HEAT TRANSFER IN THE RAREFIED GASES

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1. Introduction

Gas dynamics can be classified into certain regions as indicated in reference (1). The parameters defining these regions are the Knudsen number K the ratio of the molecular mean free path, λ, to the characteristic length of the body, a, and the parameter of non-uniformity of gas, N. The parameter N indicates the degree of the departure of molecular distribution from the one of thermal equilibrium; for example, the parameter N for high shear stress flow is given by P_{xy}/P_{xx}, where P_{xy} is shear stress and P_{xx} is normal stress.

For treating the problem of dynamics of slightly non-uniform rarefied gases, $N \ll 1$, $K \sim 1$, we have proposed a method[2] based on the Maxwell-Boltzmann equation. The essential point is that the distribution of molecules impinging upon a wall is presented by a function different from that of reflecting molecules from the wall. This method seems to be suited for the heat transfer problem, too.

In the heat transfer problem, the parameter N may be defined[3] as

$$N = \frac{q}{\rho c_v T \bar{c}}, \qquad \text{(i)}$$

where $\bar{c}$ the mean velocity of molecules, c_v the specific heat at constant volume, ρ density of gas, T temperature and q the rate of heat conduction. For the problem of heat transfer between two flat plates,

$$q = \kappa \frac{T_2 - T_1}{a}, \qquad \text{(ii)}$$

where T_2 and T_1 are the temperature of each plate, a the distance between two flat plates and κ the heat conductivity. From the kinetic theory of gas, the heat conductivity is given by

$$\kappa = 1.25\, \rho \bar{c} c_v \lambda.$$

Using these relations, the parameter N becomes

$$N = 1.25 \frac{\lambda}{a} \frac{T_2 - T_1}{T}. \qquad \text{(iii)}$$

Accordingly, when we confine ourselves to the slightly non-uniform state, the following relation must be satisfied;

$$1.25\frac{\lambda}{a}\frac{T_2-T_1}{T}\ll 1. \qquad \text{(iv)}$$

Here we remark that the above relation gives a rough criterion, because we have used the rate of heat conduction obtained by continuum theory.

In this paper we will treat the heat transfer problem between two flat plates for the case, $\frac{T_2-T_1}{T}\ll 1$. Then, for the region $\frac{\lambda}{a}\leqq 1$, we can expect that our method for slightly non-uniform gas is valid. As for the region $\frac{\lambda}{a}>1$, criterion (iv) seems to be insignificant since the relation (ii) is not exact in this case. However, there is another reason that our present theory seems to be correct even for the region, $\frac{\lambda}{a}>1$: As we see in the results of our calculation, our results for the case, $\frac{\lambda}{a}\to\infty$, agree with those of the free molecule theory which gives correct results for all values of $\frac{T_2-T_1}{T}$.

2. Fundamental Equations for Heat Transfer

The Maxwell-Boltzmann equation for the molecular velocity distribution function in a simple gas in the absence of external forces can be written in the form[4]

$$\frac{\partial f}{\partial t}+\mathbf{c}\cdot\frac{\partial f}{\partial \mathbf{r}}$$
$$=\int_0^{2\pi}\int_0^{\pi/2}\int\{f(\mathbf{r},\ \mathbf{c}',\ t)f(\mathbf{r},\ \mathbf{c}_1',\ t)-f(\mathbf{r},\ \mathbf{c},\ t)f(\mathbf{r},\ \mathbf{c}_1,\ t)\}\,VB(\theta,\ V)\sin\theta\, d\mathbf{c}_1 d\theta d\varepsilon,$$

where $f(\mathbf{r},\ \mathbf{c},\ t)$ is the space and velocity distribution function, $f(\mathbf{r},\ \mathbf{c},\ t)\,dxdydz$ being the expectation value at time t of the number of molecules, in a space interval $dxdydz$ containing the point $\mathbf{r}$, which have velocities in the interval about the velocity $\mathbf{c}$:

$$\int\ldots d\mathbf{c}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots dc_x dc_y dc_z:$$

$\mathbf{c}$ and $\mathbf{c}_1$ are the initial velocities of a pair of molecules which have final velocities $\mathbf{c}'$ and $\mathbf{c}_1'$ respectively after a collision in which the relative velocity vector turns through an angle θ, and the plane containing the initial and final relative velocity vectors make an angle ε with a plane through the final relative velocity vector and a fixed line: $V=|\mathbf{c}-\mathbf{c}_1|=|\mathbf{c}'-\mathbf{c}_1'|$ is the magnitude of the relative velocity of the colliding molecules: $B(\theta,\ V)$ is the differential scattering cross-section for scattering angle θ at relative velocity V. For molecules which are elastic spheres of diameter σ, $B(\theta,\ V)$ becomes $\sigma^2\cos\theta$.

