

# BOUNDARY CONDITIONS OF FLOWS ACCOMPANIED WITH SLIP ON WALL SURFACES

TOYOKI KOGA

(Received October 31, 1957)

## Introduction

We have two traditional methods of approach to slip of flows near surfaces of solid bodies. The first one, being initiated by Maxwell<sup>1)</sup> and developed by Millikan<sup>2)</sup> and Epstein<sup>3)</sup>, is that we do not consider in detail the feature of flows inside the layer of thickness of length of mean free path where slip appears, but calculate the values of gross-variables on the wall surfaces under consideration according to the laws of conservation of mass, momentum and energy. The second is to calculate distribution functions of molecules near walls according to the Boltzmann-Maxwell equation<sup>4)</sup>. In both methods, the states under consideration are assumed to deviate slightly from thermal equilibrium. However, as we considered in the previous paper<sup>5)</sup>, at the same time when we must consider the effect of slip, we have to expect the deviation of considered states from thermal equilibrium is considerably large. From this view point we reconsider the first approach.

## 1. Ambiguous Points in Maxwell's Method

Let us set  $x$ -axis along the flat surface of a solid plane and  $y$ -axis perpendicular to the surface. The direction of flow coincides with that of  $x$ -axis. In Fig. 1,  $AB$  is the border-line between the free molecule layer and the domain of ordinary gasdynamics. The distance between  $AB$  and the wall is assumed to be equal to the length of the mean free path  $\lambda$ . We assume that there are two kinds of molecules which are reflected from the wall surface, one which obeys Maxwell's distribution according to the temperature of the wall and the other of specular reflection from the wall. The ratio of the number of molecules of these two kinds is given by

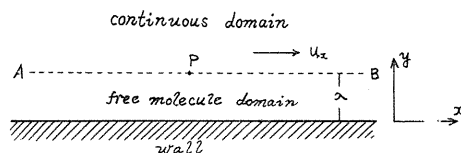


FIG. 1

$$S : (1 - S). \quad (1.1)$$

When the molecules of latter part reflect from the wall, there occurs no change in momentum component of  $x$ -direction. Consequently, those molecules belonging to this group contribute nothing to the stress ( $x$ - $y$  component of stress tensor) on the wall. On the other hand, the molecules of former part give to the wall the following amount of momentum<sup>1)</sup>:

$$S n_p m \bar{c}_p u_p / 4. \quad (1.2)$$

Here  $u_p$  is the mass velocity at  $P$  and is assumed to be *that of slip flow just on the surface*,  $\bar{c}_p$  is the mean velocity of thermal agitation,  $n_p$  the density of the number of molecules, and  $m$  the mass of a molecule. Moreover, owing to the deviation of their states from thermal equilibrium at  $P$ , they give stress to the surface as follows:

$$\frac{1}{2} S \mu \left( \frac{du}{dy} \right)_p. \quad (1.3)$$

Summing up (1.2) and (1.3), we get the entire stress on the wall and *it may be assumed that the result is equal to*  $\mu (du/dy)_p$ :

$$S \left\{ \frac{1}{2} \mu \left( \frac{du}{dy} \right)_p + \frac{1}{4} \rho \bar{c}_p u_p \right\} = \mu \left( \frac{du}{dy} \right)_p. \quad (1.4)$$

From this relation we obtain

$$u_p = \frac{2(2-S)}{S} \frac{\mu}{\rho \bar{c}} \left( \frac{du}{dy} \right)_p. \quad (1.5)$$

This is the consideration of Maxwell. Later Epstein<sup>3)</sup> studied the same problem more in detail. But the essential point is the same as that of Maxwell and also the conclusions are the same. There is a difference between Maxwell's and Epstein's theories. In the latter's theory, the distribution functions of molecules are explicitly given. In his treatment the assumption is as follows:

i) The distribution function of molecules at  $P$  in Fig. 1 is given by  $f_p(c_x, c_y, c_z)$  which includes, as parameters, the macroscopic densities of mass, of velocity, and of stress where stress\* is given by  $\mu (du/dy)$ . Of the molecules, those of negative values of  $c_y$  travel to the surface of the wall with no collision on the way.

ii) There are two kinds of reflection, one specular and the other random.

iii) The distribution of specularly reflecting molecules is presented by

$$(1-S) f_p(c_x, -c_y, c_z), \quad c_y > 0.$$

iv) The distribution of molecules reflected at random is given by

$$S f^{(0)}(c_x, c_y, c_z), \quad c_y > 0.$$

Here  $f^{(0)}$  is the Maxwell function corresponding to the temperature of the wall.

