RESEARCH REPORTS

A MATHEMATICAL TREATMENT ON THE INFLUENCE OF THE MOVING MEDIA ON A VIBRATING SYSTEM

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Introduction

When an elastic system vibrates in a medium, the vibration decays because of the emission of the sound wave into the medium as well as the dissipation of energy due to the friction in the system.

This damping effect of the medium was treated mathematically by Sakadi in a previous paper[1]. This paper deals with the influence of the moving medium on a vibrating system. The elucidation of this problem is very interesting from theoretical as well as practical point of view. However the mathematical treatment for it is very difficult at least in the most cases of practical importance. Accordingly we will confine ourselves to the following ideal cases,

1. Semi-infinite elastic body
2. Infinitely stretched membrane
3. Spherical elastic body.

Except in the last case, the media will be assumed to flow with a uniform velocity in the static state. Principal notations used are as follows:

for media

p : Pressure
ρ : Density
γ : Ratio of specific heats
ν : Coefficient of dynamical viscosity
p_{xx}, p_{xy}, p_{yy} etc.: Stress components
U: Uniform velocity
u, v, w: Small velocity components

for elastic body

σ : Density (voluminal or aerial)
λ, μ: Lamé's constants
σ_{xx}, σ_{xy} etc: Stress components
T: Tension of membrane
ξ, η, ζ: Small displacement components.

1. Semi-infinite Elastic Body

Assume that a semi-infinite elastic body and a gaseous medium occupy the regions $y<0$ and $y>0$ respectively, and the gas flows in the direction of x-axis with a uniform velocity U in the static state. The vibration is assumed to be two dimensional i.e., independent of z-coordinate.

The equations of the vibration of the elastic body are

$$\begin{cases} \sigma \dfrac{\partial^2 \xi}{\partial t^2} = (\lambda+\mu)\dfrac{\partial}{\partial x}\left(\dfrac{\partial \xi}{\partial x} + \dfrac{\partial \eta}{\partial y}\right) + \mu\Delta\xi \\ \sigma \dfrac{\partial^2 \eta}{\partial t^2} = (\lambda+\mu)\dfrac{\partial}{\partial y}\left(\dfrac{\partial \xi}{\partial x} + \dfrac{\partial \eta}{\partial y}\right) + \mu\Delta\eta. \end{cases} \tag{1}$$

If we put

$$\begin{cases} \xi = \dfrac{\partial \Phi}{\partial x} + \dfrac{\partial \Psi}{\partial y} \\ \eta = \dfrac{\partial \Phi}{\partial y} - \dfrac{\partial \Psi}{\partial x} \end{cases} \tag{2}$$

then the equations for Φ and Ψ are

$$\Delta\left(\frac{\partial^2}{\partial t^2} - a^2\Delta\right)\Phi = 0 \tag{3}$$

and

$$\Delta\left(\frac{\partial^2}{\partial t^2} - b^2\Delta\right)\Psi = 0 \tag{4}$$

where

$$a = \sqrt{\frac{\lambda+2\mu}{\sigma}}$$

and

$$b = \sqrt{\frac{\mu}{\sigma}}.$$

Let us take

$$\Phi = Ae^{i(\omega t - kx) + kry} \tag{5}$$

and

$$\Psi = Be^{i(\omega t - kx) + ksy} \tag{6}$$

where k can be taken as positive without loss of generality. Inserting (5) and (6) into (3) and (4) respectively,
we have

$$r^2 = 1 - \frac{\omega^2}{k^2a^2} \tag{7}$$

and

$$s^2 = 1 - \frac{\omega^2}{k^2b^2}. \tag{8}$$

Out of two squre roots of (7) and (8), those which have positive real parts must be chosen so as to make Φ and Ψ vanish at

$$y = -\infty.$$

From (2), (5) and (6), displacement components are given by

$$\begin{cases} \xi = (-ikAe^{kry} + ksBe^{ksy})e^{i(\omega t - kx)} \\ \eta = (krAe^{kry} + ikBe^{ksy})e^{i(\omega t - kx)}. \end{cases} \tag{9}$$

Stress components which are necessary for the subsequent analysis are

$$\begin{cases} \sigma_{xy} = \mu\left(\dfrac{\partial \eta}{\partial x} + \dfrac{\partial \xi}{\partial y}\right) = 2k^2\mu\left\{-irAe^{kry} + \left(1 - \dfrac{\omega^2}{2k^2b^2}\right)Be^{ksy}\right\}e^{i(\omega t - kx)} \\ \sigma_{yy} = -p_0 + \lambda\left(\dfrac{\partial \xi}{\partial x} + \dfrac{\partial \eta}{\partial y}\right) + 2\mu\dfrac{\partial \eta}{\partial y} \end{cases} \tag{10}$$

$$= -p_0 + 2k^2\mu\left\{\left(1-\frac{\omega^2}{2\,k^2b^2}\right)Ae^{kry} + isBe^{ksy}\right\}e^{i(\omega t-kx)}$$

where p_0 is a uniform pressure in the static state.

