A NOTE ON MEAN STRESS THEORY

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1. Introduction

It is well known that the ordinary theory of elastisity is available only for uniform and continuous bodies. But these conditions will not be satisfied in actual materials from microscopic view, because steel, for example, is composed of a number of cristals, and cast iron has many cavities or pores. Hence the usual theory will be applicable for some limitted problems.

In this paper, the author introduced an idea of mean stress, i.e., average stress value within a sphere of radius ρ, around the point considerd. Then we have the stress distribution that depends on not only the shape of the body but the ratio of ρ to its dimension. If we take the magnitude of ρ as a material specific constant, the degree of stress concentration will depend on its properties. It can be expected that the value of ρ of cast iron will be larger than that of steel, in order to explain various experimental data.

In the second section, we describe the basic relations of stresses, and offer new Hooke's law with regard to mean stresses. Especially two dimensional theory is treated in the third section, and its application to simple problems of stress concentration is given in the fourth section.

2. Fundamental Equations

Now we define the mean stess N as an average value of usual stress σ within a sphere of radius ρ, around the point considerd (x, y, z).

Then we can express the following formula,

$$N=\iiint_R \sigma dx_1 dy_1 dz_1 \Big/ \frac{4}{3}\pi\rho^3$$

where R is the region of integration, the sphere above mentioned.

If σ can be developed in Taylor series at the point,

$$\sigma=(\sigma)_{x,y,z}+\xi\left(\frac{\partial\sigma}{\partial x}\right)_{x,y,z}+\eta\left(\frac{\partial\sigma}{\partial y}\right)_{x,y,z}+\zeta\left(\frac{\partial\sigma}{\partial z}\right)_{x,y,z}+\cdots$$

where

$$\xi=x_1-x, \qquad \eta=y_1-y, \qquad \zeta=z_1-z,$$

we arrive to

$$N=\sigma+\frac{\rho^2}{10}\triangle\sigma+\frac{\rho^4}{280}\triangle\triangle\sigma+\cdots, \qquad \triangle\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}.$$

As the equilibrium equation of local stress are well known in the next form,

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_z}{\partial y}+\frac{\partial \tau_y}{\partial z}=0, \quad \text{etc.}$$

integrating these above the sphere, we can obtain the following expressions.

$$\frac{\partial N_x}{\partial x}+\frac{\partial S_z}{\partial y}+\frac{\partial S_y}{\partial z}=0, \qquad \frac{\partial S_z}{\partial x}+\frac{\partial N_y}{\partial y}+\frac{\partial S_x}{\partial z}=0, \qquad \frac{\partial S_y}{\partial x}+\frac{\partial S_x}{\partial y}+\frac{\partial N_z}{\partial z}=0.$$

where N_x, N_y, N_z are normal mean stresses, and S_x, S_y, S_z are tangential mean stresses.

Hereupon we assume the following form of Hooke's law

$$\frac{\partial u}{\partial x}=\frac{1}{E}(N_x-\nu N_y-\nu N_z), \quad \text{etc,} \qquad \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}=\frac{1}{G}S_x, \quad \text{etc.}$$

where u, v, w, are components of displacement.

Then the problem is reduced to find the solution of these equations which should satisfy the given boundary conditions. For instance, normal and tangential local stresses will be given at the boundary.

When plane stress problem is considerd, it is suitable that mean stresses within a circle instead of the sphere are used. Then we can write

$$N=\sigma+\frac{\rho^2}{8}\Delta\sigma+\frac{\rho^4}{192}\Delta\Delta\sigma+\ .\ .\ .\ , \qquad \Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