v) The sum of the two groups (iii) and (iv) is equivalent to  $f_p(c_x, c_y, c_z)$ ,  $c_y > 0$ , only with respect to the macroscopic densities of mass, momentum, energy, and stress. That is to say

---

\* Recently the present author proposed a distribution function<sup>6)</sup> taking into consideration that the gas molecules in a local cell are in a kind of equilibrium although different from thermal equilibrium, i.e. that the molecules are in equilibrium according to the condition that value of certain gross variables are given. He learned recently that Epstein<sup>3)</sup> in 1924 proposed a similar idea. There is, however, an important difference between the two ideas in that Epstein used  $\mu (du/dy)$  as the stress and the author made the stress independent variables of  $u$ .

$$\begin{aligned}
& \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \varphi f_p(c_x, c_y, c_z) dc_x dc_y dc_z \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \varphi f_p(c_x, c_y, c_z) dc_x dc_y dc_z \\
&+ (1-S) \int_{-\infty}^{+\infty} \int_0^{+\infty} \int_{-\infty}^{+\infty} \varphi f_p(c_x, -c_y, c_z) dc_x dc_y dc_z \\
&+ S \int_{-\infty}^{+\infty} \int_0^{+\infty} \int_{-\infty}^{+\infty} \varphi f^{(0)}(c_x, c_y, c_z) dc_x dc_y dc_z, \\
&\varphi = m, mc_{\xi}, mc_{\xi}c_{\eta}, \left. \begin{matrix} \xi \\ \eta \end{matrix} \right\} = x, y, z. \tag{1.6}
\end{aligned}$$

The fifth assumption may be equivalent to (1.4) of Maxwell's assumptions. In those assumptions, we find at least two ambiguous points: One is that stress is presented by  $\mu(du/dy)$ . Only when the deviation of the state from thermal equilibrium is small can we accept this assumption. It is clear that near the surface of the wall the state of the distribution of molecules may deviate considerably from Maxwell's distribution<sup>5)</sup>. Consequently it is dangerous to present the stress by  $\mu(du/dy)$ .

The other ambiguous point is as follows: There is no a priori reason for making the right-hand side equal to the left-hand side in equation (1.4) or, according to Epstein's expression, for adopting assumption (v). This consideration seems to be vital in the theory of slip. In the next section let us discuss this point more in detail.

## 2. A Proposal for the Theory of Slip Flow

The two ambiguous points excepted, as set forth in the latter part of Section 1, we adopt the same assumptions as Epstein's given in section 1:

i) Let us present the coarse-grained distribution at  $P$  in Fig. 1 as follows;

$$f_p(c_x, c_y, c_z, \rho_p, \mathbf{u}_p, \mathbf{T}_p). \tag{2.1}$$

Here  $\mathbf{T}_p$  denotes the stress at  $P$ , independently given, and is not derived from  $\mathbf{u}_p$ , and function  $f_p$  gives the following relations:

$$\begin{aligned}
\rho_p &= \iiint_{-\infty}^{+\infty} m f_p dc_x dc_y dc_z, \\
u_{p\xi} &= \iiint_{-\infty}^{+\infty} c_{\xi} f_p dc_x dc_y dc_z, \\
T_{p\xi\eta} &= \iiint_{-\infty}^{+\infty} m C_{\xi} C_{\eta} f_p dc_x dc_y dc_z, \\
C_{\xi} &= c_{\xi} - u_{\xi}, \quad \left. \begin{matrix} \xi \\ \eta \end{matrix} \right\} = x, y, z. \tag{2.2}
\end{aligned}$$

ii) There are two kinds of reflections of molecules on the surface of a solid wall, one specular and the other random. The ratio between their number is given by

$$(1-S) : S. \tag{2.3}$$

iii) The distribution of specularly reflecting molecules is given by

$$(1 - S)f_p(c_x, -c_y, c_z), \quad c_y > 0. \quad (2.4)$$

iv) The distribution of molecules reflecting at random is given by

$$Sf^{(0)}(c_x, c_y, c_z). \quad (2.5)$$

From these assumptions, we calculate the stress  $\mathbf{T}_w$  which, by measuring the force  $\mathbf{T}'_w$  acting on an elementary area of the surface of the solid wall, can be observed experimentally. This calculation is as follows:

$$\begin{aligned} -T'_{w\bar{z}\eta} &= T_{w\bar{z}\eta} \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^0 \int_{-\infty}^{+\infty} mC_{\bar{z}}C_{\eta}f_p dc_x dc_y dc_z \\ &+ S \int_{-\infty}^{+\infty} \int_0^{+\infty} \int_{-\infty}^{+\infty} mC_{\bar{z}}C_{\eta}f^{(0)} dc_x dc_y dc_z \\ &+ (1 - S) \int_{-\infty}^{+\infty} \int_0^{+\infty} \int_{-\infty}^{+\infty} mC_{\bar{z}}C_{\eta}f_p(c_x, -c_y, c_z) dc_x dc_y dc_z, \\ C_{\bar{z}} &= c_{\bar{z}} - u_{\bar{z}}, \quad \left. \begin{matrix} \bar{z} \\ \eta \end{matrix} \right\} = x, y, z. \end{aligned} \quad (2.6)$$

