BOUNDARY VALUE PROBLEMS FOR AN ELASTIC BODY WITH A PLANE CRACK

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(Received September 6, 1958)

1. Introduction

The elasticity problem of an infinite solid body with an ellipsoidal cavity under uniform stress at infinity was solved by Sadowsky and Sternberg[1] and its special case of an elliptic plane crack by Green and Sneddon.[2] In this paper, using the method of Fourier transforms, the similar problem is treated for the case when the shape of the plane crack is arbitrary and is reduced to finding potential functions with given values on the crack surface. Further, when the crack is elliptic, special solutions are given in explicit forms.

2. Fourier Transforms of the Equations

Let the elastic body occupy the whole space outside the crack and put:

x_j: rectangular coordinates,
u_j: components of displacement,
σ: Poisson's ratio,

$$\theta = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3},$$

then the equations of elasticity are

$$\frac{\partial \theta}{\partial x_i} + (1-2\sigma)\,\Delta u_i = 0. \tag{1}$$

We take the crack S on the $x_2 - x_3$ plane and apply the method of Fourier transforms.[3] Let $f(x_1, x_2, x_3)$ be a function outside S and put:

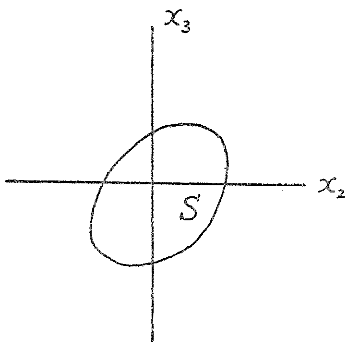

FIG. 1

$$\frac{1}{(2\pi)^{3/2}} \iiint_{-\infty}^{\infty} f(x_1, x_2, x_3)\, e^{i(\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3)}\, dx_1 dx_2 dx_3 = f'(\xi_1, \xi_2, \xi_3). \tag{2}$$

Then, under suitable assumptions on $f(x_1, '', '')$, we have:

$$\frac{1}{(\ \prime\prime\ )^{\prime\prime}}\iiint f'(\xi_1,\ \xi_2,\ \xi_3)\,e^{-i(\xi_1 x_1+\ \prime\prime\ +\ \prime\prime)}\,d\xi_1 d\xi_2 d\xi_3 = f(x_1,\ \prime\prime,\ \prime\prime), \tag{3}$$

$$\frac{1}{(\ \prime\prime\ )^{\prime\prime}}\iiint \frac{\partial f}{\partial x_j}\,e^{i(\xi_1 x_1+\ \prime\prime\ +\ \prime\prime)}\,dx_1 dx_2 dx_3 = -\,i\xi_j f'(\xi_1,\ \prime\prime,\ \prime\prime)$$

$$-\,\delta_{1j}\frac{1}{(\ \prime\prime\ )^{\prime\prime}}\iint_S \{f(+0,\ x_2,\ x_3)-f(-0,\ x_2,\ x_3)\}\,e^{i(\xi_2 x_2+\xi_3 x_3)}\,dx_2 dx_3, \tag{4}$$

$$\frac{1}{(\ \prime\prime\ )^{\prime\prime}}\iiint \Delta f\cdot e^{i(\ \prime\prime\ )}\,dx_1 dx_2 dx_3 = -\,(\xi_1^2+\xi_2^2+\xi_3^2)\cdot f'(\xi_1,\ \prime\prime,\ \prime\prime)$$

$$-\frac{1}{(\ \prime\prime\ )^{\prime\prime}}\iint_S \left[\left(\frac{\partial f}{\partial x_1}\right)^+ - \left(\ \prime\prime\ \right)^-\right] e^{i(\xi_2 x_2+\ \prime\prime)}\,dx_2 dx_3$$

$$+\,\frac{i\xi_1}{(\ \prime\prime\ )^{\prime\prime}}\iint_S (f^+ - f^-)\,e^{i(\ \prime\prime\ )}\,dx_2 dx_3. \tag{5}$$