Since we treat the steady one dimensional problem in this paper the left hand side of the Maxwell-Boltzmann equation becomes $c_y\cdot\frac{df}{dy}$. In the following sections, for convenience, we use the distribution function in the form $f(\mathbf{r}, \mathbf{c}, t) = f$, $f(\mathbf{r}, \mathbf{c}', t) = f'$, $f(\mathbf{r}, \mathbf{c}'_1, t) = f'_1$ and $f(\mathbf{r}, \mathbf{c}_1, t) = f_1$.

Consider the steady heat transfer in the rarefied gas in contact with a flat wall, where y is the distance normal to the wall and the heat flux normal to the wall is q. And we assume the components of heat flux vector parallel to the wall does not exsist. Then q is independent of y; $dq/dy = 0$.

Grad[5] expresses the distribution function in Hermite polynomials. For the case in which the stress and mean velocity vanish, it becomes

$$f = \frac{p}{m(2\pi)^{3/2}(RT)^{5/2}} e^{-\frac{c^2}{2RT}} \left\{1 - \frac{qc_y}{2pRT} + \frac{qc_y c^2}{10p(RT)^2}\right\}, \tag{1}$$

where m is the mass of a molecule of the gas; R, the universal gas constant; p, the pressure of the gas; and T, the temperature.

However, Grad's distribution function is inadequate to satisfy the boundary conditions, as we indicated in the previous paper,[2] because on the boundary the incident molecules and reflecting ones have different distributions. To satisfy the boundary conditions, we must express the distribution function by the two different functions for impinging molecules and reflecting molecules respectively. Considering that the distributin function can be expressed by Eq. (1) sufficiently apart from the boundary, we assume f to be of the form

$$f = f^+ + f^-$$

where

$$f = f^+ \quad \text{for} \quad c_y > 0,$$

$$f = f^- \quad \text{for} \quad c_y < 0, \tag{2}$$

with

$$f^\pm = \frac{p^\pm}{m(2\pi)^{3/2}(RT^\pm)^{5/2}} e^{-\frac{c^2}{2RT^\pm}} \left\{1 - \frac{q^\pm c_y}{2p^\pm RT^\pm} + \frac{q^\pm c_y c^2}{10p^\pm (RT^\pm)^2}\right\}.$$

In these expressions, superscripts $+$ and $-$ on a function are for the reflecting molecules and for the incident molecules, respectively, and these superscripts will be used with the same meaning hereafter. Here we put

$$T^\pm = T_0(1 + t^\pm),$$

$$p^\pm = p_0(1 + k^\pm),$$

where T_0 is a standard temperature and p_0 is a standard pressure. We assume that $t \ll 1$ and $k^\pm \ll 1$. Then neglecting the second and higher order terms of $t^\pm$, $k^\pm$, $q^\pm$, Eq. (2) approximately becomes

$$f^\pm = \frac{p_0}{m(2\pi)^{3/2}(RT_0)^{5/2}} e^{-\frac{c^2}{2RT_0}} \left\{1 + k^\pm - \frac{5}{2}t^\pm + \frac{t^\pm c^2}{2RT_0} - \frac{q^\pm c_y}{2p_0 RT_0} + \frac{q^\pm c_y c^2}{10p_0(RT_0)^2}\right\}, \tag{3}$$

where the parameters k^+, k^-, t^+, t^-, q^+ and q^- are the unknown functions of y.

The pressure p_0 and heat flux q are related to the contractions of the moment tensor of the distribution function, that is,

$$m\int c^2 f d\mathbf{c} = 3p_0,$$

$$m\int c_y c^2 f d\mathbf{c} = q.$$

It is clear that the density ρ is given by

$$m\int f d\mathbf{c} = \rho.$$

Since we assume that the macroscopic velocity is 0,

$$m\int c_y f d\mathbf{c} = 0.$$

Substituting Eq. (2) into the above expressions, we get

$$15(k^+ + k^-) + \frac{2\sqrt{2}}{p_0\sqrt{RT_0}\sqrt{\pi}}(q^+ - q^-) = 0, \tag{4}$$

$$(k^+ - k^-) + \frac{1}{2}(t^+ - t^-) + \frac{\sqrt{\pi}}{4\sqrt{2RT_0}p_0}(q^+ + q^-) = \frac{q\sqrt{\pi}}{2p_0\sqrt{2RT_0}}, \tag{5}$$

$$(k^+ - k^-) = \frac{1}{2}(t^+ - t^-), \tag{6}$$

$$5(k^+ + k^-) - 5(t^+ + t^-) - \frac{1}{\sqrt{\pi}\sqrt{2RT_0}p_0}(q^+ - q^-) = -10t. \tag{7}$$

