MATHEMATICAL THEORY OF INITIAL STRESS IN AN ELASTIC SOLID BODY

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Preliminaries

The theory of equations of motion of an elastic solid body with initial stress was considered by some authors, for example, M. A. Biot.[1,2] We attemped to attack the same problem and obtained the results, which are, in particular, different in the expressions of boundary conditions from those hitherto obtained. In the 1st part of this paper the theory was developed by Sakadi and in the 2nd part the calculation in some important cases was carried out by the 2 authors.

PART I

I. Notations

We now assume that the initial stress of finite magnitude is distributed in the solid and that the strain from this initial state is small of the 1st order and consider only the quantities of the 1st order magnitude. Let x_1, x_2 and x_3 be the rectangular coordinate system, and further put:

y_j : coordinates of a material point P in unstrained state,

x_j : coordinates of the same particle in strained state at time t, the position being P',

$x_j - y_j = \xi_j(t, x) = \eta_j(t, y)$: displacement components, small quantities of the 1st order,

$\rho^0(y)$: density in unstrained state,

$A^0_{ij}(y)$: components of initial stress at P, being considered as finite quantities,

$X^0_j(y)$: body force per unit mass at P,

$A_{ij}(t, x)$: components of stress at P' for the same material portion, the differences $A_{ij}(t, x) - A^0_{ij}(y)$ being of the 1st order,

$X_j(t, x)$: body force per unit mass at P',

$$\sigma_{ij} = \frac{1}{2}\left(\frac{\partial \xi_j}{\partial x_i} + \frac{\partial \xi_i}{\partial x_j}\right), \quad \sigma_{kk} = \frac{\partial \xi_k}{\partial x_k}, \quad \omega_{ij} = \frac{1}{2}\left(\frac{\partial \xi_j}{\partial x_i} - \frac{\partial \xi_i}{\partial x_j}\right).$$

II. Expressions for Stress Tensor in Strained State

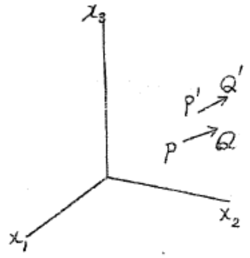

Fig. 1.

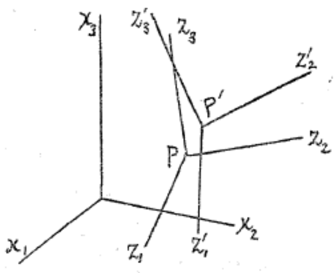

Fig. 2.

We consider an arbitrary material point $Q(y+dy)$ near P in unstrained state and in the strained state this point will be removed into $Q'(x+dx)$ with

$$\left.\begin{aligned} dx_i &= dy_i + \frac{\partial \eta_i}{\partial y_j} dy_j = (\delta_{ij} + \lambda_{ij}) dy_j, \\ \lambda_{ij} &= \frac{\partial \eta_i}{\partial y_j}. \end{aligned}\right\} \cdots\cdots \text{(II, 1)}$$

(II, 1) represent a linear transformation of 'Vektorkörper' $\vec{PQ}(dy)$ into 'Vektorkörper' $\vec{P'Q'}(dx)$ where Q and Q' take arbitrary points near P and P' respectively. As well known,[3] there exist 3 mutually perpendicular axes through P, composed of material points (z_1, z_2, z_3 coordinate system), which, after deformation into strained state, remain also orthogonal to each other with origin $P'(z_1', z_2', z_3'$ coordinate system), and (II, 1) can be decomposed into rotation and 'pure deformation':

	x_1	x_2	x_3
z_1	a_{11}	a_{12}	a_{13}
z_2	a_{21}	a_{22}	a_{23}
z_3	a_{31}	a_{32}	a_{33}
z_1'	a_{11}'	a_{12}'	a_{13}'
z_2'	a_{21}'	a_{22}'	a_{23}'
z_3'	a_{31}'	a_{32}'	a_{33}'

1) z_j: z-coordinates of material point $Q(y+dy)$,

$$z_i = a_{ij}\, dy_j,$$

2) z_j: z'-coordinates of the same material point after parallel displacement and rotation from z-coordinate system into z'-system,

3) z_j': z'-coordinates of the same point after the pure deformation along the 3 axes (z_1', z_2', z_3'), occupying the final position Q',

$z_j' = (1+l^j) z_j$, (here by the terms $l^j z_j$ we do not of course mean summation with respect to j),

4) $dx_i = a_{ji}' z_j'$, $a_{ij}' = a_{ij} + da_{ij}$.

