RESEARCH REPORTS

ON THE BENDING OF A CYLINDRICAL BODY BY ITS OWN WEICHT

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I. Fundamental Equations

The theory of bending of a cylindrical elastic body, placed horizontally and fixed at one end, by its own weight, has been developed in Love's "Mathematical Theory of Elasticity" 4th edition, p. 349-364. We solved the problem for the 2 cases of an elliptic cylinder and a circular tube.

Let

x, y, z: rectangular co-ordinates, z-axis being taken along the cylinder through the centre of mass of the cross section and x- and y-axes into the directions of principal moment of inertia of the cross section. We further assume that the origin is at the fixed end and y- and z-axes are horizontal and x-axis is directed vertically downwards and that the cross section is symmetrical with respect to x-axis.

According to the theory referred above we put

$\varepsilon_0,\ \kappa_0,\ \kappa_1,\ \kappa_2$: constants,
$e_{xx}^{(0)},\ e_{yy}^{(0)},\ e_{xy}^{(0)},\ \chi$: unknown functions of x and y only,
$e=\kappa_1 z+\kappa_2 z^2,\quad \psi=\varepsilon_0-\kappa_0 x+2\kappa_2(\chi+xy^2),$

and assume the following forms for the components of strain:

$$\left.\begin{aligned}&e_{xx}=\sigma e x+e_{xx}^{(0)},\quad e_{yy}=\sigma e x+e_{yy}^{(0)},\quad e_{xy}=e_{xy}^{(0)},\quad e_{zz}=\psi-ex,\\&e_{xz}=\frac{de}{dz}\left\{\frac{\partial\chi}{\partial x}+\frac{\sigma}{2}x^2+\left(1-\frac{\sigma}{2}\right)y^2\right\},\\&e_{yz}=\frac{de}{dz}\left\{\frac{\partial\chi}{\partial y}+(2+\sigma)xy\right\}.\end{aligned}\right\}\tag{1}$$

Among the 6 conditions of compatibility, 5 conditions are satisfied identically and the 1 becomes:

$$\frac{\partial^2 e_{xx}^{(0)}}{\partial y^2}+\frac{\partial^2 e_{yy}^{(0)}}{\partial x^2}=\frac{\partial^2 e_{xy}^{(0)}}{\partial x\partial y},\tag{2}$$

$$D=e_{xx}+e_{yy}+e_{zz}=\psi-(1-2\sigma)ex+e_{xx}^{(0)}+e_{yy}^{(0)},$$

$$S=\frac{\lambda}{\mu}D+2\sigma e x=\frac{2\sigma}{1-2\sigma}\{\psi+e_{xx}^{(0)}+e_{yy}^{(0)}\}.$$

The components of stress are

$$\left.\begin{aligned}&X_x=\lambda D+2\,\mu e_{xx}=\mu S+2\,\mu e_{xx}^{(0)},\quad Y_y=\mu S+2\,\mu e_{yy}^{(0)},\\&X_y=\mu e_{xy},\quad X_z=\mu e_{xz},\quad Y_z=\mu e_{yz},\\&Z_z=\mu S+2\,\mu\{\psi-(1+\sigma)ex\},\end{aligned}\right\}\tag{3}$$

and the equations of equilibrium:

$$\left.\begin{aligned}&\frac{\partial S}{\partial x}+2\frac{\partial e_{xx}^{(0)}}{\partial x}+\frac{\partial e_{xy}^{(0)}}{\partial y}+\frac{\partial e_{xz}}{\partial z}+\frac{\rho g}{\mu}=0,\\&\frac{\partial e_{xy}^{(0)}}{\partial x}+\frac{\partial S}{\partial y}+2\frac{\partial e_{yy}^{(0)}}{\partial y}+\frac{\partial e_{yz}}{\partial z}=0,\end{aligned}\right\}\tag{4}$$

$$\frac{\partial e_{xz}}{\partial x}+\frac{\partial e_{yz}}{\partial y}-2(1+\sigma)\frac{de}{dz}x=0.\tag{5}$$