Here the temperature T of the gas at any point has been expressed as $T = T_0(1+t)$. For convenience of later calculation, we rewrite these equations as follows;

$$q^+ + q^- = 2q - \frac{8\sqrt{2RT_0}p_0}{\sqrt{\pi}}(k^+ - k^-), \tag{8}$$

$$q^+ - q^- = -\frac{15p_0\sqrt{RT_0}\sqrt{\pi}}{2\sqrt{2}}(k^+ + k^-), \tag{9}$$

$$t^+ - t^- = 2(k^+ - k^-), \tag{10}$$

$$t^+ + t^- = \frac{7}{4}(k^+ + k^-) + 2t. \tag{11}$$

On the other hand, we can derive the following equations from the Maxwell-Boltzmann equation. Multiplying both sides of the Maxwell-Boltzmann equation by mc_yc^2, and integrating them over the domain where $c_y > 0$ and over the domain where $c_y < 0$, we have the following equations;

$$\frac{5}{2}RT_0p_0\frac{dk^+}{dy} + \frac{5}{2}RT_0p_0\frac{dt^+}{dy} + \frac{9}{5}\frac{\sqrt{2RT_0}}{\sqrt{\pi}}\frac{dq^+}{dy} = I^+, \tag{12}$$

$$\frac{5}{2}RT_0p_0\frac{dk^-}{dy} + \frac{5}{2}RT_0p_0\frac{dt^-}{dy} - \frac{9}{5}\frac{\sqrt{2RT_0}}{\sqrt{\pi}}\frac{dq^-}{dy} = I^-, \tag{13}$$

where

$$I^+ = \int_0^{2\pi}\int_0^{\pi/2}\iint_{c_y>0} mc_y^2(f'f_1' - ff_1)\, V\cos\theta\sin\theta\sigma^2 d\mathbf{c}d\mathbf{c}_1 d\theta d\varepsilon, \tag{14}$$

$$I^- = \int_0^{2\pi}\int_0^{\pi/2}\iint_{c_y<0} mc_y^2(f'f_1' - ff_1)\, V\cos\theta\sin\theta\sigma^2 d\mathbf{c}d\mathbf{c}_1 d\theta d\varepsilon: \tag{15}$$

$$\int_{c_y>0} \ldots d\mathbf{c} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} \ldots dc_y\, dc_x\, dc_z, \quad \int_{c_y<0} \ldots d\mathbf{c} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{0} \ldots dc_y dc_x dc_z.$$

Similarly, multiplying by mc^2 and integrating them, we get

$$\frac{2\sqrt{2RT_0}p_0}{\sqrt{\pi}}\frac{dk^+}{dy} + \frac{\sqrt{2RT_0}p_0}{\sqrt{\pi}}\frac{dt^+}{dy} + \frac{1}{2}\frac{dq^+}{dy} = R^+, \tag{16}$$

$$-\frac{2\sqrt{2RT_0}p_0}{\sqrt{\pi}}\frac{dk^-}{dy} - \frac{\sqrt{2RT_0}p_0}{\sqrt{\pi}}\frac{dt^-}{dy} + \frac{1}{2}\frac{dq^-}{dy} = R^-, \tag{17}$$

where

$$R^+ = \int_0^{2\pi}\int_0^{\pi/2}\iint_{c_y>0} mc^2(f'f_1' - ff_1)\, V\cos\theta\sin\theta\sigma^2 d\mathbf{c}d\mathbf{c}_1 d\theta d\varepsilon, \tag{18}$$

$$R^- = \int_0^{2\pi}\int_0^{\pi/2}\iint_{c_y<0} mc^2(f'f_1' - ff_1)\, V\cos\theta\sin\theta\sigma^2 d\mathbf{c}d\mathbf{c}_1 d\theta d\varepsilon. \tag{19}$$

From these resulting equations, we can make the equations $\{(12)+(13)\}$, $\{(12)-(13)\}$ and $\{(16)-(17)\}$. Here we note that equation $\{(16)+(17)\}$ coincides with the equation $dq/dy=0$. Thus we have seven independent equations (8), (9), (10), (11), $\{(12)+(13)\}$, $\{(12)-(13)\}$ and $\{(16)-(17)\}$. Now we can solve this set of equations for the functions k^+, k^-, t^+, t^-, q^+, q^- and t. The final object of our calculation is to express t as a function of y. But before doing this, we must express the collision integrals as the functions of k^+, k^-, t^+, t^-, q^+, q^-, t.