3. Two dimensional problems

For plane stress problems, using the Airy's stress function F

$$N_x=\frac{\partial^2 F}{\partial y^2}, \qquad N_y=\frac{\partial^2 F}{\partial x^2}, \qquad S_z=-\frac{\partial^2 F}{\partial x \partial y},$$

and elliminating u, v from the equations, we obtain the fundamental equatin of F,

$$\Delta\Delta F=0.$$

Since stress function for local stress can be also defined by the following relation,

$$\sigma_x=\frac{\partial^2 f}{\partial y^2}, \qquad \sigma_y=\frac{\partial^2 f}{\partial x^2}, \qquad \tau_2=-\frac{\partial^2 f}{\partial x \partial y},$$

f will be related to F by the equation,

$$F=f+\frac{\rho^2}{8}\Delta f, \quad \text{or} \quad f=F-\frac{\rho^2}{8}\Delta F.$$

4. Examples

a) *The effect of a circular hole on stress distribution in an infinite plate*

An infinite plate is submitted to an uniform tension of magnitude T in the x direction. If a circular hole is made in the plate, the stress distribution in the neighborhood of the hole will be changed, and its stress concentration is quite important in practice.

After the ordinary theory, we obtain

$$F = A_0 r^2 + B_0 \log r + (A_1 r^2 + B_1 r^4 + C^1/r^2 + D_1)\cos 2\theta$$

where r, θ are components of polar coordinate, and its origin is taken at the centre of the hole. Then we have stress components

$$N_r = 2A_0 + B_0/r^2 - (2A_1 + 6C_1/r^4 + 4D_1/r^2)\cos 2\theta$$
$$N_\theta = 2A_0 - B_0/r^2 + (2A_1 + 12B_1 r^2 + 6C_1/r^4)\cos 2\theta$$
$$S = (2A_1 + 6B_1 r^2 - 6C_1/r^4 - 2D_1/r^2)\sin 2\theta$$

As boundary conditions at $r \to \infty$ are

$$N_r = T\cos^2\theta, \qquad S = -T\sin\theta\cos\theta,$$

we obtain

$$A_0 = T/4, \quad A_1 = T/4, \quad B_1 = 0.$$

And considering the conditions at the edge of the hole ($r=a$) we can determine other constants, i.e.,

$$B_0 = -\frac{T}{2}a^2, \qquad D_1 = \frac{T}{2}a^2, \qquad C_1 = -\frac{T}{4}\left(1+\frac{\rho^2}{a^2}\right)a^4.$$

Since mean stresses are significant in this case only at $r > a+\rho$, we must discuss the magnitude of stress concentration at $r=a+\rho$. Then it can be seen that N_θ becomes the greatest when $\theta=\pi/2$ or $3\pi/2$.

At these points

$$(N_\theta)_{\left(\substack{\theta=2/\pi \\ r=a+\rho}\right.} = T\frac{3+5\lambda+8\lambda^2+4\lambda^3+\lambda^4}{(1+\lambda)^4} = kT$$

where $\lambda = \rho/a$.

When $\lambda \to 0$, the results coincide with the ordinary theory, and decrease as λ increases. This aspect can be seen easily in the table below.

λ	0	0.1	0.2	0.5	1.0	∞
k	3	2.448	2.100	1.593	1.313	1

b) Rotating disks

The strength of rotating circular disks is of great practical inportance. If the thickness of the disk is small in comparison with its radius, the problem can be solved by two dimensional analysis.

In this case equations of equilibrium are

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_z}{\partial y} + \gamma\omega^2 x = 0, \qquad \frac{\partial\tau_z}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \gamma\omega^2 y = 0,$$

where γ=specific gravity, ω=angular velocity.

The local stress function are defined by

$$\sigma_x = \frac{\partial^2 f}{\partial y^2} - \frac{\gamma\omega^2}{2}r^2, \qquad \sigma_y = \frac{\partial^2 f}{\partial x^2} - \frac{\gamma\omega^2}{2}r^2, \qquad \tau_2 = -\frac{\partial^2 f}{\partial x \partial y},$$

then mean stresses become

$$N_x=\frac{\partial^2 f}{\partial y^2}+\frac{\rho^2}{8}\Delta\frac{\partial^2 f}{\partial y^2}+\ldots-\frac{\gamma\omega^2}{2}r^2-\frac{\rho^2\gamma\omega^2}{4},\qquad N_y=\frac{\partial^2 f}{\partial x^2}+\frac{\rho^2}{8}\Delta\frac{\partial^2 f}{\partial x^2}+\ldots$$

$$-\frac{\gamma\omega^2}{2}r^2-\frac{\rho^2\gamma\omega^2}{4},\qquad S_z=-\frac{\partial^2 f}{\partial x\partial y}-\frac{\rho^2}{8}\Delta\frac{\partial^2 f}{\partial x\partial y}-\ldots,$$