From (5) we have

$$\varDelta\chi=0.\tag{6}$$

The boundary conditions for lateral surface are

$$\left.\begin{aligned}(S+2\,e_{xx}^{(0)})\cos(\nu,\ x)+e_{xy}^{(0)}\cos(\nu,\ y)=0,&\\e_{xy}^{(0)}\cos(\nu,\ x)+(S+2\,e_{yy}^{(0)})\cos(\nu,\ y)=0,&\end{aligned}\right\}\tag{7}$$

$$e_{xz}\cos(\nu,\ x)+e_{yz}\cos(\nu,\ y)=0,\tag{8}$$

here ν denotes the outward normal at the boundary.

From (8) we obtain

$$\frac{\partial\chi}{\partial\nu}=-\left\{\frac{\sigma}{2}x^2+\left(1-\frac{\sigma}{2}\right)y^2\right\}\cos(\nu,\ x)-(2+\sigma)xy\cos(\nu,\ y).\tag{9}$$

(6) and (9) determine χ excepting the additive constant, and this constant can be put into ε_0. Instead of the remaining $e_{xx}^{(0)}$, $e_{yy}^{(0)}$ and $e_{xy}^{(0)}$ we use the following functions T_1, T_2 and T_3:

$$\left.\begin{aligned}&T_1=S+2\,e_{xx}^{(0)}+\frac{\rho g}{\mu}x+2\,\kappa_2\left\{\chi+\frac{\sigma}{6}\,(x^3-3\,xy^2)+xy^2\right\},\\&T_2=S+2\,e_{yy}^{(0)}+2\,\kappa_2\left\{\chi-\frac{1}{3}\left(1+\frac{\sigma}{2}\right)(x^3-3\,xy^2)\right\},\\&T_3=e_{xy}^{(0)}.\end{aligned}\right\}\tag{10}$$

Then (2) and (4) become respectively:

$$\sigma\varDelta(T_1+T_2)+2\frac{\partial^2 T_3}{\partial x\partial y}-\frac{\partial^2 T_1}{\partial y^2}-\frac{\partial^2 T_2}{\partial x^2}=0,\tag{11}$$

$$\left.\begin{aligned}\frac{\partial T_1}{\partial x}&=-\frac{\partial T_3}{\partial y},\\\frac{\partial T_2}{\partial y}&=-\frac{\partial T_3}{\partial x}.\end{aligned}\right\}\tag{12}$$

Putting (12) into (11) we have

$$\varDelta(T_1+T_2)=0,\tag{13}$$

and (7) can be written:

$$\left.\begin{aligned} T_1\cos(\nu,\ x)+T_3\cos(\nu,\ y)&=\left\{\frac{\rho g}{\mu}x+2\kappa_2\Big[\chi+\frac{\sigma}{6}(x^3-3xy^2)+xy^2\Big]\right\}\cos(\nu,\ x),\\ T_3\cos(\nu,\ x)+T_2\cos(\nu,\ y)&=2\kappa_2\left\{\chi-\frac{1}{3}\Big(1+\frac{\sigma}{2}\Big)(x^3-3xy^2)\right\}\cos(\nu,\ y).\end{aligned}\right\}\tag{14}$$

From (18) we have

$$T_1+T_2=-2\Omega,\quad \Delta\Omega=0,\tag{15}$$

and put

$$T_3=x\frac{\partial\Omega}{\partial y}+\Omega_1.$$

Inserting these expressions into (12) we have

$$\frac{\partial T_2}{\partial x}=-\frac{\partial^2}{\partial x^2}(x\Omega)+\frac{\partial\Omega_1}{\partial y},$$
$$\frac{\partial T_2}{\partial y}=-\frac{\partial^2}{\partial x\partial y}(x\Omega)-\frac{\partial\Omega_1}{\partial x}.$$