3. Calculations of the Collision Terms

First, we shall consider I^+. Substituting Eq. (2) into $f'f_1' - ff_1$, we obtain

$$\begin{aligned}
f'f_1' - ff_1 = {} & \frac{1}{10p_0(RT_0)^2}\left(\frac{q^+ + q^-}{2}\right)(c_{y1}'c_1'^2 + c_y'c'^2 - c_{y1}c_1^2 - c_y c^2) f_m f_{m1} \\
& + \left(\frac{k^+ - k^-}{2} - \frac{5}{2}\frac{t^+ - t^-}{2}\right)(2f_m'^+ f_{m1}'^+ - 2f_m'^- f_{m1}'^- - 2f_m^+ f_{m1}^+ + 2f_m^- f_{m1}^-) \\
& + \frac{1}{2RT_0}\left(\frac{t^+ - t^-}{2}\right)\{c'^2(f_m'^+ f_{m1}' - f_m'^- f_{m1}') + c_1'^2(f_m' f_{m1}'^+ - f_m' f_{m1}'^-) \\
& \qquad - c^2(f_m^+ f_{m1} - f_m^- f_{m1}) - c_1^2(f_m f_{m1}^+ - f_m f_{m1}^-)\} \\
& - \frac{1}{2p_0 RT_0}\left(\frac{q^+ - q^-}{2}\right)\{c_y'(f_m'^+ f_{m1}' - f_m'^- f_{m1}') + c_{y1}'(f_m' f_{m1}'^+ - f_m' f_{m1}'^-) \\
& \qquad - c_y(f_m^+ f_{m1} - f_m^- f_{m1}) - c_{y1}(f_m f_{m1}^+ - f_m f_{m1}^-)\} \\
& + \frac{1}{10p_0(RT_0)^2}\left(\frac{q^+ - q^-}{2}\right)\{c_y'c'^2(f_m'^+ f_{m1}' - f_m'^- f_{m1}') + c_{y1}'c_1'^2(f_m' f_{m1}'^+ - f_m' f_{m1}'^-) \\
& \qquad - c_y c^2(f_m^+ f_{m1} - f_m^- f_{m1}) - c_{y1}c_1^2(f_m f_{m1}^+ - f_m f_{m1}^-)\},
\end{aligned} \tag{20}$$

where

$$f_m^+ = f_m \quad \text{for} \quad c_y > 0, \qquad f_m^+ = 0 \quad \text{for} \quad c_y < 0,$$

$$f_m^- = f_m \quad \text{for} \quad c_y < 0, \qquad f_m^- = 0 \quad \text{for} \quad c_y > 0,$$

$$f_m = \frac{p_0}{m(2\pi)^{3/2}(RT_0)^{5/2}} e^{-\frac{c^2}{2RT_0}},$$

To obtain Eq. (20), we have used the laws of conservation of energy and momentum,

$$\frac{m}{2}(c'^2+c_1'^2)=\frac{m}{2}(c^2+c_1^2),$$
$$m(\mathbf{c}'+\mathbf{c}_1')=m(\mathbf{c}+\mathbf{c}_1).$$

Substituting Eq. (20) into (14), we have

$$I^+=\left(\frac{q^++q^-}{2}\right)\frac{32\sqrt{2}}{5\sqrt{RT_0}}\frac{p_0\sigma^2}{m}A^++\left(\frac{k^+-k^-}{2}-\frac{5}{2}\frac{t^+-t^-}{2}\right)\frac{32p_0^2\sigma^2}{m}L^+(1,\ c_yc^2)$$
$$+\left(\frac{t^+-t^-}{2}\right)\frac{32p_0^2\sigma^2}{m}L^+(c^2,\ c_yc^2)-\left(\frac{q^+-q^-}{2}\right)\frac{16\sqrt{2}p_0\sigma^2}{m\sqrt{RT_0}}L^+(c_y,\ c_yc^2)$$
$$+\left(\frac{q^+-q^-}{2}\right)\frac{32p_0\sigma^2}{5m}L^+(c_yc^2,\ c_yc^2). \tag{21}$$

where the notations $L^+(\phi(\mathbf{c}),\ \varphi(\mathbf{c}))$ and A^+ are defined by