On the other hand, the linearized equations for the gas are

$$\rho_0\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)u = -\frac{\partial p}{\partial x} + \frac{\rho_0\nu}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \rho_0\nu\Delta u \tag{11}$$

$$\rho_0\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)v = -\frac{\partial p}{\partial y} + \frac{\rho_0\nu}{3}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \rho_0\nu\Delta v \tag{12}$$

$$\frac{1}{c^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\frac{p}{\rho_0}\right) + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{13}$$

where, c is the velocity of sound in the static state.

Now we assume that the velocity components u and v can be derived from a scalar potential φ i.e.

$$u = \frac{\partial\varphi}{\partial x}, \qquad v = \frac{\partial\varphi}{\partial y}. \tag{14}$$

In connection with this assumption, we assume that the gas does not adhere to the surface of the elastic body but slides along it.

From (11), (12), (13) and (14), we obtain

$$\frac{p-p_0}{\rho_0} = -\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \frac{4\nu}{3}\Delta\right)\varphi \tag{15}$$

and

$$\frac{1}{c^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\frac{p}{\rho_0}\right) = -\Delta\varphi. \tag{16}$$

Elimination of p from (15) and (16) leads to the following equation for φ

$$\frac{1}{c^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \frac{4\nu}{3}\Delta\right)\varphi = \Delta\varphi. \tag{17}$$

In accordance with (5) and (6), we take

$$\varphi = Ce^{i(\omega t-kx)-kqy}. \tag{18}$$

Substituting (18) into (17), we get

$$q^2 = 1 - \frac{\frac{1}{c^2k}(\omega - kU)^2}{1 + \frac{4i\nu}{3c^2}(\omega - kU)} \tag{19}$$

where we must take $\Re(q) > 0$ so as to make φ vanish at $y = \infty$.

The velocity components and the stress components are given by

$$\begin{cases} u = -ikCe^{i(\omega t-kx)-kqy} \\ v = -kqCe^{i(\omega t-kx)-kqy} \end{cases} \tag{20}$$

$$\begin{cases} p_{xy} = \rho_0 \nu \left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} \right) = 2\, i \rho_0 \nu k^2 q C e^{i(\omega t - kx) - kqy} \\ p_{yy} = -p - \dfrac{2}{3} \rho_0 \nu \left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} \right) + 2\, \rho_0 \nu \dfrac{\partial v}{\partial y} \\ \qquad = -p_0 + \rho_0 \left\{ i(\omega - kU) + 2\, \nu k^2 \right\} C e^{i(\omega t - kx) - kqy}. \end{cases} \tag{21}$$

Boundary conditions on the surface of the elastic body $y=0$ are

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + U \dfrac{\partial}{\partial x} \right) \eta = v \\ \sigma_{xy} = p_{xy} \\ \sigma_{yy} = p_{yy}. \end{cases} \tag{22}$$

Inserting (9), (10), (20) and (21) into (22), and eliminating A, B and C, we obtain the equation for ω

$$q \left\{ \left(1 - \frac{\omega^2}{2\, k^2 b^2} \right)^2 - rs \right\} - i \frac{\rho_0 \nu}{\mu} q (\omega - kU) \left(1 - \frac{\omega^2}{2\, k^2 b^2} - rs \right) + \frac{\rho_0 r}{4\, \mu k^2} \frac{\omega^2}{k^2 b^2} (\omega - kU) \left\{ (\omega - kU) - 2\, i \nu k^2 \right\} = 0. \tag{23}$$

In order to rewrite this in a convenient non-dimensional form, we introduce the following notations.

$$\frac{\omega}{kb} = \zeta, \quad \frac{\rho_0}{\sigma} = \varepsilon, \quad \frac{4\, k\nu}{3\, b} = \delta, \quad \frac{U}{b} = m, \quad \frac{b}{c} = n, \quad e = \frac{b^2}{a^2} < \frac{1}{2}$$

then (23), (7), (8) and (19) are transformed into

$$q \left\{ \left(1 - \frac{\zeta^2}{2} \right)^2 - rs \right\} - \frac{3 i \varepsilon \delta}{4} q (\zeta - m) \left\{ \left(1 - \frac{\zeta^2}{2} \right) - rs \right\} + \frac{\varepsilon}{4} \zeta^2 r (\zeta - m) \left\{ (\zeta - m) - \frac{3}{2} i \delta \right\} = 0 \tag{23'}$$

$$r^2 = 1 - e\zeta^2 \tag{7'}$$

$$s^2 = 1 - \zeta^2 \tag{8'}$$

and

$$q^2 = 1 - \frac{n^2 (\zeta - m)^2}{1 + i n^2 \delta (\zeta - m)} \tag{9'}$$

respectively.