Hence

$$\Delta\Omega_1=0.\tag{16}$$

and let

$$f(z)=\Omega_1+i\Omega_2$$

be the regular function, then we have:

$$\left.\begin{aligned} T_2&=-\frac{\partial}{\partial x}(x\Omega)-\Omega_2,\\ T_1&=x\frac{\partial\Omega}{\partial x}-\Omega+\Omega_2,\\ T_3&=x\frac{\partial\Omega}{\partial y}+\Omega_1,\end{aligned}\right\}\tag{17}$$

here the additive constant in T_2 being assumed to be put into Ω_2.

In the domain of the cross section χ is one-valued as is seen from (1), e_{zz}, hence so are T_1, T_2 and T_3 from (10), and Ω, Ω_1 and Ω_2 from (15) and (17). When the domain is simply connected we can further transform (17) in the following way. Putting

$$iF'(z)=f(z),\quad F(z)=\Omega'+i\Phi,\tag{18}$$

we obtain

$$\left.\begin{aligned} T_1&=\frac{\partial}{\partial x}(x\Omega+\Omega')-2\Omega,\\ T_2&=-\frac{\partial}{\partial x}(x\Omega+\Omega'),\\ T_3&=\frac{\partial}{\partial y}(x\Omega+\Omega'),\end{aligned}\right\}\tag{19}$$

and the left sides of (14) can also be written in

$$\left.\begin{aligned}&-\Omega\cos(\nu,\ x)+\frac{\partial\Omega}{\partial\nu}x+\frac{\partial\Omega'}{\partial\nu},\\&\frac{\partial}{\partial s}(x\Omega+\Omega')\end{aligned}\right\}\tag{20}$$

respectively, where s is taken along the boundary.

When the length of the cylinder is l, the conditions at $z=l$:

$$X_z=Y_z=0$$

are fulfilled by

$$\kappa_1=-2\,\kappa_2 l,\tag{21}$$

but in general Z_z does not vanish, and we can determine ε_0 and κ_0 by:

$$\int Z_z dxdy=0\tag{22}$$

and

$$\int Z_z x dxdy=0\tag{23}$$

while κ_2 is determined from (14).

II. Elliptic Cylinder

Let the boundary of the cross section be:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

then

$$\chi=-\frac{a^2\{2(1+\sigma)a^2+b^2\}}{3\,a^2+b^2}x+\frac{1}{3}\frac{2\,a^2+b^2+\frac{\sigma}{2}(a^2-b^2)}{3\,a^2+b^2}(x^3-3\,xy^2).$$

We assume the following expressions for Ω and Ω':

$$\left.\begin{aligned}&\Omega=k_1x+k_2(x^3-3\,xy^2),\\&\Omega'=k_1'+k_2'(x^2-y^2)+k_3'(x^4-6\,x^2y^2+y^4),\end{aligned}\right\}\tag{24}$$

then from (20) and (14) we obtain the results:

$$\left.\begin{aligned}&k_1=\frac{\kappa_2}{6}\frac{12(1+\sigma)a^8+3(9+7\,\sigma)a^6b^2+(65+53\,\sigma)a^4b^4+3(7-11\,\sigma)a^2b^6+3(1-7\,\sigma)b^8}{(3\,a^2+b^2)(a^4+2\,a^2b^2+b^4)},\\&k_2=\frac{\kappa_2}{6}\frac{(1+\sigma)a^6+(3+\sigma)a^4b^2+(11-5\,\sigma)a^2b^4+(1+3\,\sigma)b^6}{(3\,a^2+b^2)(a^4+2\,a^2b^2+b^4)},\\&k_1'=b^4k_3',\quad k_2'=2\,b^2k_3',\\&k_3'=-\frac{1-2\,\sigma}{6}\kappa_2\cdot\frac{a^2+3\,b^2}{3\,a^2+b^2}\cdot\frac{a^2b^2}{a^4+2\,a^2b^2+5\,b^4},\end{aligned}\right\}\tag{25}$$

with

$$\frac{\rho g}{\mu} = \kappa_2 a^2(1+\sigma), \tag{26}$$

and the last formula (26) corresponds to (36) in Love, p. 356.