We first assume that

$$u_1:\ \text{odd},\quad u_2,\ u_3,\ \theta:\ \text{even functions of } x_1, \tag{6}$$

and transform (1) and $\Delta\theta=0$ by using (4) and (5), then, taking (6) into account, we obtain:

$$\left.\begin{aligned} &-i\xi_1\theta' + (1-2\sigma)\{-(\xi_1^2+\xi_2^2+\xi_3^2)\,u_1' + i\xi_1 F(\xi_2,\ \xi_3)\} = 0,\\ &-i\xi_j\theta' + (\quad \prime\prime \quad)\{-(\qquad \prime\prime \qquad)\,u_j' - F_j\ (\ \ \prime\prime\ \ )\} = 0,\\ &\hspace{18em}(j=2,3)\\ &(\xi_1^2+\xi_2^2+\xi_3^2)\,\theta' = -(1-2\sigma)\,G, \end{aligned}\right\} \tag{7}$$

with

$$F(\xi_2,\ \xi_3) = \frac{1}{(\ \prime\prime\ )^{\prime\prime}}\iint_S (u_1^+ - u_1^-)\,e^{i(\xi_2 x_2+\ \prime\prime)}\,dx_2 dx_3,$$

$$F_j(\ \ \prime\prime\ \ ) = \frac{1}{(\ \prime\prime\ )^{\prime\prime}}\iint_S \left[\left(\frac{\partial u_j}{\partial x_1}\right)^+ - \left(\ \prime\prime\ \right)^-\right] e^{i(\ \prime\prime\ )}\,dx_2 dx_3,\ (j=2,3)$$

$$(1-2\sigma)\,G(\ \prime\prime\ ) = \frac{1}{(\ \prime\prime\ )^{\prime\prime}}\iint_S \left[\left(\frac{\partial \theta}{\partial x_1}\right)^+ - \left(\ \prime\prime\ \right)^-\right] e^{i(\ \prime\prime\ )}\,dx_2 dx_3.$$

We solve (7) and apply (3) and perform the integration with respect to ξ_1 by the method of complex integration, then we obtain the results:

$$\left.\begin{aligned} &u_j = \Omega_j + x_1\frac{\partial \Psi}{\partial x_j}, \qquad (j=1,2,3)\\ &\frac{\partial \Psi}{\partial x_1} = -\frac{1}{3-4\sigma}\left(\frac{\partial \Omega_1}{\partial x_1} + \frac{\partial \Omega_2}{\partial x_2} + \frac{\partial \Omega_3}{\partial x_3}\right), \end{aligned}\right\} \tag{8}$$

where

$$\Psi = \mp \frac{\pi}{2(\,''\,)''} \iint_{-\infty}^{\infty} \frac{G}{\rho^2} e^{-\rho|x_1| - i(\xi_2 x_2 + \,''\,)} d\xi_2 d\xi_3,$$

$$\Omega_1 = \pm \frac{\pi}{(\,''\,)''} \iint F e^{-\,''\,-\,''} d\xi_2 d\xi_3, \qquad (x_1 \gtrless 0)$$

$$\Omega_j = -\frac{\pi}{(\,''\,)''} \iint \frac{F_j}{\rho} e^{\,''} d\xi_2 d\xi_3$$

$$+ \frac{i\pi}{2(\,''\,)''} \iint \frac{G}{\rho^3} \xi_j e^{\,''} d\xi_2 d\xi_3, \qquad (j = 2, 3)$$

$$\rho = \sqrt{\xi_2^2 + \xi_3^2},$$

$$\Delta\Psi = \Delta\Omega_1 = \Delta\Omega_2 = \Delta\Omega_3 = 0. \qquad (9)$$

When we next assume that

u_1: even, u_2, u_3, θ: odd functions of x_1,

we obtain also the same results (8), hence (8) with (9) is the most general solution of (1) for the region outside S.