$$\left.\begin{aligned}L^+(\phi(\mathbf{c}),\ \varphi(\mathbf{c}))&=\int_0^{2\pi}\int_0^{\pi/2}\iint_{c_y>0}\varphi(\mathbf{c})\{\phi(\mathbf{c}')(g'^+g_1'-g'^-g)+\phi(\mathbf{c}_1')(g'g_1'^+-g'g_1'^-)\\&\qquad-\phi(\mathbf{c})(g^+g_1-g^-g_1)-\phi(\mathbf{c}_1)(gg_1^+-gg_1^-)\}V\cos\theta\sin\theta d\mathbf{c}d\mathbf{c}_1d\theta d\varepsilon,\\A^+&=\int_0^{2\pi}\int_0^{\pi/2}\iint_{c_y>0}c_yc^2(c_y'c'^2+c_{y1}'c_1'^2-c_{y1}c_1^2-c_yc^2)gg_1V\cos\theta\sin\theta d\mathbf{c}d\mathbf{c}_1d\theta d\varepsilon,\end{aligned}\right\}\tag{22}$$

where

$$g=\frac{1}{(2\pi)^{3/2}}e^{-c^2}.$$

The integral $L^+(\phi(\mathbf{c}),\ \varphi(\mathbf{c}))$ and A^+ have no dimension.

Similarly, we have

$$I^-=\left(\frac{q^++q^-}{2}\right)\frac{32\sqrt{2}p_0\sigma^2}{5m\sqrt{RT_0}}A^-+\left(\frac{k^+-k^-}{2}-\frac{5}{2}\frac{t^+-t^-}{2}\right)\frac{32p_0^2\sigma^2}{m}L^-(1,\ c_yc^2)$$
$$+\left(\frac{t^+-t^-}{2}\right)\frac{32p_0^2\sigma^2}{m}L^-(c^2,\ c_yc^2)-\left(\frac{q^+-q^-}{2}\right)\frac{16\sqrt{2}p_0\sigma^2}{m\sqrt{RT_0}}L^-(c_y,\ c_yc^2)$$
$$+\left(\frac{q^+-q^-}{2}\right)\frac{32p_0\sigma^2}{5m\sqrt{RT_0}}L^-(c_yc^2,\ c_yc^2), \tag{23}$$

where $L^-(\phi(\mathbf{c}),\ \varphi(\mathbf{c}))$ and A^- are given by changing the integrating region of $L^+(\phi(\mathbf{c}),\ \varphi(\mathbf{c}))$, A^+ into $c_y<0$. Substituting (8), (9), (10) and (11) into (21) and (23), we have

$$I^\pm=\frac{32\sqrt{2}p_0\sigma^2A^+}{5m\sqrt{RT_0}}q+(k^+-k^-)\frac{p_0\sigma^2}{m}\left\{-\frac{256}{5\sqrt{\pi}}A^\pm-64L^\pm(1,\ c_yc^2)+32L^\pm(c^2,\ c_yc^2)\right\}$$
$$+(k^+-k^-)\frac{60\sqrt{\pi}p_0^2\sigma^2}{\sqrt{2}m}\left\{\sqrt{2}L^\pm(c_y,\ c_yc^2)-\frac{2}{5}L^\pm(c_yc^2,\ c_yc^2)\right\}. \tag{24}$$

Using Eq. (20), the collision integrals R^+ and R^- can be expressed by

$$R^{\pm}=\frac{32p_0\sigma^2B^{\pm}}{5m\sqrt{RT_0}}q+(k^+-k^-)\frac{16\sqrt{2}p_0^2\sigma^2}{m\sqrt{RT_0}}\left\{-\frac{8}{5\sqrt{\pi}}B^{\pm}-2L^{\pm}(1,c^2)+L^{\pm}(c^2,c^2)\right\}$$
$$+(k^++k^-)\frac{60p_0^2\sigma^2\sqrt{\pi}}{\sqrt{2}\sqrt{RT_0}m}\left\{L^{\pm}(c_y,c^2)-\frac{5}{2}L^{\pm}(c_yc^2,c^2)\right\}, \tag{25}$$

where

$$B^{\pm}=\int_0^{2\pi}\int_0^{\pi/2}\iint_{\substack{c_y>0\\c_y<0}}c^2(c'_{y1}c_1'^2+c'_yc'^2-c_{y1}c_1^2-c_yc^2)gg_1V\cos\theta\sin\theta\,d\mathbf{c}d\mathbf{c}_1d\theta d\varepsilon. \tag{26}$$

4. A Solution of Equations

The next relations about $L^{\pm}(\phi(\mathbf{c}),\varphi(\mathbf{c}))$ and $A^{\pm}$ in I^+ and I^- are verified with ease:

$$\left.\begin{aligned}&A^+=A^-,\\&L^+(1,c_yc^2)=L^-(1,c_yc^2),\\&L^+(c^2,c_yc^2)=L^-(c^2,c_yc^2),\\&L^+(c_y,c_yc^2)=-L^-(c_y,c_yc^2),\\&L^+(c_yc^2,c_yc^2)=-L^-(c_yc^2,c_yc^2).\end{aligned}\right\} \tag{27}$$