Here we assume that ε and δ are small numbers of the same order. Then the roots of (23′) are nearly equal to those of the equation

$$f(\zeta) = \left(1 - \frac{\zeta^2}{2} \right)^2 - rs = 0 \tag{24}$$

or to those of the equation

$$q = 0. \tag{25}$$

These equations correspond to the free vibrations of the elastic body (Rayleigh's wave)[2] and the gas respectively.

For the purpose of this paper we may omit the discussion of the latter roots.

As is easily seen, the equation (24) has always two real roots with the same absolute value ζ_0 depending upon the parameter e contained in r and, ζ_0 decreases continually from a certain value smaller than and very near to unity to $\sqrt{3-\sqrt{5}}$, as e increases from 0 to $\frac{1}{2}$. We can see also that f' (ζ_0) is always positive.

The velocity of propagation of Rayleigh wave is given by

$$v_0 = \zeta_0 b. \tag{26}$$

Considering U i.e. m variable as for its sign as well as its magnitude, we may omit the discussion of the root of (23) which is nearly equal to $-\zeta_0$ and put

$$\zeta = \zeta_0 + \zeta' \tag{27}$$

where ζ' is a small number of the order of ε.

Neglecting small quantities of higher order, the first approximation for ζ' is

$$\zeta' = -\frac{\varepsilon\zeta_0^2 r_0(\zeta_0 - m)^2}{4f'(\zeta_0)q_0} \tag{28}$$

where
$$r_0 = \sqrt{1-e\zeta_0^2} > 0 \tag{29}$$

and
$$q_0^2 = 1 - n^2(\zeta_0 - m)^2. \tag{30}$$

Here comes a ramification into existence, namely $q_0>0$, $q_0^2<0$ and $q_0^2=0$

1° $q_0^2>0$, i.e. $v_0 - c < U < v_0 + c$.

In this case, q_0 is a real number. The postulation $\Re(q)>0$ can be secured by taking
$$q_0 = q_1 = \sqrt{1-n^2(\zeta_0-m)^2} > 0. \tag{31}$$

It can be observed from (28) that the frequency of the vibration decreases from that of free vibration, and the stability of the vibration is almost neutral. Further approximation to the second order reveals us that $\Im(\zeta')$ i.e. $\Im(\zeta)$, being a small quantity of the order of $\delta\varepsilon$, is positive or negative according as $U<v_0$ or $U>v_0$. This means that the vibration is slightly damped or slightly excited according as the uniform velicity U is smaller or greater than v_0.

2^0 $q_0^2<0$, i.e. $U<v_0-c$ or $U>v_0+c$.

In this case q_0 is purely imaginary.

Precisely
$$q_0 = \pm iq_2 = \pm i\sqrt{n^2(\zeta_0-m)^2-1}. \tag{32}$$

In contrast to the preceeding case, it is not evident a priori which of the two alternatives $q_0 = \pm iq_2$ can satisfy $\Re(q)>0$.

From (32), (28), and (7), we have

$$\zeta' = \pm i\frac{\varepsilon\zeta_0^2 r_0(\zeta_0 - m)^2}{4f'(\zeta_0)q_2} \tag{33}$$

and
$$q = \pm iq_2 + \frac{n^2(\zeta_0 - m)^3}{2q_2}\left(\pm n^2\delta - \frac{\varepsilon\zeta_0^2 r_0}{2f'(\zeta_0)q_2}\right) \tag{34}$$

Hence it can be seen that, for $U>v_0+c$, i.e. $\zeta_0-m<0$ $-$ sign certainly satisfies $\Re(q)>0$, and under certain circumstances $+$ sign does too, but, for $U<v_0-c$ i.e. $\zeta_0-m>0$, $+$ sign does only under certain circumstannces and $-$ sign certainly does not. Precise situation for the case of $U<v_0-c$ is this: Except in the case where q_2 is sufficiently great, i.e., v_0-c-U is sufficiently great, we can find no solution satifying $\Re(q)>0$.

As this seems a serious breakdown of the theory, some appropriate interpretation is needed. The following seems adaptable.

Suppose that we neglect the viscosity of the gas, then the breakdown is general for $U<v_0-c$. Therefore the introduction of the viscosity for the prevention of this breakdown is only partial in effect. Though, in fact, we treated the viscosity only approximately for the sake of convenience, it does not alter the essential character of the present situation. Conceivably there will be no escape from this situation within the scope of linear theory.