By (24), (25) and (26) we have the expressions for $X_x, \ldots$:

$$\left.\begin{aligned}
X_x &= \mu\kappa_2 \cdot \frac{1}{3(a^4+2a^2b^2+5b^4)(3a^2+b^2)}[\{3(1+\sigma)a^6+3(3+\sigma)a^4b^2+(19+13\sigma)a^2b^4 \\
&\qquad +3(3-\sigma)b^6\}(a^2-x^2)x-6(1-2\sigma)(a^6+a^4b^2+2a^2b^4)xy^2], \\
Y_y &= \qquad {}'' \qquad [\{(1+\sigma)a^6+(3+\sigma)a^4b^2+(5+7\sigma)a^2b^4 \\
&\qquad -(1-7\sigma)b^6\}3(b^2-y^2)x-2(1-2\sigma)b^2(3a^4+b^4)x^3], \\
X_y &= \qquad {}'' \qquad [2(1-2\sigma)a^2b^2(a^2+3b^2)(b^2-y^2)y-3\{(1+\sigma)a^6 \\
&\qquad +(1+5\sigma)a^4b^2+(5+7\sigma)a^2b^4+(1+3\sigma)b^6\}x^2y], \\
X_z &= \mu\kappa_2 \cdot \frac{2(z-l)}{3a^2+b^2} \cdot [-\{2(1+\sigma)a^2+b^2\}(a^2-x^2)+a^2(1-2\sigma)y^2], \\
Y_z &= \mu\kappa_2 \frac{4(z-l)}{3a^2+b^2}\{(1+\sigma)a^2+\sigma b^2\}xy, \\
Z_z &= E\{\varepsilon_0-\kappa_0 x-\kappa_2 z(z-2l)x\} \\
&\quad +\mu\kappa_2 \frac{x}{3(3a^2+b^2)(a^4+2a^2b^2+5b^4)}[-\{3(8+15\sigma+7\sigma^2)a^8+6(10+16\sigma+7\sigma^2)a^6b^2 \\
&\quad +2(72+118\sigma+52\sigma^2)+6(10+6\sigma-3\sigma^2)+3\sigma(1-7\sigma)\}+\{(8+7\sigma-\sigma^2)a^6 \\
&\quad +(20+13\sigma-\sigma^2)a^4b^2+(48+29\sigma+5\sigma^2)a^2b^4+(20-\sigma-3\sigma^2)b^6\}x^2 \\
&\quad +3\{(4-\sigma+\sigma^2)a^6+(8+\sigma+\sigma^2)a^4b^2+5(4+\sigma-\sigma^2)a^2b^4+(11\sigma+3\sigma^2)b^6\}y^2].
\end{aligned}\right\} \tag{27}$$

The condition (22) is satisfied by $\varepsilon_0 = 0$.

In the special case of circular cylinder, (27) agrees with the results in Love p. 362.

III. Circular Tube

Let the domain be

$$a_0 \leqq r = \sqrt{x^2+y^2} \leqq a_1,$$

then

$$\chi = -\left(\frac{3}{4}+\frac{\sigma}{2}\right)\left(a_0^2+a_1^2+\frac{a_0^2a_1^2}{x^2+y^2}\right)x+\frac{1}{4}(x^3-3xy^2).$$