Accordingly the equations $\{(12)+(13)\}$ and $\{(12)-(13)\}$ become

$$\frac{5p_0RT_0}{2}\frac{d}{dy}(k^++k^-)+\frac{5p_0RT_0}{2}\frac{d}{dy}(t^++t^-)+\frac{9}{5}\frac{\sqrt{2RT_0}}{\sqrt{\pi}}\frac{d}{dy}(q^+-q^-)$$
$$=\frac{64\sqrt{2}p_0\sigma^2A^+}{5m\sqrt{RT_0}}q+\alpha\frac{p_0^2\sigma^2}{m}(k^+-k^-), \tag{28}$$

$$\frac{5p_0RT_0}{2}\frac{d}{dy}(k^+-k^-)+\frac{5p_0RT_0}{2}\frac{d}{dy}(t^+-t^-)+\frac{9}{5}\frac{\sqrt{2RT_0}}{\sqrt{\pi}}\frac{d}{dy}(q^++q^-)$$
$$=(k^++k^-)\frac{p_0^2\sigma^2}{m}\beta, \tag{29}$$

where

$$\alpha=\left\{-\frac{512}{5\sqrt{\pi}}A^+-128L^+(1,c_yc^2)+64L^+(c^2,c_yc^2)\right\},$$
$$\beta=\frac{60\sqrt{\pi}}{\sqrt{2}}\left\{2\sqrt{2}L^+(c_y,c_yc^2)-\frac{4}{5}L^+(c_yc^2,c_yc^2)\right\}.$$

Similarly, using the relations

$$\begin{aligned}&B^+=-B^-\\&L^+(1,c^2)=-L^-(1,c^2),\\&L^+(c^2,c^2)=-L^-(c^2,c^2),\\&L^+(c_y,c^2)=L^-(c_y,c^2),\\&L^+(c_yc^2,c^2)=L^-(c_yc^2,c^2),\end{aligned}$$

the equation $\{(16)-(17)\}$ becomes

$$\frac{2\sqrt{2}p_0\sqrt{RT_0}}{\sqrt{\pi}}\frac{d}{dy}(k^+ + k^-) + \frac{p_0\sqrt{2RT_0}}{\sqrt{\pi}}\frac{d}{dy}(t^+ + t^-) + \frac{1}{2}\frac{d}{dy}(q^+ - q^-)$$
$$= \frac{64p_0\sigma^2B^+}{5mRT_0}q + (k^+ - k^-)\frac{p_0^2\sigma^2}{m\sqrt{RT_0}}\gamma, \tag{30}$$

where

$$\gamma = 32\sqrt{2}\left\{-\frac{8}{5\sqrt{\pi}}B^+ - 2L^+(1, c^2) + L^+(c^2, c^2)\right\}.$$

Eliminating $(t^+ + t^-)$, $(t^+ - t^-)$, $(q^+ + q^-)$ and $(q^+ - q^-)$ from these equations with the aid of Eqs. (8), (9), (10) and (11), we have

$$-\frac{53}{8}p_0RT_0\frac{d}{dy}(k^+ + k^-) + 5p_0RT_0\frac{dt}{dy} = \frac{64\sqrt{2}p_0\sigma^2A^+}{5m\sqrt{RT_0}}q + \alpha\frac{p_0^2\sigma^2}{m}(k^+ - k^-), \tag{31}$$

$$\left(\frac{15}{2} - \frac{144}{5\pi}\right)p_0RT_0\frac{d}{dy}(k^+ - k^-) = \beta\frac{p_0^2\sigma^2}{m}(k^+ + k^-), \tag{32}$$

$$\frac{15}{8}\frac{p_0\sqrt{RT_0}\sqrt{2}}{\sqrt{\pi}}(2-\pi)\frac{d}{dy}(k^+ + k^-) + \frac{2p_0\sqrt{2RT_0}}{\sqrt{\pi}}\frac{dt}{dy}$$
$$= \frac{64p_0\sigma^2B^+}{5mRT_0}q + \frac{\gamma p_0^2\sigma^2}{m\sqrt{RT_0}}(k^+ - k^-). \tag{33}$$

We consider the heat transfer problem between two plates placed at $y = -a$ and $y = a$. In this case, the boundary condition at $y = -a$ can be written

$$f^+ = nf^-(-c_y) + \frac{kp_0}{m(2\pi)^{3/2}(RT_w)^{5/2}}e^{-\frac{c^2}{2RT_w}}.$$