And we take the mean stress function as

$$N_x=\frac{\partial^2 F}{\partial y^2}-\frac{\gamma\omega^2}{2}r^2-\frac{\rho^2\gamma\omega^2}{4},\qquad N_y=\frac{\partial^2 F}{\partial x^2}-\frac{\gamma\omega^2 r^2}{2}-\frac{\rho^2\gamma\omega^2}{4},\qquad S_2=-\frac{\partial^2 F}{\partial x\partial y},$$

now

$$F=f+\frac{\rho^2}{8}\Delta f+\ldots.$$

Finally we obtain the equation of F,

$$\Delta\Delta F=2(1-\nu)\gamma\omega^2.$$

This can be easily solved in this case,

$$F=A\log r+Br^2\log r+Cr^2+D+\frac{(1-\nu)\gamma\omega^2}{32}r^4.$$

Considering the boundary conditions $\sigma_r=0$ at $r=a$, $r=b$, where a and b are inner and outer radius respectively, we can determine integration constants.

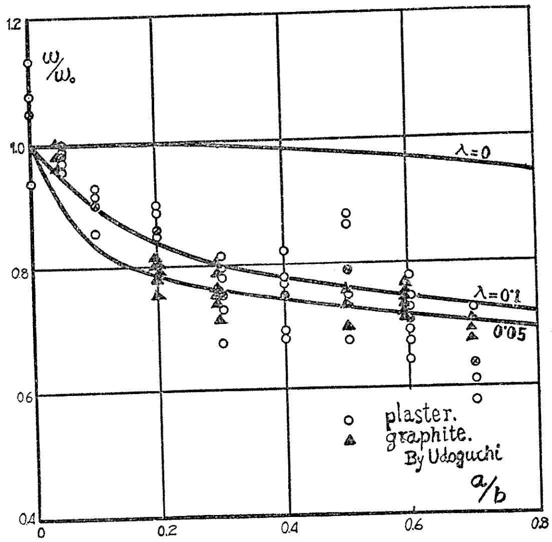
The tangential stress at $r=a+\rho$ becomes,

$$(N_0)_{a+\rho}=\frac{3+\nu}{8}\gamma\omega^2 b^2\left[1+\alpha^2-\frac{1+\nu}{3+\nu}\lambda^2-\frac{1+3\nu}{3+\nu}(\lambda+\alpha)^2+\frac{\alpha^2}{(\lambda+\alpha)^2}\right]$$

where

$$\alpha=a/b\qquad \lambda=\rho/b$$

Apparently $(N_0)_{a+\rho}$ is a function of α, λ, some numerical results are shown in the figure below, on which the ordinate ω/ω_0 is critical angular velocity ratio when the maximum mean stress approaches that of the disk $\alpha=0$. Experimental data



by Prof. Udoguchi (1) are also plotted.

It will be noted that the theoretical curves for $\lambda=0.05$, 0.1 give good agreement with the experiment.

5. Conclusion

Being introduced the concept of mean stress, degree of stress concentration becomes to depend on material constant ρ, and in this paper two simple examples are treated. Hence we can solve various problems in the almost same way as the ordinary theory. And the author believes that many interesting phenomena are able to be explained by the mean stress theory; for instance, rupture of brittle materials, yielding of mild steel, fatigue limit of a grooved test piece and so on.

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Reference

1) T. Udoguchi: Jap. Soc. Mech. Eng. Vol. **55**, No. 402, p. 474-480; Science of Machine Vol. 5, No. 1, '53-Jan. p. 11-18.