From the form of the expressions in (14) we may assume:

$$\begin{gathered}
\Omega = \Re(\alpha_1 z+\alpha_2 z^{-1}+\alpha_3 z^3+\alpha_4 z^{-3}), \\
f(z) = \Omega_1+i\Omega_2 = i(2\beta_1 z+\beta_0 z^{-1}+4\beta_3 z^3-2\beta_2 z^{-3})
\end{gathered} \tag{28}$$

with real coefficients $\alpha_1, \ldots, \beta_0, \ldots$. Hence if we put

$$F(z) \equiv \Omega'+i\Phi = \beta_1 z^2+\beta_0 \log z+\beta_3 z^4+\beta_2 z^{-2},$$

$$\Omega' = \beta_0 \log r + \beta_1(x^2 - y^2) + \beta_2 \frac{x^2 - y^2}{r^4} + \beta_3(x^4 - 6x^2y^2 + y^4), \tag{29}$$

we have as in (18)

$$f(z) = iF'(z),$$

and (19) and (20) hold good also in this not simply connected case.

Inserting (28) and (29) into (14), we obtain the results:

$$\left.\begin{aligned}
&\alpha_4 = 0, \quad \alpha_3 = \frac{\kappa_2}{12}, \quad \beta_3 = -\frac{1}{24}\left(\frac{1}{2} - \sigma\right)\kappa_2, \\
&\alpha_2 = \kappa_2 a_0^2 a_1^2 \left\{\gamma + \left(1 + \frac{\sigma}{2}\right)\right\}, \\
&\beta_0 = \kappa_2 a_0^2 a_1^2 \gamma, \\
&\beta_1 = \frac{\kappa_2}{2(a_0^2 + a_1^2)}\left\{a_0^2 a_1^2 \gamma - \frac{1}{6}\left(\frac{1}{2} - \sigma\right)(a_0^4 + a_0^2 a_1^2 + a_1^4)\right\}, \\
&\beta_2 = -\frac{\kappa_2 a_0^4 a_1^4}{2(a_0^2 + a_1^2)}\left\{\gamma + \frac{1}{6}\left(\frac{1}{2} - \sigma\right)\right\}, \\
&\alpha_1 = \frac{2\kappa_2}{a_0^2 + a_1^2}\left\{-a_0^2 a_1^2 \gamma + \frac{1}{3}\left(1 + \frac{\sigma}{4}\right)(a_0^4 + a_1^4) + \frac{1}{3}\left(\frac{7}{4} + \sigma\right) a_0^2 a_1^2\right\},
\end{aligned}\right\} \tag{30}$$

$$\frac{\rho g}{\mu} = \kappa_2 (1 + \sigma)(a_0^2 + a_1^2)$$

here γ is an arbitrary constant.

The results for X_x, . . . are:

