EXPANSION OF $P_l^m(\mu)e^{\xi\mu}$ IN A SERIES OF $P_n^m(\mu)$

Ichizo NINOMIYA

Department of Applied Physics

(Received October 21, 1952)

1. Introduction

I was confronted with this problem of expanding $P_l^m(\mu)e^{\xi\mu}$ in a series of $P_n^m(\mu)$ in my previous work, "On the Viscous Flow Past a Rotating Sphere at Small Reynolds Numbers."[1] There I found the general expression of the expansion coefficients, and utilized it for the practical computation in that work. But I avoided detailed account of this expansion in order not to digress far from the physical subject, contenting myself only with the specification of the expansion coefficients.

In this paper I am going to treat the expansion in detail.

2. Expansion of $e^{\xi\mu}$ in a Series of $P_n(\mu)$

It is a known result that

$$e^{\xi\mu} = \sum_{n=0}^{\infty}(2n+1)F_n(\xi)P_n(\mu), \tag{1}$$

where

$$F_n(\xi) = \left(\frac{\pi}{2\xi}\right)^{1/2} I_{n+1/2}(\xi). \tag{2}$$

From this definition (2) of $F_n(\xi)$, the following recurrence formulae can be easily derived from those of modified Bessel functions.

$$\frac{F_n}{\xi} = \frac{1}{2n+1}(F_{n-1} - F_{n+1}). \tag{3.1}$$

$$F_n' = \frac{1}{2n+1}(nF_{n-1} + (n+1)F_{n+1}). \tag{3.2}$$

These formulae will prove to be useful in the subsequent analysis.

Now let us write

$$P_l^m(\mu)e^{\xi\mu} = \sum_{n=m}^{\infty}(2n+1)F_{l,n}^m(\xi)P_n^m(\mu), \quad l \geqq m \geqq 0, \tag{4}$$

where

$$P_n^m(\mu) = (1-\mu^2)^{m/2}\frac{d^m P_n(\mu)}{d\mu^m}, \quad n \geqq m \geqq 0. \tag{5}$$

Then it is necessary that (4) should reduce to (1) for $l=0$; i.e.,

$$F_{0,n}^0 = F_n \tag{6}$$

3. General Expression of the Expansion Coefficients $F^m_{l,n}$

As specified in the above mentioned work, the general expression of the coefficients $F^m_{l,n}$ are given, with some inessential changes of notations, as

$$F^m_{l,n} = \frac{(n-m)!}{(n+m)!}\,\frac{m!}{2^m(2m)!}\sum_{r=m}^{n\cap l}\frac{(2r)!}{r!(r-m)!}\,\frac{(2(l+m-r))!}{(l-r)!(l+m-r)!}\,\frac{(2(n+m-r))!}{(n-r)!(n+m-r)!}\times$$
$$\times\frac{(n+l-r)!(n+l+m-r)!}{(2(n+l-r)+1)!}(2(n+l+m-2r)+1)\frac{F_{n+l+m-2r}}{\xi^m},$$
$$n\cap l \geqq m \geqq 0, \qquad (7)$$

where $n\cap l$ denotes the smaller one between n and l.

Now it seems worth while to list some results easily seen from (7). Firstly, (7) is consistent with (6). I.e., for $l=0$ (7) reduces to (6). Secondly, (7) includes Goldstein's results[2] as the two special cases for $m=0$ and $m=1$. Precisely

$$(2n+1)F^0_{l,n} = \Psi_{n,l}$$

and

$$(2n+1)F^1_{l,n} = X_{n,l}.$$

Thirdly, as a relation between $F^m_{l,n}$ and $F^m_{l,n}$ we have

$$\frac{(n+m)!}{(n-m)!}F^m_{l,n} = \frac{(l+m)!}{(l-m)!}F^m_{n,l}. \qquad (8)$$

4. Recurrence Formulae for $F^m_{l,n}$

From (7) and (3) the following recurrence formulae can be verified after some elementary calculations.