This equation shows that a fraction n of the incident molecules is specularly reflected and the remaining molecules are absorbed by the wall and re-emitted with a Maxwellian distribution at the temperature of the wall T_w. If we take the temperature of gas at $y = 0$ as a standard temperature and assume that $t_w \ll 1$, from the above equation, we have

$$f_m\left\{1 + k^+ - \frac{5}{2}t^+ + \frac{t^+c^2}{2RT_0} - \frac{q^+c_y}{2p_0RT_0} + \frac{q^+c_yc^2}{10p_0(RT_0)^2}\right\}$$
$$= nf_m\left\{1 + k^- - \frac{5}{2}t^- + \frac{t^-c^2}{2RT_0} + \frac{q^-c_y}{2p_0RT_0} - \frac{q^-c_yc^2}{10p_0(RT_0)^2}\right\}$$
$$+ kf_m\left\{1 - \frac{5}{2}t_w + \frac{t_wc^2}{2RT_0}\right\}.$$

Comparison of both sides yields that

$$\left.\begin{aligned} k^+ &= n + nk^- + k - 1, \\ t^+ &= nt^- + kt_w, \\ q^+ &= -nq^-. \end{aligned}\right\} \tag{34}$$

Similarly, we have the conditions at $y = a$,

$$\left.\begin{aligned} k^- &= n + nk^+ + k - 1, \\ t^- &= nt^+ - kt_w, \\ q^- &= -nq^+. \end{aligned}\right\} \tag{35}$$

In the first equations of (34) and (35), we assum that k^+, k^- are so small that they can be neglected. Then we have $k=1-n$. Using Eqs. (8), (9), (10), (11), (34) and (35), the boundary conditions for $(k^+ + k^-)$ and $(k^+ - k^-)$ become

$$q = \frac{4\sqrt{2RT_0}p_0}{\sqrt{\pi}}(k^+ - k^-) + \frac{15\sqrt{\pi}(1-n)}{4\sqrt{2}(1+n)}p_0\sqrt{RT_0}(k^+ + k^-), \tag{36}$$

$$t = \frac{1+n}{1-n}(k^+ - k^-) - \frac{7}{8}(k^+ + k^-) - t_w, \qquad \text{at} \quad y = -a, \tag{37}$$

$$q = \frac{4\sqrt{2RT_0}p_0}{\sqrt{\pi}}(k^+ - k^-) - \frac{15\sqrt{\pi}}{4\sqrt{2}}\frac{1-n}{1+n}p_0\sqrt{RT_0}(k^+ + k^-), \tag{38}$$

$$t = -\frac{1+n}{1-n}(k^+ - k^-) - \frac{7}{8}(k^+ + k^-) + t_w, \qquad \text{at} \quad y = a. \tag{39}$$

Solving Eqs. (31), (32) and (33), we obtain

$$k^+ + k^- = C_1\cosh\left(\frac{p_0\sigma^2 W}{mRT_0}y\right) + C_2\sinh\left(\frac{p_0\sigma^2 W}{mRT_0}y\right), \tag{40}$$

$$k^+ - k^- = \frac{\beta}{W\left(\frac{15}{2} - \frac{144}{5\pi}\right)}\left[C_1\sinh\left(\frac{p_0\sigma^2 W}{mRT_0}y\right) + C_2\cosh\left(\frac{p_0\sigma^2 W}{mRT_0}y\right)\right] - \frac{Mq}{p_0\sqrt{RT_0}}, \tag{41}$$

$$t = \frac{q\sigma^2}{5m(RT_0)^{3/2}}\left(\frac{64\sqrt{2}}{5}A^+ - \alpha M\right)y + \left\{\frac{53}{40} + \frac{\alpha\beta}{5W^2\left(\frac{15}{2} - \frac{144}{5\pi}\right)}\right\} \times$$
$$\left\{C_1\cosh\left(\frac{p_0\sigma^2 W}{mRT_0}y\right) + C_2\sinh\left(\frac{p_0\sigma^2 W}{mRT_0}y\right)\right\} + C_3, \tag{42}$$

where

$$W = \sqrt{\frac{\beta(5\sqrt{\pi}\gamma - 2\sqrt{2}\alpha)10\pi}{(75\pi - 288)(250 - 72\pi)}}, \qquad M = \frac{256A^+ - 320\sqrt{\pi}B^+}{10\sqrt{2}\alpha - 25\sqrt{\pi}\gamma}.$$

The constants of integration, C_1, C_2 are determined from the boundary conditions (36) and (38) as follows;

$$C_1 = 0$$

$$C_2 = \frac{q}{p_0\sqrt{RT_0}}\left\{\frac{\sqrt{2}(\sqrt{\pi} + 4\sqrt{2}M)}{\frac{80\sqrt{\pi}\beta}{W(75\pi - 288)}\cosh\left(\frac{p_0\sigma^2 Wa}{mRT_0}\right) + \frac{15\pi}{4}\frac{n-1}{n+1}\sinh\left(\frac{p_0\sigma^2 Wa}{mRT_0}\right)}\right\} \tag{43}$$

Noting that T_0 is the temperature at $y=0$, we have

$$t = 0 \qquad \text{at} \qquad y = 0$$

which yields $C_3 = 0$.