At any rate, $+$ sign for $U<v_0-c$ and $-$ sign $U>v_0+c$ will be reasonable from the physical point of view. The former means the damping and the latter the instability of the vibration. Even the following interpretation is not inconceivable: The theoretical breakdown $\Re(q)<0$ is an exaggerated expression for the rapidity of the accumulation of energy emitted from the elastic body into the medium

3° $q_0=0$, i.e. $U=v_0-c$ or $U=v_0+c$.

In this case (28) can not be used. Putting $\zeta=\zeta_0+\zeta'$, and $m=\zeta_0\pm\frac{1}{n}$ in (9′) and (23′), we obtain

$$q^2=\pm n(2\zeta'-i\delta) \tag{35}$$

and

$$n^{\frac{1}{2}}\left\{\pm(2\zeta'-i\delta)\right\}^{\frac{1}{2}}f'(\zeta_0)\zeta'=-\frac{\varepsilon\zeta_0^2 r_0}{4n^2}. \tag{36}$$

Squaring (36), we obtain a cubic equation for ζ'

$$P(\zeta')=2\zeta'^3-i\delta\zeta'^2\mp\kappa\varepsilon^2=0 \tag{37}$$

where, for the sake of brevity, we put

$$\kappa=\frac{\zeta_0^4 r_0^2}{16\,n^5 f'^2(\zeta_0)}.$$

It will be observed that two out of the three roots of (37), have positive imaginary parts, and that two of the three roots of (37) have positive or negative real parts according as $U=v_0-c$ or $U_0=v_0+c$. Rough configurations of the roots can be inferred from those of the equation (37) with $\delta=0$. Fig. 1 is a schematical diagram of the configurations of roots. Corresponding configurations of q's estimated from (36) are shown in Fig. 2. These configurations are maintained in a narrow neighbourhood (of the order of ε) of $U=v_0\pm c$. Among these three roots, 1 is the continuation of the root which we adopted as physically adequate and 2 is the continuation of the root which we discarded as physically inadequate, while 3 is the continuation of the root whose discussions we omitted as irrelevant for the present purpose.

The orders of the magnitude of ζ''s and q's are $\varepsilon^{2/3}$ and $\varepsilon^{1/3}$ respectively. This extraordinary magnitude of ζ''s compared with those in other cases means a sort of resonance effect. Precisely, we have a sharp peak of damping effect and exciting effect for $U=v_0-c$ and $U=v_0+c$ respectively.

The theoretical unfavorableness $\Re(q)>0$ for $U=v_0-c$ will be interpreted as before.

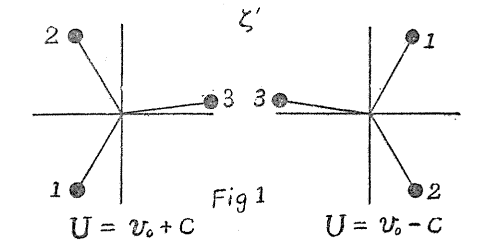

Fig 1

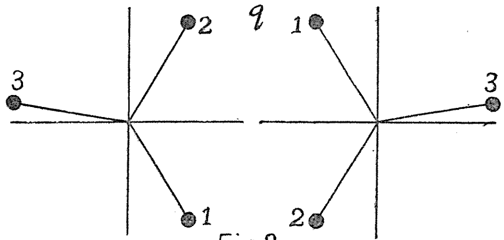

Fig 2

2. Infinitely Stretched Membrane

Let a membrane occupy the plane $y=0$, and two gaseous media in the both sides of the membrane flow in the direction of x-axis with uniform velocities U and U' respectively in the static state. The vibration is again assumed to be two dimentional. The analysis of the vibration of gas in the region $y>0$ is literally the same as that of §1, so we will not repeat it. Similar analysis is available among the corresponding notations with prime in the region $y<0$. Precisely we put

$$\varphi' = C'e^{i(\omega t-kx)+kq'y}$$

corresponding to

$$\varphi = Ce^{i(\omega t-kx)-kqy}.$$

This requires the condition $\Re(q')>0$ for the real part of q'.

On the other hand the equation of the vibration of membrane is given by

$$\sigma\frac{\partial^2\xi}{\partial t^2} - T\frac{\partial^2\xi}{\partial x^2} = p_{yy} - p'_{yy} \tag{1}$$

at $y=0$, where ξ is small normal displacement. The boundary conditions at $y=0$ are the combination of (1) with the following:

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + U\dfrac{\partial}{\partial x}\right)\xi = v \\ \left(\dfrac{\partial}{\partial t} + U'\dfrac{\partial}{\partial x}\right)\xi = v'. \end{cases} \tag{2}$$