$$\begin{cases} F^{m\prime}_{l,n} = \dfrac{1}{2l+1}((l+m)F^m_{l-1,n} + (l-m+1)F^m_{l+1,n}). & (9.1)\\[2ex] F^{m\prime}_{l,n} = \dfrac{1}{2l+1}((l-m)F^m_{n,l-1} + (l+m+1)F^m_{n,l+1}). & (9.2)\end{cases}$$

$$\begin{cases} (n-m)(n+m+1)F^{m+1}_{l,n} = (l-m)(l+m+1)F^m_{l,n} \\ \qquad + \dfrac{\xi}{2l+1}((l-m)(l-m+1)F^m_{l+1,n} - (l+m)(l+m+1)F^m_{l-1,n}). & (10.1)\\[2ex] F^{m+1}_{n,l} = F^m_{n,l} + \dfrac{\xi}{2l+1}(F^m_{n,l+1} - F^m_{n,l-1}). & (10.2)\end{cases}$$

$$\begin{cases} F^{m-1}_{l,n} = F^m_{l,n} + \dfrac{\xi}{2l+1}(F^m_{l+1,n} - F^m_{l-1,n}). & (11.1)\\[2ex] (n+m)(n-m+1)F^{m-1}_{n,l} = (l+m)(l-m+1)F^m_{n,l} \\ \qquad + \dfrac{\xi}{2l+1}((l+m)(l+m+1)F^m_{n,l+1} - (l-m)(l-m+1)F^m_{n,l-1}). & (11.2)\end{cases}$$

Here a proviso should be added. That is: If it happens that the condition of (7) breaks down for any term $F^\mu_{\lambda,\nu}$ in these formulae, i.e., if it happens that $\mu \geqq \nu\cap\lambda \geqq 0$, the term in question should be omitted. Thus putting $n=0$ and $m=0$ in (10.2) and

(9.2), we obtain (3.1) and (3.2) respectively. Now we see that, in each group of the above formulae, the first one can be transformed into the second one and *vice versa* by (8), so that, as a matter of fact, it suffices to verify only one of them, say, the first one.

Incidentally we may add some results of interest from the formalistic point of view. Let us define $F_{l,n}^{-m}$ with

$$P_l^{-m}(\mu)e^{\xi\mu} = \sum_{n=m}^{\infty}(2n+1)F_{l,n}^{-m}(\xi)P_n^{-m}(\mu), \quad l \geqq m \geqq 0, \tag{4'}$$

where

$$P_n^{-m}(\mu) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(\mu), \quad n \geqq m \geqq 0. \tag{5'}$$

Then we obtain

$$F_{l,n}^{-m} = \frac{(n+m)!}{(n-m)!} \frac{(l-m)!}{(l+m)!} F_{l,n}^m, \quad l \cap n \geqq m \geqq 0. \tag{12}$$

As this is symmetric with respect to m and $-m$, we may extend the domain of m $l \cap n \geqq m \geqq 0$ to $l \cap n \geqq |m|$. This together with (8) enables us to interpret (10) and (11) as the four alternative expressions of the single proposition; e.g., (11.1). By the way we obtain from (12) and (8)

$$F_{l,n}^{-m} = F_{n,l}^m, \quad l \cap n \geqq |m|. \tag{13}$$

5. Convergency of the Series $\sum_{n=m}^{\infty}(2n+1)F_{l,n}^m P_n^m(\mu)$

Before proceeding further it is necessary to test convergency of the series $S_l^m = \sum_{n=m}^{\infty}(2n+1)F_{l,n}^m P_n^m(\mu)$ with the coefficients given by (7).

Concerning $P_n^m(\mu)$ the following estimation is available

$$|P_n^m(\mu)| < (1-\mu^2)^{-(m/2+1/4)}\left(\frac{4\pi}{n}\right)^{1/2}\frac{(n+m)!}{n!} \tag{14}$$
[3]

provided $-1<\mu<1$, $n \geqq 1$, $n-m+1>0$, and $m \geqq 0$.

Next we proceed to the estimation of $F_{l,n}^m$.