Here the  $u$ 's are given by (2.7) and it should be emphasized that  $\mathbf{T}_w$  is different from  $\mathbf{T}_p$ , because the fifth of Epstein's assumptions is excluded. We can measure experimentally  $\mathbf{T}'_w$  but not  $\mathbf{T}_p$ . In the same way the velocity of flow just on the surface of a solid body is given by

$$\begin{aligned} u_{w\bar{z}} &= \int_{-\infty}^{+\infty} \int_{-\infty}^0 \int_{-\infty}^{+\infty} c_{\bar{z}}f_p dc_x dc_y dc_z \\ &+ S \int_{-\infty}^{+\infty} \int_0^{+\infty} \int_{-\infty}^{+\infty} c_{\bar{z}}f^{(0)} dc_x dc_y dc_z \\ &+ (1 - S) \int_{-\infty}^{+\infty} \int_0^{+\infty} \int_{-\infty}^{+\infty} c_{\bar{z}}f_p(c_x, -c_y, c_z) dc_x dc_y dc_z. \end{aligned} \quad (2.7)$$

Here we know that according to the law of mass conservation

$$u_{wy} = 0, \quad (2.8)$$

although we do not necessarily apply the laws of momentum and energy conservation. Further, the density of mass just on the wall surface is obtained as follows:

$$\begin{aligned} \rho_w &= \int_{-\infty}^{+\infty} \int_{-\infty}^0 \int_{-\infty}^{+\infty} mf_p dc_x dc_y dc_z \\ &+ S \int_{-\infty}^{+\infty} \int_0^{+\infty} \int_{-\infty}^{+\infty} mf^{(0)} dc_x dc_y dc_z \\ &+ (1 - S) \int_{-\infty}^{+\infty} \int_0^{+\infty} \int_{-\infty}^{+\infty} mf_p(c_x, -c_y, c_z) dc_x dc_y dc_z. \end{aligned} \quad (2.9)$$

According to the results (2.6), (2.7) and (2.9),  $\rho_w$ ,  $\mathbf{u}_w$ ,  $\mathbf{T}_w$  may be given in terms of  $\rho_p$ ,  $\mathbf{u}_p$ ,  $\mathbf{T}_p$  or vice versa. Here those which we can measure experimentally on the wall are not  $\rho_p$ ,  $\mathbf{u}_p$ ,  $\mathbf{T}_p$  but  $\rho_w$ ,  $\mathbf{u}_w$ ,  $\mathbf{T}_w$ . However the variables in terms of

which the macroscopic law of the external field of flow is given, are the continuation of  $\rho_p$ ,  $\mathbf{u}_p$ ,  $\mathbf{T}_p$ . This means that when we treat the gas flow as a continuous field presented by the equation of continuity, equation of momentum, etc., the boundary condition of the field must be given by the values of  $\rho_p$ ,  $\mathbf{u}_p$ ,  $\mathbf{T}_p$ .

### Summary and Conclusion

It is emphasized that there is no a priori reason that we, as Maxwell and others did, assume the coincidence between the respective values of density, velocity and stress of gas flow just on the surface of a wall and their values at the point apart from the wall the length of mean free path.

Instead of that assumption, we derived the equations which present the relations between the two sets of values of our gross variables, i.e. of density, velocity and stress.

According to these equations, we can give the boundary conditions of the field of gas outside the slip layer in terms of the values of gross variables measured experimentally just on the wall.

### References

- 1) E. H. Kennard: *Kinetic Theory of Gases* (McGraw-Hill Book Company, New York and London, 1938), First Edition, p. 295.
- 2) R. A. Millikan: *Phys. Rev.* **21**, 217 (1923).
- 3) P. S. Epstein: *Phys. Rev.* **23**, 710 (1924).
- 4) C. S. Wang Chang and G. E. Uhlenbeck: University of Michigan, Eng. Res. Inst., Project M 999, Sept., 1953, Project 1999, June, 1954.
- 5) T. Koga: Published in the present issue.
- 6) T. Koga: *J. Chem. Phys.* **22**, 1633 (1954).