$$\begin{aligned}
X_x = {} & \mu\kappa_2 \left\{\frac{5 + 2\sigma}{12}(a_0^2 + a_1^2) + \frac{a_0^2 a_1^2}{a_0^2 + a_1^2}\left(\gamma + \frac{1 - 2\sigma}{12}\right)\left(1 - \frac{a_0^2 a_1^2}{(x^2 + y^2)^2}\right) + \left(\gamma + \frac{2 + 3\sigma}{2}\right)\frac{a_0^2 a_1^2}{x^2 + y^2}\right\} x \\
& - \mu\kappa_2 \left\{\frac{5 + 2\sigma}{12} + 2\left(\gamma + \frac{\sigma + 2}{2}\right)\frac{a_0^2 a_1^2}{(x^2 + y^2)^2}\right\} x^3 - \frac{\mu\kappa_2}{4}(1 - 2\sigma) x y^2, \\
Y_y = {} & \mu\kappa_2 \left[\left\{\frac{1 + 2\sigma}{4}(a_0^2 + a_1^2) + 3\left(\gamma + \frac{1 - 2\sigma}{12}\right)\frac{a_0^2 a_1^2}{a_0^2 + a_1^2}\right\} - 3\left(\gamma + \frac{1}{6}\right)\frac{a_0^2 a_1^2}{x^2 + y^2}\right. \\
& \qquad \left. + \frac{a_0^4 a_1^4}{a_0^2 + a_1^2}\left(\gamma + \frac{1 - 2\sigma}{12}\right)\frac{1}{(x^2 + y^2)^2}\right] x \\
& - \mu\kappa_2 \left\{\frac{1 + 2\sigma}{4} - 2\left(\gamma + \frac{1 - 2\sigma}{12}\right)\frac{a_0^3 a_1^4}{(a_0^2 + a_1^2)(x^2 + y^2)^3}\right\} x y^2 \\
& - \mu\kappa_2 \left\{\frac{1 - 2\sigma}{12} - 2\left(\gamma + \frac{2 + \sigma}{2}\right)\frac{a_0^2 a_1^2}{(x^2 + y^2)^2} + \left(2\gamma + \frac{1 - 2\sigma}{6}\right)\frac{a_0^4 a_1^4}{(a_0^2 + a_1^2)(x^2 + y^2)^3}\right\} x^3, \\
Z_z = {} & E\{\varepsilon_0 - \kappa_0 x - z(z - 2l)\kappa_2 x\} \\
& - \mu\kappa_2 \left\{\frac{1}{3}(9 + 13\sigma + 4\sigma^2)(a_0^2 + a_1^2) - 4\sigma\left(\gamma + \frac{1 - 2\sigma}{12}\right)\frac{a_0^2 a_1^2}{a_0^2 + a_1^2}\right. \\
& \qquad \left. + 2\left(\sigma\gamma + \frac{3 + 4\sigma + \sigma^2}{2}\right)\frac{a_0^2 a_1^2}{x^2 + y^2}\right\} x + \mu\kappa_2\left(1 + \frac{\sigma}{2}\right)(x^3 + x y^2), \\
X_y = {} & \mu\kappa_2 \left\{\frac{1 - 2\sigma}{12}(a_0^2 + a_1^2) - \frac{a_0^2 a_1^2}{a_0^2 + a_1^2}\left(\gamma + \frac{1 - 2\sigma}{12}\right) + \gamma \frac{a_0^2 a_1^2}{x^2 + y^2}\right.
\end{aligned}$$

$$+\frac{a_0^4a_1^4}{(a_0^2+a_1^2)(x^2+y^2)^2}\left(\gamma+\frac{1-2\sigma}{12}\right)\Big\}y$$

$$-\mu\kappa_2\left\{\frac{1+2\sigma}{4}+\frac{2a_0^2a_1^2}{(x^2+y^2)^2}\left(\gamma+\frac{2+\sigma}{2}\right)-\frac{2a_0^4a_1^4}{(a_0^2+a_1^2)(x^2+y^2)^3}\left(\gamma+\frac{1-2\sigma}{12}\right)\right\}xy^2$$

$$-\mu\kappa_2\left\{\frac{1-2\sigma}{12}+\frac{2a_0^4a_1^4}{(a_0^2+a_1^2)(x^2+y^2)^3}\left(\gamma+\frac{1-2\sigma}{12}\right)\right\}y^3,$$

$$X_z=\kappa_2\mu(z-l)\Big\{-\left(\frac{3}{2}+\sigma\right)(a_0^2+a_1^2-x^2)+\frac{1}{2}(1-2\sigma)y^2-\frac{1}{2}(3+2\sigma)\frac{a_0^2a_1^2}{x^2+y^2}$$

$$+(3+2\sigma)\frac{a_0^2a_1^2}{(x^2+y^2)^2}x^2\Big\},$$

$$Y_z=\kappa_2\mu(z-l)\left\{(3+2\sigma)\frac{a_0^2a_1^2}{(x^2+y^2)^2}+(1+2\sigma)\right\}xy.$$

The conditions (22) and (23) determines $\varepsilon_0=0$ and κ_0, but γ is yet arbitrary and may be computed, for example, by

$$\iint Z_z x^3 dx dy=0.$$

Of course, in the special case $a_0=0$, we obtain the known result.