To begin with we have

$$|(2n+1)F_n(\xi)| = \frac{2^n n!}{(2n)!}|\xi|^n\left(1+\frac{\xi^2}{2(2n+3)}+\frac{\xi^4}{2\cdot4(2n+3)(2n+5)}+\cdots \textit{ ad inf.}\right)$$
$$< \frac{2^n n!}{(2n)!}|\xi|^n \sum_{r=0}^{\infty}\frac{\xi^{2r}}{r!(4n)^r},$$

i.e.,

$$|(2n+1)F_n(\xi)| < \frac{2^n n!}{(2n)!}|\xi|^n \exp\frac{\xi^2}{4n}, \quad n>0. \tag{15}$$

Therefore for $n>l$ we have

$$|(2n+1)F_{l,n}^m| < \frac{(n-m)!}{(n+m)!}\frac{m!}{2^m(2m)!}(2n+1)\sum_{r=m}^{l}A_r^m B_{l,r}^m C_{l,n;r}^m \times$$
$$\times \exp\frac{\xi^2}{4(n+l+m-2r)}\cdot|\xi|^{n+l-2r} \tag{16}$$

where, for the sake of brevity, we put

$$A_r^m = \frac{(2r)!}{r!\,(r-m)!},$$

$$B_{l,r}^m = \frac{(2(l+m-r))!}{(l-r)!\,(l+m-r)!},$$

and

$$C_{l,n;r}^m = \frac{(2(n+m-r))!}{(n-r)!\,(n+m-r)!}\,\frac{(n+l-r)!\,(n+l+m-r)!}{(2(n+l-r)+1)!}\,\frac{2^{n+l+m-2r}(n+l+m-2r)!}{(2(n+l+m-2r))!}.$$

The maxima of A_r^m, $B_{l,r}^m$, and $C_{l,n;r}^m$, with respect to r, can be easily found, taking the ratio of successive terms in each series. Thus we have

$$\mathrm{Max}_r\, A_r^m = A_l^m = \frac{(2l)!}{l!\,(l-m)!},$$

$$\mathrm{Max}_r\, B_{l,r}^m = B_{l,m}^m = \frac{(2l)!}{l!\,(l-m)!},$$

and for sufficiently large n, say, $n>N$

$$\mathrm{Max}_r\, C_{l,n;r}^m = C_{l,n;l}^m = \frac{2^{n-l+m}n!\,(n+m)!}{(n-l)!\,(2n+1)!}.$$

Evidently we have also

$$\mathrm{Max}_r \exp\frac{\xi^2}{4(n+l+m-2r)} = \exp\frac{\xi^2}{4(n-l+m)}.$$

Substituting these maxima for the corresponding terms on the right hand side in (16), we obtain

$$|(2n+1)F_{l,n}^m| < \frac{m!}{(2m)!}\left(\frac{(2l)!}{l!\,(l-m)!}\right)^2\cdot\sum_{r=0}^{l-m}\xi^{2r}\cdot\exp\frac{\xi^2}{4(n-l+m)}\cdot\frac{n!\,(n-m)!}{(2n)!}\,\frac{(2|\xi|)^{n-l}}{(n-l)!},\quad n>N. \tag{17}$$

Combining (17) with (14) we obtain

$$|(2n+1)F_{l,n}^m\, P_n^m(\mu)| < K_l^m(\mu,\,\xi_0)N^{-1/2}\exp\frac{\xi^2}{4(N-l+m)}\cdot\frac{(N+m)!\,(N-m)!}{(2N)!}\,\frac{(2\xi_0)^{n-l}}{(n-l)!}. \tag{18}$$

with the conditions

$$|\xi|<\xi_0,\quad -1<\mu<1,\quad m\geqq 0,\quad \text{and}\quad n>N$$

and the abbreviation

$$K_l^m(\mu,\,\xi_0) = (4\pi)^{1/2}(1-\mu^2)^{-(m/2+1/4)}\frac{m!}{(2m)!}\left(\frac{(2l)!}{l!\,(l-m)!}\right)^2\sum_{r=0}^{l-m}\xi_0^{2r}.$$

Hence we conclude that the series S_l^m is absolutely and uniformly convergent in any interval $|\xi|<\xi_0$ with respect to ξ.