3. Boundary Value Problems

(a) When the components of displacement are given on the two sides of S, then

$$(\Omega_j)_{x_1 = \pm 0} = (u_j)_{\,''}, \qquad (j = 1, 2, 3)$$

are known on S and the potential functions Ω_1, Ω_2, Ω_3 can be determined uniquely. When specially $(u_2)_{\,''} = (u_3)_{\,''} = 0$, we put into (8)

$$\Omega_2 = \Omega_3 = 0, \; \Omega_1 = -(3 - 4\sigma)\Psi,$$

and we have:

$$u_j = \left\{ -(3 - 4\sigma)\delta_{1j} + x_1 \frac{\partial}{\partial x_j} \right\} \Psi, \qquad (j = 1, 2, 3)$$

with

$$\Delta\Psi = 0, \; (\Psi)_{x_1 = \pm 0} = -\frac{1}{3 - 4\sigma} (u_1)_{\,''}.$$

In the case of an elliptic crack:

$$\frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} = 1,$$

let

$$\frac{x_1^2}{\lambda} + \frac{x_2^2}{a^2 + \lambda} + \frac{x_3^2}{b^2 + \lambda} = 1, \qquad \lambda(x_1, x_2, x_3) > 0,$$

$$\Delta(s) = \sqrt{s(a^2 + s)\;(b^2 + s)}$$

and take

$$\Psi = \Psi^{(1)} = x_1 \int_\lambda^\infty \frac{ds}{s\Delta(s)}, \tag{10}$$

then

$$(u_1)_{x_1=\pm 0} = -(3-4\sigma)\,(\Psi^{(1)})_{\prime\prime} = \mp \frac{2(3-4\sigma)}{ab} \sqrt{1-\left(\frac{x_2^2}{a^2}+\frac{x_3^2}{b^2}\right)}.$$

The components of stress on the crack surface are given by:

$$\widehat{x_1x_1} = -4(1-\sigma)\mu\left(\frac{\partial\Psi^{(1)}}{\partial x_1}\right)_{\prime\prime} = 4(1-\sigma)\mu\int_0^\infty\left(\frac{1}{ab}-\frac{1}{\sqrt{(a^2+s)\,(b^2+s)}}\right)\frac{ds}{s^{3/2}} = \text{const.},$$

$$\widehat{x_1x_j} = -2(1-2\sigma)\mu\left(\frac{\partial\Psi^{(1)}}{\partial x_j}\right)_{\prime\prime}$$

$$= \mp\frac{4(1-2\sigma)}{ab}\mu\frac{\partial}{\partial x_j}\sqrt{\prime\prime}, \qquad (j=2,3)$$

with μ: rigidity.

This is one particular solution. Other simple solutions are given by taking for $\Psi^{1)}$

$$x_1x_2\int_\lambda^\infty\frac{ds}{s(a^2+s)\Delta(s)},\quad x_1x_3\int_\lambda^\infty\frac{ds}{s(b^2+s)\Delta(s)},\quad x_1x_2x_3\int_\lambda^\infty\frac{ds}{(\Delta(s))^3},$$

$$x_1\left(\frac{x_1^2}{\theta^{(i)}}+\frac{x_2^2}{a^2+\theta^{(i)}}+\frac{x_3^2}{b^2+\theta^{(i)}}-1\right)\int_\lambda^\infty\frac{ds}{s(s-\theta^{(i)})^2\Delta(s)} \qquad (i=1,2),$$

where $\theta^{(i)}$'s are roots of

$$5\,\theta^2+4(a^2+b^2)\theta+3\,a^2b^2=0.$$

The values of u_1 on the crack are given, respectively, by:

$$\mp\frac{2(3-4\sigma)}{a^3b}x_2\sqrt{1-\left(\frac{x_2^2}{a^2}+\frac{x_3^2}{b^2}\right)},\quad \mp\frac{2(3-4\sigma)}{ab^3}x_3\sqrt{\prime\prime},\quad \mp\frac{2(\prime\prime)}{a^3b^3}x_2x_3\sqrt{\prime\prime},$$

$$\mp\frac{2(\prime\prime)}{ab(\theta^{(i)})^2}\left(\frac{x_2^2}{a^2+\theta^{(i)}}+\frac{x_3^2}{b^2+\theta^{(i)}}-1\right)\sqrt{\prime\prime}.$$

These solutions cannot be adopted separately, because they take positive and negative values on one side of the crack (this is impossible physically), and must be combined linearly with (10) to make $(u_1)_{x_1=+0}$ positive.