Putting

$$\xi = Ae^{i(\omega t-kx)} \tag{3}$$

and using (20), (21) of §1 and the corresponding expressions for primed quantities, we obtain

$$\begin{cases} p_0 = p_0' \\ (k^2T-\sigma\omega^2)A = \rho_0\left\{i(\omega-kU)+2\nu k^2\right\}C - \rho_0'\left\{i(\omega-kU')+2\nu' k^2\right\}C' \\ i(\omega-kU)A = -kqC \\ i(\omega-kU')A = kq'C. \end{cases} \tag{4}$$

Elimination of A, C, and C' leads to the equation for ω.

$$kqq'(\sigma\omega^2 - k^2T) + q'\rho_0(\omega - kU)\{(\omega - kU) - 2\,i\nu k^2\} \\ + q\rho_0'(\omega - kU')\{(\omega - kU') - 2\,i\nu' k^2\} = 0. \tag{5}$$

Let us introduce the following notations,

$$\sqrt{\frac{T}{\sigma}} = b, \quad \frac{\omega}{bk} = \zeta, \quad \frac{\rho_0}{\sigma k} = \varepsilon, \quad \frac{\rho_0'}{\sigma k} = \varepsilon', \quad \frac{U}{b} = m, \quad \frac{U'}{b} = m',$$

$$\frac{4}{3}\frac{k\nu}{b} = \delta, \quad \frac{4}{3}\frac{k\nu'}{b} = \delta', \quad \frac{b}{c} = n, \quad \frac{b}{c'} = n'$$

then (5) can be rewritten as

$$qq'(\zeta^2 - 1) + \varepsilon q'(\zeta - m)\left\{(\zeta - m) - \frac{3i}{2}\delta\right\} + \varepsilon' q(\zeta - m')\left\{(\zeta - m') - \frac{3i}{2}\delta'\right\} = 0 \tag{6}$$

where
$$q^2 = 1 - \frac{n^2(\zeta - m)^2}{1 + in^2\delta(\zeta - m)}$$

and
$$q'^2 = 1 - \frac{n'^2(\zeta - m')^2}{1 + in'^2\delta'(\zeta - m')}.$$

As we have many parameters in (6), it will be very complicated to take care of all the possibilities exhaustively. Therefore we will make some simplifying assumptions, and treat several interesting cases only.

I. $\rho_0 = \rho_0'$, $\nu = \nu'$, $c = c'$, $U = U'$.

The equation to be solved is

$$q(\zeta^2 - 1) + 2\varepsilon(\zeta - m)\left\{(\zeta - m) - \frac{3i}{2}\delta\right\} = 0 \tag{7}$$

where ε and δ are assumed to be small quantities of the same order.

The roots of (7) relevant for the present purpose are the one which are nearly equal to the roots of $\zeta^2 - 1 = 0$, i.e. $\zeta_0 = \pm 1$. We may put $\zeta = 1 + \zeta'$ for the same reason as that in § 1. Then the first approximation for ζ' is given by

$$\zeta' = -\frac{\varepsilon(1 - m)^2}{q_0} \tag{8}$$

where
$$q_0^2 = 1 - n^2(1 - m)^2.$$

From here on, all the discussion is qualitatively the same as that of § 1. The following correspondences

$$\zeta_0 \to 1, \qquad \frac{\zeta_0^2 r_0}{4f'(\zeta_0)} \to 1, \qquad v_0 \to b$$

suffice to make the results of §1 available for the present section.

II. Now we restore the equation (6). The ε's and δ's are assumed to be small quantities of the same order. Putting $\zeta = 1 + \zeta'$, the first approximation for ζ' is

$$\zeta' = -\frac{\varepsilon(1-m)^2}{2\,q_0} - \frac{\varepsilon'(1-m')^2}{2\,q_0'}. \tag{9}$$

The situations where q_0^2 and $q_0'^2$ are both positive, or one of them is positive and the other is negative, are qualitatively similar to those of I where q_0^2 (of *1*) is positive or negative respectively. Accordingly we will confine ourselves to the case where both q_0^2 and $q_0'^2$ are negative.
Let us put

$$q_0 = \pm iq_2 = \pm i\sqrt{n^2(1-m)^2-1}$$

and

$$q_0' = \pm' iq_2' = \pm' i\sqrt{n'^2(1-m')^2-1}$$

then ζ' is given by

$$\zeta' = \pm i\frac{\varepsilon(1-m)^2}{2\,q_2} \quad \pm' i\varepsilon'\frac{(1-m')^2}{2\,q_2'} \tag{10}$$

The q's can be easily obtained as:

$$q = \pm iq_2 + \frac{n^2(1-m)}{2\,q_2}\left\{\pm n^2(1-m)^2\delta - \frac{\varepsilon(1-m)^2}{q_2} \mp \left(\pm'\frac{\varepsilon'(1-m')^2}{q_2'}\right)\right\} \tag{11}$$

and $$q' = \pm' iq_2' + \frac{n'^2(1-m')}{2\,q_2'}\left\{\pm' n'^2(1-m')^2\delta' \mp'\left(\pm\frac{\varepsilon(1-m)^2}{q_2}\right) - \frac{\varepsilon'(1-m')^2}{q_2'}\right\}.$$

