \delta'_z \delta_z \sum_{y^* \in \mathbf{L}^*} \delta'_{y^*}(2^{n-1}\delta_{y^*0} - f_{y^*}) = \pm 2^{n-1},$$

because of $\delta'_y = \delta'_z \delta_{y^*}$. Taking $\delta'_z = \delta_z$ for every $z \in \mathbf{L}_f$, we have

$$2^{n-1} - 2^k \sum_{y^* \in \mathbf{L}^*} \delta'_{y^*}(2^{n-1}\delta_{y^*0} - f_{y^*}) = \pm 2^{n-1}.$$

Here let us put

$$2^{n-1}\delta_{y^*0} - f_{y^*} = 2^{n-k-1}\delta_{y^*0} - f^*_{y^*}, \tag{6.4}$$

then the above expression will become

$$\sum_{y^* \in \mathbf{L}^*} \delta'_{y^*} f^*_{y^*} = \pm 2^{n-k-1}.$$

Since $\delta'_{y^* \oplus z^*} = \delta'_{y^*}\delta'_{z^*}$ can be easily seen, we conclude that $\{f^*_{y^*}\}$, $y^* \in \mathbf{L}^*$ represents a function f^* of $n-k$ variables $x_1^*, x_2^*, \ldots, x_{n-k}^*$. Clearly f^* has the same order as f and is irreducible. Thus it has been shown that the study of reducible functions can be reduced in a sense to the study of irreducible functions with fewer variables.

In considering reducible functions, it is frequently convenient to assume that they are of the standard form, where a reducible function f is of the standard form if and only if $f \leqq y_i'$ for every odd linear function y_i of a base of its associated group $\mathbf{L}_f$. For such a f, (6.1) and (6.3) will become

$$f_y = 2^{n-1}\delta_{y0} - 0(f), \qquad y \in \mathbf{L}_f \tag{6.1'}$$

and

$$2^{n-1}\delta_{y0} - f_y = 2^{n-1}\delta_{y^*0} - f_{y^*}, \qquad y = z \oplus y^*, \qquad z \in \mathbf{L}_f, \qquad x^* \in \mathbf{L}^* \tag{6.3'}$$

respectively. Here it may be noted that this definition of the standard form is compatible with the previous one concerning functions of odd orders given in Section 5.

Now we shall show that every reducible function can be transformed into the standard form by a suitable complementation operator. In order to be able to transform a reducible function f such that $f \leqq y_i^{\varepsilon_i}$, $i = 1, 2, \ldots, k$ into the standard form, it has only to show the existence of a complementation operator transforming $\{y_1, y_2, \ldots, y_k\}$ into $\{y_1^{\bar{\varepsilon}_1}, y_2^{\bar{\varepsilon}_2}, \ldots, y_k^{\bar{\varepsilon}_k}\}$. In fact the following theorem holds.

Theorem 6.1: Let $\{y_1, y_2, \ldots, y_n\}$ be any base of $\mathbf{L}_n$, then there exists a unique complementation operator transforming $\{y_1, y_2, \ldots, y_n\}$ into $\{y_1^{\varepsilon_1}, y_2^{\varepsilon_2}, \ldots, y_n^{\varepsilon_n}\}$ for every possible $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$.

Proof: Let the fundamental product $x_1^{\varepsilon_1}x_2^{\varepsilon_2} \ldots x_n^{\varepsilon_n}$ be put in correspondence with the complementation operator priming only those variables x_i such that $\varepsilon_i = 1$. Thus there arises a one-to-one correspondence between fundamental products and complementation operators. Now it will be seen that in this correspondence the complementation operators which transform an odd linear function y into its complement y' correspond to fundamental products p such that $p \leqq y$. Therefore the complementation operator in question is given by that which corresponds to the fundamental product $y_1^{\bar{\varepsilon}_1}y_2^{\bar{\varepsilon}_2} \ldots y_n^{\bar{\varepsilon}_n}$.