Combining 1)–4) and, comparing with (II, 1) we obtain:

$$\begin{aligned} dx_i &= a_{ji}' z_j' = a_{ji}'(1+l^j) z_j \\ &= a_{ji}' a_{jk}(1+l^j) dy_k = (\delta_{ik} + l^j a_{ji} a_{jk} + a_{jk}\, da_{ji}) dy_k \\ &= (\delta_{ij} + \lambda_{ij}) dy_j, \end{aligned}$$

$$l^k a_{ki} a_{kj} + a_{kj}\, da_{ki} = \lambda_{ij},$$

$$a_{kj}\, da_{ki} = \frac{1}{2}(\lambda_{ij} - \lambda_{ji}) = \omega_{ji}. \qquad \cdots\cdots\cdots \text{(II, 2)}$$

Now we can express the components of stress at P' in the following way: Using the transformation formulae for tensor components we have:

$$A_{ij}(t, x; x) = a_{ki}' a_{lj}' A_{kl}(t, x; z')$$

where in $A_{ij}(t, x; x)$ and $A_{kl}(t, x; z')$, (t, x) means the point P' in x coordinate system, and $(;x)$ and $(;z')$ mean that we are taking the coordinate systems (x_j) and (z'_j) respectively and similarly for A^0_{ij}:

$$A^0_{kl}(y;z)=a_{ki}\,a_{lj}\,A^0_{ij}(y;x).$$

The connexion between $A_{kl}(t,x;z')$ and $A^0_{kl}(y;z)$ is given by:

$$A_{kl}(t,x;z')=A^0_{kl}(y;z)+A'_{kl}(t,x;z'),$$

here the terms $A'_{kl}(t,x;z')$ are due from the pure deformation along z'-coordinate axes. Further we have:

$$A^0_{ij}(y;x)=A^0_{ij}(x;x)-\xi_k\frac{\partial A^0_{ij}(x;x)}{\partial x_k}.$$

Combination of these 4 relations gives:

$$\begin{aligned}A_{ij}(t,x)&=A_{ij}(t,x;x)=a'_{ki}a'_{lj}A_{kl}(t,x;z')\\&=a'_{ki}a'_{lj}(A^0_{kl}(y;z)+A'_{kl}(t,x;z'))\\&=a'_{ki}a'_{lj}a_{kr}a_{ls}A^0_{rs}(y;x)+A'_{ij}(t,x;x)\\&=A^0_{ij}(y;x)+a_{ks}da_{kj}A^0_{is}(y;x)+a_{ks}da_{ki}A^0_{sj}(y;x)+A'_{ij}(t,x;x)\\&=A^0_{ij}(y)+\omega_{sj}A^0_{is}(y)+\omega_{si}A^0_{sj}(y)+A'_{ij}(t,x), \qquad\cdots\cdots\cdots(\text{II}, 3)\\&=A^0_{ij}(x)-\xi_k\frac{\partial A^0_{ij}(x)}{\partial x_k}+\omega_{sj}A^0_{is}(x)+\omega_{si}A^0_{sj}(x)+A'_{ij}(t,x),\\&\qquad\qquad\cdots\cdots\cdots(\text{II}, 4)\end{aligned}$$

here $A'_{ij}(t,x)=A'_{ij}(t,x;x)$ depend only on $\sigma_{kl}(t,x)$ and when the solid is isotropic we have:

$$A'_{ij}=A'_{ij}(t,x)=\lambda\sigma_{kk}\delta_{ij}+2\mu\,\sigma_{ij}.$$

III. Equations of Motion and Boundary Conditions

Let

$$\rho(t,x)=\rho^0(y)(1-\sigma_{kk})=\rho^0(x)-\xi_k\frac{\partial\rho^0(x)}{\partial x_k}-\rho^0(x)\sigma_{kk}$$

be the density in P' and put

$$X_i(t,x)=X^0_i(y)+X'_i(t,x)=X^0_i(x)-\xi_k\frac{\partial X^0_i(x)}{\partial x_k}+X'_i(t,x),$$

then the equations of equilibrium in unstrained state and those of motion are:

$$\rho^0(x)X^0_i(x)+\frac{\partial A^0_{ij}(x)}{\partial x_j}=0, \qquad\cdots\cdots\cdots(\text{III}, 1)$$

$$\rho(t,x)X_i(t,x)+\frac{\partial A_{ij}(t,x)}{\partial x_j}=\rho^0(x)\frac{\partial^2\xi_i(t,x)}{\partial t^2}.$$