As is discussed in § 1, physically reasonable choices of signs are as follows:

$+$	for $U<b-c$	i.e. $1-m>0$
$-$	for $U>b-c$	i.e. $1-m<0$
$+'$	for $U'<b-c'$	i.e. $1-m'>0$
$-'$	for $U'>b-c$	i.e. $1-m'<0$

The cases where $1-m$ and $1-m'$ are both positive or negative are again qualitatively similar to those of *1* where $1-m$ is positive or negative respectively.

Therefore we will take the case where one of them is positive and the other is negative, e.g. $1-m<0$ and $1-m'>0$
The equations (10) and (11) become

$$\zeta' = -i\frac{\varepsilon(1-m)^2}{2\,q^2} + i\frac{\varepsilon'(1-m')^2}{2\,q_2'} \tag{10'}$$

$$q = -iq_2 + \frac{n^2(1-m)}{2\,q_2}\left\{-n^2(1-m)^2\delta - \frac{\varepsilon(1-m)^2}{q_2} + \frac{\varepsilon'(1-m')^2}{q_2'}\right\}$$

$$q' = iq_2' + \frac{n'^2(1-m')}{2\,q_2'}\left\{n'^2(1-m')^2\delta' + \frac{\varepsilon(1-m)^2}{q_2} - \frac{\varepsilon'(1-m')^2}{q_2'}\right\}. \tag{11}$$

Assume that $\frac{\varepsilon(1-m)^2}{q_2} > \frac{\varepsilon'(1-m')^2}{q_2'}$ then $\Im(\zeta')<0$, $\Re(q)>0$ and $\Re(q')>0$ i.e., the vibration is unstable. The absorption of the energy from the gas in the region $y>0$ exceeds the emission of the energy into the gas in the region $y<0$. Assume, on the contrary, that $\frac{\varepsilon(1-m)^2}{q^2} < \frac{\varepsilon'(1-m')^2}{q_2'}$, then $\Im(\zeta')>0$, but whih respect to

$\Re(q)$ and $\Re(q')$ we face again the similar theoretical unfavorableness $\Re(q)<0$ and $\Re(q')<0$ under certain circumstances.

The vibration is stable i.e., the emission of the energy into the gas in the region $y<0$ exceeds the absorption of the energy from the gas in the region $y<0$.

Assume, at last, that $\frac{\varepsilon(1-m)^2}{q_2} = \frac{\varepsilon'(1-m')^2}{q_2'}$ then

$$\Im(\zeta')=0 \quad \text{and} \quad \Re(q)>0 \quad \text{and} \quad \Re(q')>0$$

i.e. the vibration is almost neutral and almost the same as free vibration. The emission and the absorption of energy compensate for each other perfectly.

III. $c=c'=\infty$ i.e. $n=n'=0$ and $\nu=\nu'=0$

This case is only of theoretical interest. Owing to the above assumption q's reduce to 1. The general equation (6) can be simplfied as

$$\zeta^2-1+\varepsilon(\zeta-m)^2+\varepsilon'(\zeta-m')^2=0. \tag{12}$$

This is a quadratic equation in ζ, and can be solved generally.
The roots are

$$\zeta=\frac{1}{1+\varepsilon+\varepsilon'}(\varepsilon m+\varepsilon' m'\pm\sqrt{D}) \tag{13}$$

where the discriminant D is

$$D=1+\varepsilon+\varepsilon'-(\varepsilon m^2+\varepsilon' m'^2)-\varepsilon\varepsilon'(m-m')^2. \tag{14}$$

1) $m \neq m'$

Taking k sufficiently small, D can be made negative. Hence, we see that the vibration is unstable for sufficiently long wave length.

2) $m=m'$

$$D=1+\varepsilon+\varepsilon'-(\varepsilon+\varepsilon')m^2.$$

If $|m|\leqq|$ i.e. $|U|\leqq b$, then D is positive for every value of k. Thus the vibration is always neutral. If, on the contrary, $|m|>1$ i.e. $|U|>b$ then D becomes negative for sufficiently small values of k. Thus the vibration is unstable for sufficiently long wave length.