6. Legitimacy of Term by Term Differentiation of the Series $\sum_{n=m}^{\infty}(2n+1)F_{l,n}^m P_n^m(\mu)$

Now let us examine the legitimacy of term by term differentiation of the series S_l^m with respect to ξ and μ respectively.

(i) *Legitimacy of Term by Term Differentiation with Respect to* ξ.

Differentiate each term of the series S_l^m with respect to ξ, then from (9.1) we obtain.

$$\sum_{n=m}^{\infty}(2n+1)F_{l,n}^{m\prime}P_n^m(\mu)=\frac{1}{2l+1}\sum_{n=m}^{\infty}(2n+1)((l+m)F_{l-1,n}^m+(l-m+1)F_{l+1,n}^m)P_n^m(\mu).$$

We can separate the series on the right hand side into two series with the coefficients $(2n+1)F_{l-1,n}^m$ and $(2n+1)F_{l+1,n}^m$, because both the series are absolutely convergent according to **5**.

In addition they are also uniformly convergent in any interval $|\xi|<\xi_0$ with respect to ξ. Hence term by term differentiation of the series S_l^m with respect to ξ is legitimate, whether it represents $P_l^m(\mu)e^{\xi\mu}$ or not.

(ii) *Legitimacy of Term by Term Differentiation with Respect to* μ.

We can not establish the legitimacy in the similar way as in the case (i), because we don't know whether the series of $P_n^m(\mu)$ like S_l^m is uniformly convergent with respect to μ or not, so far as the result in **5** is concerned. In this case, on the contrary, we should prove $S_l^m=P_l^m(\mu)e^{\xi\mu}$ first. The legitimacy will then be established *ex post* from (10.1) or (11.1). There some appropriate proofs of absolute convergency will be needed, but these will be feasible in the same manner as in **5**, *mutatis mutandis.*

7. Proof for $P_l^m(\mu)e^{\xi\mu}=\sum_{n=m}^{\infty}(2n+1)F_{l,n}^mP_n^m(\mu)$

With all the preliminaries settled, we now proceed to the proof of the fact that $S_l^m=P_l^m(\mu)e^{\xi\mu}$. We are going to prove this in two steps.

(i) *Proof for* $P_m^m(\mu)e^{\xi\mu}=\sum_{n=m}^{\infty}(2n+1)F_{m,n}^mP_n^m(\mu).$ (19)

Because

$$P_m^m(\mu)=\frac{(2m)!}{2^m m!}(1-\mu^2)^{m/2}$$

and

$$F_{m,n}^m=\frac{(2m)!}{2^m m!}\frac{F_n}{\xi^m}$$

from (7), we may prove
$$e^{\xi\mu}=\sum_{n=m}^{\infty}(2n+1)\frac{F_n}{\xi^m}\frac{d^mP_n(\mu)}{d\mu^m} \tag{20}$$

instead of (19). First, it is clear that (20) holds for $m=0$ because of (1). Next, assume that (20) holds for $m=k$, where k is a certain non negative integer; i.e.,

$$e^{\xi\mu}=\sum_{n=k}^{\infty}(2n+1)\frac{F_n}{\xi^k}\frac{d^kP_n(\mu)}{d\mu^k}. \tag{21}$$

Then from (3.1)

$$\sum_{n=k+1}^{\infty}(2n+1)\frac{F_n}{\xi^{k+1}}\frac{d^{k+1}P_n(\mu)}{d\mu^{k+1}}=\sum_{n=k+1}^{\infty}\frac{1}{\xi^k}(F_{n-1}-F_{n+1})\frac{d^{k+1}P_n(\mu)}{d\mu^{k+1}}.$$

Since both the series $\sum_{n=k+1}^{\infty}\frac{F_{n-1}}{\xi^k}\frac{d^{k+1}P_n(\mu)}{d\mu^{k+1}}$ and $\sum_{n=k+1}^{\infty}\frac{F_{n+1}}{\xi^k}\frac{d^{k+1}P_n(\mu)}{d\mu^{k+1}}$ are absolutely convergent as can be easily verified, we have

$$\sum_{n=k+1}^{\infty}(2n+1)\frac{F_n}{\xi^{k+1}}\frac{d^{k+1}P_n(\mu)}{d\mu^{k+1}}=\sum_{n=k}^{\infty}\frac{F_n}{\xi^k}\frac{d^k}{d\mu^k}\Big(\frac{dP_{n+1}(\mu)}{d\mu}-\frac{dP_{n-1}(\mu)}{d\mu}\Big)$$
$$=\sum_{n=k}^{\infty}(2n+1)\frac{F_n}{\xi^k}\frac{d^kP_n(\mu)}{d\mu^k}=e^{\xi\mu}.$$

Therefore (20) holds for $m=k+1$. Thus by induction (20) and, in consequence, (19) are proved.