From the boundary condition (37) and Eq. (42), we have the relation between t_w and q:

$$t_w = \frac{-q}{p_0\sqrt{RT_0}}\left\{\frac{\sqrt{2}(\sqrt{\pi}+4\sqrt{2}M)}{\frac{80\pi\beta}{W(75\pi-288)}\cosh\left(\frac{Wa}{\sqrt{2}\pi\lambda}\right)+\frac{15\pi}{4}\frac{n-1}{n+1}\sinh\left(\frac{Wa}{\sqrt{2}\pi\lambda}\right)}\right\}$$

$$\times\left\{\left[\frac{11}{5}+\frac{2\pi\alpha\beta}{W^2(75\pi-288)}\right]\sinh\left(\frac{Wa}{\sqrt{2}\pi\lambda}\right)+\left(\frac{n-1}{n+1}\right)\frac{10\pi\beta}{W(75\pi-288)}\cosh\left(\frac{Wa}{\sqrt{2}\pi\lambda}\right)\right\}$$

$$-\frac{q\sigma^2 a}{5m(RT_0)^{3/2}}\left(\frac{64\sqrt{2}A^+}{5}-\alpha M\right)+\left(\frac{n+1}{n-1}\right)\frac{Mq}{p_0\sqrt{RT_0}}. \tag{44}$$

where

$$\lambda = mRT_0/\sqrt{2}\pi p_0\sigma^2.$$

This result is valid in the entire region between continuum and free molecule flow. In the limitting case of free molecule flow, $\lambda/a\to\infty$, Eq. (44) becomes

$$t_w = -\left\{\frac{1}{p_0\sqrt{RT_0}}\left(\frac{n+1}{n-1}\right)\frac{\sqrt{2\pi}}{8}\right\}q. \tag{45}$$

If $e^{-a/\lambda}$ can be neglected as compared with $e^{a/\lambda}$, Eq. (44) becomes

$$t_w = \frac{-q}{p_0\sqrt{RT_0}}\left\{\frac{\sqrt{2}(\sqrt{\pi}+4\sqrt{2}M)}{\frac{80\beta}{W(75\pi-288)}+\frac{15\pi}{4}\frac{n-1}{n+1}}\right\}\left\{\left(\frac{11}{5}+\frac{2\pi\alpha\beta}{W^2(75\pi-288)}\right)\right.$$

$$\left.+\left(\frac{n+1}{n-1}\right)\frac{10\pi\beta}{W(75\pi-288)}\right\}-\frac{64\sqrt{2}}{25}\frac{\sigma^2 aq}{m(RT_0)^{3/2}}A^+. \tag{46}$$

This is an asymptote of Eq. (44) and corresponds to the slip flow approximation. And in the continuum region, $\lambda/a\to 0$, we have

$$t_w = -qa\left(\frac{64\sqrt{2}A^+}{25}\frac{\sigma^2}{m(RT_0)^{3/2}}\right).$$

5. Concluding Remarks

In this paper we have calculated the temperature distribution between two parallel plates. In this case, we have obtained the relations between the temperature difference and heat flux. This expression seems to be valid in all regions of aerodynamics, which agrees with the free molecular flow solution[6] for $\lambda/a\to\infty$. Our expressions have very difficult integrals $A^\pm$, $L^\pm(\phi(\mathbf{c}), \varphi(\mathbf{c}))$ and $B^\pm$. But we think it is easy to determine them from the experimental results.

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References

1) T. Koga, *The Boltzmann-Maxwell Equation as the Fundamental Equation of Gasdynamics*, This issue.

2) K. Takao, *On the Flow of Rarefied Gases near a Flat Wall,* To be published in another journal.
3) T. Koga, Appendix of Reference (1).
4) S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases,* Camblidge University Press, London, 1952.
5) H. Grad, *On the Kinetic Theory of Rarefied Gases,* Communications on Pure and Applied Mathematics, **2** (1949), 331.
6) E. H. Kennard, *The Kinetic Theory of Gases,* McGraw-Hill Book Company, Inc., New York, 1938.