3. Spherical Elastic Body in a Medium

Let an elastic body occupy the region $r<a$ and a gaseous medium which is at rest, occupy the region $r>a$ in the static state, where $r=\sqrt{x^2+y^2+z^2}$. The equations of the vibration of the elastic body are

$$\begin{cases} \sigma\frac{\partial^2\xi}{\partial t^2}=(\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial\xi}{\partial x}+\frac{\partial\eta}{\partial y}+\frac{\partial\zeta}{\partial z}\right)+\mu\Delta\xi \\ \sigma\frac{\partial^2\eta}{\partial t^2}=(\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial\xi}{\partial x}+\frac{\partial\eta}{\partial y}+\frac{\partial\zeta}{\partial z}\right)+\mu\Delta\eta \\ \sigma\frac{\partial^2\zeta}{\partial t^2}=(\lambda+\mu)\frac{\partial}{\partial z}\left(\frac{\partial\xi}{\partial x}+\frac{\partial\eta}{\partial y}+\frac{\partial\zeta}{\partial z}\right)+\mu\Delta\zeta. \end{cases} \tag{1}$$

When the vibration is radial, the displacement components ξ, η, ζ can be derived from a scalar potential Φ i.e.,

$$\xi=\frac{\partial\Phi}{\partial x},\qquad \eta=\frac{\partial\Phi}{\partial y},\qquad \zeta=\frac{\partial\Phi}{\partial z} \tag{2}$$

and especially the radial displacement ξ_r is given by

$$\xi_r=\frac{\partial\Phi}{\partial r}. \tag{3}$$

The equation for Φ can be obtained from (1), and (2) as

$$\Delta\left(\sigma\frac{\partial^2}{\partial t^2}-(\lambda+2\mu)\Delta\right)\Phi=0. \tag{4}$$

As a particular solution, we will take

$$\Phi=A\frac{e^{i\omega t}}{r}\sin\frac{\omega r}{b} \tag{5}$$

where
$$b=\sqrt{\frac{\lambda+2\mu}{\sigma}}.$$

The radial displacement and the stress component are given by

$$\xi_r=\frac{\partial\Phi}{\partial r}=Ae^{i\omega t}\left(\frac{\omega}{br}\cos\frac{\omega r}{b}-\frac{1}{r^2}\sin\frac{\omega r}{b}\right) \tag{6}$$

and

$$\begin{aligned}\sigma_{rr}&=-p_0+\lambda\left(\frac{\partial\xi}{\partial x}+\frac{\partial\eta}{\partial y}+\frac{\partial\xi}{\partial z}\right)+2\mu\frac{\partial\xi_r}{\partial r}\\ &=-p_0+4\mu Ae^{i\omega t}\left\{\left(\frac{1}{r^3}-\frac{\sigma\omega^2}{4\mu r}\right)\sin\frac{\omega r}{b}-\cos\frac{\omega r}{b}\right\}\end{aligned} \tag{7}$$

where p_0 is a uniform pressure in the static state.

On the other hand, the equation of the vibration for gas are

$$\left\{\begin{aligned}\rho_0\frac{\partial u}{\partial t}&=-\frac{\partial p}{\partial x}+\frac{\rho_0\nu}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\rho_0\nu\Delta u\\ \rho_0\frac{\partial v}{\partial t}&=-\frac{\partial p}{\partial y}+\frac{\rho_0\nu}{3}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\rho_0\nu\Delta v\\ \rho_0\frac{\partial w}{\partial t}&=-\frac{\partial p}{\partial z}+\frac{\rho_0\nu}{3}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\rho_0\nu\Delta w\\ \frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{p}{\rho_0}\right)&+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0.\end{aligned}\right. \tag{8}$$

As the vibration is radial, the velocity components u, v, w can be derived from a scalar potential φ i.e.

$$u=\frac{\partial\varphi}{\partial x},\qquad v=\frac{\partial\varphi}{\partial y},\qquad w=\frac{\partial\varphi}{\partial z} \tag{9}$$

and especially the radial velocity u_r is

$$u_r = \frac{\partial \varphi}{\partial r}. \tag{10}$$

From (7) anp (8), we get

$$p - p_0 = -\rho_0\left(\frac{\partial}{\partial t} - \frac{4\nu}{3}\Delta\right)\varphi \tag{11}$$

and

$$\frac{1}{c^2\rho_0}\frac{\partial p}{\partial t} = -\Delta\varphi.$$

Consequently the equation for φ is

$$\frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} - \frac{4\nu}{3}\Delta\right)\varphi = \Delta\varphi. \tag{12}$$