(ii) *Proof for* $P_l^m(\mu)e^{\xi\mu}=\sum_{n=m}^{\infty}(2n+1)F_{l,n}^mP_n^m(\mu)$. (22)

First, from (19), this holds for $l=m$. Differentiating (19) with respect to ξ and applying (9.1), we obtain

$$\mu P_m^m(\mu)e^{\xi\mu}=\sum_{n=m}^{\infty}(2n+1)F_{m,n}^{m\prime}P_n^m(\mu)=\frac{1}{2m+1}\sum_{n=m}^{\infty}(2n+1)F_{m+1,n}^mP_n^m(\mu).$$

Here term by term differentiation is justified because of 6. But

$$\mu P_m^m(\mu)=\frac{1}{2m+1}P_{m+1}^m(\mu),$$

so that

$$P_{m+1}^m(\mu)e^{\xi\mu}=\sum_{n=m}^{\infty}(2n+1)F_{m+1,n}^mP_n^m(\mu).$$

I.e., (22) holds for $l=m+1$. Next, assume that (22) holds for $l\leqq k\geqq m+1$. Then particuiarly

$$P_k^m(\mu)e^{\xi\mu}=\sum_{n=m}^{\infty}(2n+1)F_{k,n}^mP_n^m(\mu),$$

and

$$P_{k-1}^m(\mu)e^{\xi\mu}=\sum_{n=m}^{\infty}(2n+1)F_{k-1,n}^mP_n^m(\mu),$$

and in consequence

$$\Big(\mu P_k^m(\mu)-\frac{k+m}{2k+1}P_{k-1}^m(\mu)\Big)e^{\xi\mu}=\sum_{n=m}^{\infty}(2n+1)\Big(F_{k,n}^{m\prime}-\frac{k+m}{2k+1}F_{k-1,n}^m\Big)P_n^m(\mu).$$

Here again term by term differentiation is justified as before.
Hereupon

$$\mu P_k^m(\mu)-\frac{k+m}{2k+1}P_{k-1}^m(\mu)=\frac{k-m+1}{2k+1}P_{k+1}^m(\mu)$$

and from (9.1)

$$F_{k,n}^{m\prime}-\frac{k+m}{2k+1}F_{k-1,n}^m=\frac{k-m+1}{2k+1}F_{k+1,n}^m$$

so that

$$P_{k+1}^m(\mu)e^{\xi\mu} = \sum_{n=m}^{\infty}(2n+1)F_{k+1,n}^m P_n^m(\mu).$$

I.e., (22) holds for $l=k+1$. This, by induction, completes the proof at last.

8. Expansion of $P_l^m(\mu)e^{i\xi\mu}$ in a Series of $P_n^m(\mu)$

The transition to the similar case of $P_l^m(\mu)^{i\xi\mu}$ can be immediately accomplished by merely formal replacement of ξ with $i\xi$. The most important aspect in this connection is the fact that the convergency of the series is not affected at all. Indeed it is even improved considerably. We could develop totally parallel arguments in this new case. But we might reasonably spare this rather unpleasant task.

Concluding this paper, I wish to express my profound gratitude to Prof. Z. Sakadi for his continual encouragement and valuable criticism.

References

1) Ninomiya, I.: Research Reports of the Faculty of Engineering, Nagoya University, **IV-2**, p. 139 (1951) (in Japanese).
2) Goldstein, S.: Proc. Roy. Soc. of London, A, Vol. **123**, p. 225 (1929).
3) Hobson, E. W.: The Theory of Spherical and Ellipsoidal Harmoics, p. 313 (1931).