7. Isomorphic and Linear Transformations of Switching Functions

In the early part of Section 4, we have found that every function of the form $y_1^{\varepsilon_1}y_2^{\varepsilon_2} \ldots y_n^{\varepsilon_n}$ is a fundamental product and conversely every fundamental product can be expressed in this form where $\{y_1, y_2, \ldots, y_n\}$ is any set of n independent odd linear functions. Hence a transformation $x_i \rightarrow y_i^{\varepsilon_i}$, $i = 1, 2, \ldots, n$ gives rise to a permutation of fundamental products. The extention of this permutation in the same sense as in the case of symmetric transformations to general switching functions will be defined as an isomorphic transformation. Clearly an isomorphic transformation has the Boolean invariancy. Any two functions are said to be isomorphic if and only if they can be transformed from the one to the other by an isomorphic transformation. Since an isomorphic transformation transforms linear functions into linear functions, its effects on the coordinates of a function can be given in the same form as in Theorem 5.1. That is: Let an isomorphic transformation be denoted by a symbol τ, then we have

$$\tau(f)_y = f_{\tau^{-1}(y)}. \tag{7.1}$$

The isomorphic transformation $x_i \to y_i^{\varepsilon_i}$ can be performed in two steps, $x_i \to y_i$ in the first step, and $y_i \to y_i^{\varepsilon_i}$ in the second step. The effects of the first step is a permutation and those of the second step are changes of signs of coordinates. Obviously an isomorphic transformation is a symmetric transformation if and only if it preserves the lengths of linear functions. Here we remember that there exists one more kind of transformation with the similar effects on the coordinates of functions. That is the linear additions of linear functions $f \to f \oplus z^{\varepsilon}$ (see (3.2)). For the sake of briefness, we call this as a linear transformation. A linear transformation is not an isomorphic transformation, because it has no Boolean invariancy.

In view of the simple characters of the coordinate representations of isomorphic and linear transformations, we introduce two more concepts, *i.e.* those of "prototype" and "family" for the classification of switching functions. Their definitions are as follows.

"Two functions are of the same prototype (family) if and only if they can be transformed from the one to the other by an isomorphic (symmetric) transformation and (or) a linear transformation".

Thus we have 4 concepts for the classification of switching functions, *i.e.*, type, genus, family and prototype. Comparing their definitions and taking account of the fact that the symmetric transformation and the complementation are the special cases of isomorphic transformations and linear transformations respectively, it will be observed that they form an increasing sequence in the broadness of concepts in the above listed order.

Concerning the significance of the concepts of type and genus, we have argued in Section 5. The significance of the concept of prototype consists in the fact that switching functions can be classified into a rather small number of prototypes. In fact, 256 (65,536) switching functions of 3 (4) variables are classified into only 3 (8) prototypes, as will be reported shortly.

On the other hand, the significance of the concept of family consists in the following theorem.

Theorem 7.1: The sets of the genera represented by the functions obtained from two given functions by linear transformations are equal if and only if they are of the same family.

Proof: "If" part; Let f and g be functions of the same family, then there are a linear function y^{δ} and a symmetric transformation σ such that $\sigma(f \oplus y^{\delta}) = g$. Therefore, for each linear function z^{ε}, we have $\sigma(f \oplus y^{\delta} \oplus \sigma^{-1}(z^{\varepsilon})) = g \oplus z^{\varepsilon}$. Since every linear function can be expressed in the form of $y^{\delta} \oplus \sigma^{-1}(z^{\varepsilon})$, the "if" part is proved.

"Only if" part; If we assume the conclusion of the theorem for two given functions f and g, then, for each linear function y^{δ}, there are a linear function z^{ε} and a symmetric transformation σ such that $\sigma(f \oplus y^{\delta}) = g \oplus z^{\varepsilon}$, *i.e.*, $\sigma(f \oplus y^{\delta} \oplus \sigma^{-1}(z^{\varepsilon})) = g$. This means that f and g are of the same family.

The above theorem shows that the process of the classification of switching functions into genera will be almost finished by the classification into families, since every genus will be represented by a function obtained fom an arbitrarily chosen representative of a certain only one family by a certain linear transformation. As a matter of fact, no serious difficulty is involved in the process of classifying switching functions into families in the case of 4 or less than 4 variables, as will be reported shortly.

Summary

A theory of the coordinate representation of switching functions was developed in a fully generel form. As we have seen, the method of the coordinate representation seems to be rather awkward for the practical manipulation of switching functions, but it possesses a peculiar potentiality for the study of the structure of switching functions. No applications are contained in this work, but some of them will be shortly reported by the author. As regards the originality of this work, the author owes the starting idea (Theorem 2.1) to D. E. Muller but the development thereafter is independent to his work.

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