As a particular solution, we will take

$$\varphi = B\frac{e^{i\omega(t-r/\beta c)}}{r} \tag{13}$$

where

$$\beta^2 = 1 + \frac{4\,i\nu\omega}{3\,c^2} \tag{14}$$

and

$$\Im\left(\frac{\omega}{\beta}\right) < 0.$$

Then the radial velocity and the stress component are given by

$$u_r = \frac{\partial\varphi}{\partial r} = -Be^{i\omega(t-r/\beta c)}\left(\frac{i\omega}{\beta c r} + \frac{1}{r^2}\right) \tag{15}$$

and

$$p_{rr} = -p - \frac{2}{3}\rho_0\nu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\sigma z}\right) + 2\,\rho_0\nu\frac{\partial u_r}{\sigma r}$$

$$= -p_0 + \rho_0 Be^{i\omega(t-r/\beta c)}\left(\frac{i\omega}{r} + \frac{4\,i\nu\omega}{\beta c^2 r^2} + \frac{4\,\nu}{r^3}\right). \tag{16}$$

Boundary conditions at $r=a$ are as follows:

$$\begin{cases} \dfrac{\partial \xi_r}{\partial t} = u_r \\ \sigma_{rr} = p_{rr}. \end{cases} \tag{17}$$

Inserting (6), (7), (15) and (16) into (17), and eliminating A and B, we obtain the following equation for ω.

$$\left(1 + \frac{i\omega a}{\beta c}\right)\left\{\left(1 - \frac{\sigma\omega^2 a^2}{4\,\mu}\right)\tan\frac{\omega a}{b} - \frac{\omega a}{b}\right\}$$
$$+ \frac{i\rho_0\omega}{4\,\mu}\left(\frac{\omega a}{b} - \tan\frac{\omega a}{b}\right)\left\{i\omega a^2 + 4\,\nu\left(1 + \frac{i\omega a}{\beta c}\right)\right\} = 0. \tag{18}$$

Introducing the notations

$$\frac{\omega a}{b} = \zeta, \quad \frac{\sigma b^2}{4\,\mu} = \frac{\lambda + 2\,\mu}{4\,\mu} = e > \frac{1}{2}, \quad \frac{\rho_0 b^2}{4\,\mu} = \frac{(\lambda + 2\,\mu)\rho_0}{4\,\mu\sigma} = \varepsilon, \quad \frac{b}{c} = n, \quad \frac{4\,\nu}{3\,ba} = \delta$$

(18) can be rewritten as:

$$\left(1+i\frac{n\zeta}{\beta}\right)(1-e\zeta^2-\zeta\cot\zeta) = -\varepsilon\zeta(1-\zeta\cot\zeta)\left\{\zeta-3i\delta\left(1+\frac{in\zeta}{\beta}\right)\right\} \tag{19}$$

where

$$\beta^2 = 1+in^2\delta\zeta. \tag{20}$$

Assume that ε and δ are small quantities of the same order, then we can put

$$\zeta = \zeta_0 + \zeta' \tag{21}$$

where ζ_0 is a root of the equation

$$f(\zeta) = 1 - e\zeta^2 - \zeta\cot\zeta = 0 \tag{22}$$

and ζ' is a small quantity of the order of ε.

The equation (22) has infinite positive roots $\zeta_0^{(n)}$ (n=1, 2, . . .), and it will be observed that each of them lies in the interval $\left(n-\frac{1}{2}\right)\pi < \zeta_0^{(n)} < n\pi$.

The first approximation for ζ' can be given from (19), (21), (22) by

$$\zeta' = -\frac{\varepsilon e\zeta_0^3}{(e^2\zeta_0^2+1-3e)(1\pm \mathrm{in}\zeta_0)} = -\frac{\varepsilon e\zeta_0^3(1\mp in\zeta_0)}{(e^2\zeta_0^2+1-3e)(1+n^2\zeta_0^2)} \tag{23}$$

where

$$f'(\zeta_0) = e^2\zeta_0^2+1-3e = (e\zeta_0-1)^2 + 2e\left(\zeta_0-\frac{3}{2}\right) > 0$$

because of

$$\zeta_0 > \frac{\pi}{2} > \frac{3}{2}.$$

The $\pm$ signs in (23) correspond to

$$\beta = \pm\left(1+\frac{i}{2}n^2\delta\zeta_0\right) \tag{24}$$

which must be determined so as to secure $\Im\left(\frac{\zeta}{\beta}\right) < 0$.

From (23) and (24), we have

$$\Im\left(\frac{\zeta}{\beta}\right) = \frac{1}{1+\frac{n^4}{4}\delta^2\zeta_0^2}\left\{\mp\frac{n^2}{2}\delta\zeta_0^2 + \frac{\varepsilon en\zeta_0^4}{(e^2\zeta_0^2+1-3e)(1+n^2\zeta_0^2)}\right\}. \tag{25}$$

Unfortunately the form of (25) prevents to secure $\Im\left(\frac{\zeta}{\beta}\right)$ under all circumstances.

In spite of this theoretical breakdown, the physically reasonable choice must be + sign. Then $\Im(\zeta')>0$ in (23) shows the damping effect of the medium and $\Re(\beta)>0$ in (24) shows the emission of the energy into the medium.

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References

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