(b) When

u_1: odd, u_2, u_3: even functions of x_1

and the shearing stress components $\widehat{x_1x_2}$, $\widehat{x_1x_3}$ vanish on S, then we have from (8):

$$\left(\frac{\partial u_1}{\partial x_j}+\frac{\partial u_j}{\partial x_1}\right)_{x_1=0} = \left(\frac{\partial\Omega_1}{\partial x_j}+\frac{\partial\Omega_j}{\partial x_1}+\frac{\partial\Psi}{\partial x_j}\right)_{\prime\prime} = (P_j)_{\prime\prime}=0. \qquad (j=2,3)$$

Here the potential functions P_2 and P_3 vanish on the whole plane $x_1 = 0$, and hence vanish identically in the whole space. Putting the odd function of x_1

$$\Omega_1 + \Psi = (1-2\sigma)\frac{\partial Q}{\partial x_1},$$

we obtain from (8) the results:

$$\left.\begin{aligned} u_1 &= \left\{\ 2(1-\sigma) - x_1\frac{\partial}{\partial x_1}\right\}\frac{\partial Q}{\partial x_1}, \\ u_j &= \left\{\ -(1-2\sigma) - x_1\frac{\partial}{\partial x_1}\ \right\}\frac{\partial Q}{\partial x_j}, \\ &\qquad\qquad\qquad\qquad (j = 2, 3) \\ \Delta Q &= 0. \end{aligned}\right\} \tag{11}$$

(b 1) When u_1 is given on S:

$$\left(\frac{\partial Q}{\partial x_1}\right)_{x_1=+0} = \frac{1}{2(1-\sigma)}\,(u_1)_{\prime\prime} = \alpha(x_2,\ x_3),$$

we have:

$$Q = -\frac{1}{2\pi}\iint_S \frac{\alpha(\xi_2,\ \xi_3)}{r}\,d\xi_2 d\xi_3.$$

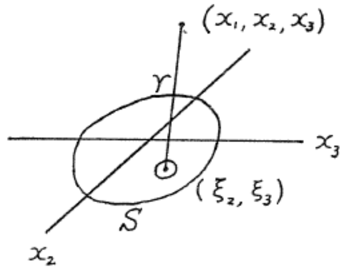

FIG. 2

(b 2) When the normal pressure p is preassigned, we have on S:

$$\left(\frac{\partial^2 Q}{\partial x_1^2}\right)_{x_1=0} = \frac{1}{2\mu}\widehat{x_1 x_1} = -\frac{1}{2\mu}p(x_2,\ x_3).$$

When the crack is elliptic and we take, for example,

$$Q = -\frac{1}{2}\int_\lambda^\infty \left(\frac{x_1^2}{s} + \frac{x_2^2}{a^2+s} + \frac{x_3^2}{b^2+s} - 1\right)\frac{ds}{\Delta(s)},$$

then

$$\frac{\partial Q}{\partial x_1} = -\Psi^{(1)},$$

where $\Psi^{(1)}$ is given by (10) and from (11) we have:

$$u_1 = \left\{-2(1-\sigma) + x_1\frac{\partial}{\partial x_1}\right\}\Psi^{(1)},$$

$$u_j = (1-2\sigma)x_j\int_\lambda^\infty \frac{ds}{(a^2+s)\,\Delta(s)} + x_1\frac{\partial \Psi^{(1)}}{\partial x_j}. \qquad (j = 2, 3)$$

The residual boundary condition for this solution is either

$$(u_1)_{x_1=\pm 0} = \mp \frac{4(1-\sigma)}{ab}\sqrt{1-\left(\frac{x_2^2}{a^2}+\frac{x_3^2}{b^2}\right)}$$

or

$$p(x_2,\ x_3) = -2\mu\left(\frac{\partial^2 Q}{\partial x_1^2}\right)_{x_1=0}$$
$$= -2\mu\int_0^\infty\left(\frac{1}{ab} - \frac{1}{\sqrt{(a^2+s)\ (b^2+s)}}\right)\frac{ds}{s^{3/2}}$$
$$= \text{const.}$$

This is the same solution as offered in the paper of Green and Sneddon.

References

1) M. A. Sadowsky and E. Sternberg, Journ. Appl. Mech., **16** (1949), p. 149-157.
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