From these 2 systems of equations and (II, 4) we obtain:

$$\begin{aligned}\frac{\partial A_{ij}}{\partial x_j}-\frac{\partial A^0_{ij}}{\partial x_j}&=-\frac{\partial\xi_k}{\partial x_j}\frac{\partial A^0_{ij}}{\partial x_k}-\xi_k\frac{\partial^2A^0_{ij}}{\partial x_j\partial x_k}+\frac{\partial\omega_{sj}}{\partial x_j}A^0_{is}+\omega_{sj}\frac{\partial A^0_{is}}{\partial x_j}+\frac{\partial\omega_{si}}{\partial x_j}A^0_{js}\\&\quad+\omega_{si}\frac{\partial A^0_{js}}{\partial x_j}+\frac{\partial A'_{ij}}{\partial x_j}\\&=-\sigma_{jk}\frac{\partial A^0_{ij}}{\partial x_k}+\xi_k\frac{\partial}{\partial x_k}(\rho^0(x)X^0_i(x))+\frac{\partial\omega_{jk}}{\partial x_k}A^0_{ij}+\frac{\partial\omega_{ji}}{\partial x_k}A^0_{kj}-\omega_{ki}\rho^0X^0_k+\frac{\partial A'_{ij}}{\partial x_j},\end{aligned}$$

$$\rho X_i - \rho^0 X_i^0 = -\left(\xi_k \frac{\partial \rho^0}{\partial x_k} + \sigma_{kk}\rho^0\right) X_i^0 + \rho^0\left(-\xi_k \frac{\partial X_i^0}{\partial x_k} + X_i'\right),$$

$$\rho^0 \frac{\partial^2 \xi_i}{\partial t^2} = \rho^0(X_i' - \sigma_{kk} X_i^0 + \omega_{ij} X_j^0) - \sigma_{jk} \frac{\partial A_{ij}^0}{\partial x_k} + \frac{\partial \omega_{jk}}{\partial x_k} A_{ij}^0 + \frac{\partial \omega_{ji}}{\partial x_k} A_{kj}^0 + \frac{\partial A_{ij}'}{\partial x_j}. \qquad \cdots\cdots\cdots (III, 2)$$

(III, 2) correspond to the equations (4, 7) in Biot[1] or (20) in Biot,[2] the additional terms

$$\rho^0 \xi_j \frac{\partial X_i^0}{\partial x_j},$$

in them seem to be due from the different definition of X_j'.

For the boundary conditions we proceed as in[4] and[5]:

$y_j = f_j(u, v)$: boundary surface of unstrained state,

$x_j = y_j + \eta_j(t, y) = y_j + \xi_j(t, f) = x_j(t, u, v)$: boundary surface of strained state,

$$q_1^0 = \begin{vmatrix} \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} \end{vmatrix} = \frac{\partial(f_2, f_3)}{\partial(u, v)}, \quad q_2^0,\ q_3^0,$$

$n_j^0 = \dfrac{q_j^0}{\sqrt{q_k^{02}}}$: direction cosines of the outward normal to the boundary of unstrained state, (the parameters u and v are to be taken suitably so that $\dfrac{q_j^0}{\sqrt{\prime\prime}}$ represent the outward normal)

$$\beta_j(t, u, v) = \xi_j(t, f),$$

where we put $f_k(u, v)$ instead of x_k in $\xi_j(t, x)$,

$$q_1(t, u, v) = \begin{vmatrix} \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} \\ \frac{\partial x_3}{\partial u} & \frac{\partial x_3}{\partial v} \end{vmatrix} = \frac{\partial(f_2, f_3)}{\partial(u, v)} + \frac{\partial(\beta_2, f_3)}{\partial(u, v)} + \frac{\partial(f_2, \beta_3)}{\partial(u, v)}$$

$$= q_1^0 + q_1', \quad q_2 = q_2^0 + q_2', \quad q_3 = q_3^0 + q_3',$$

$$n_j = \frac{q_j}{\sqrt{q_k^2}} = n_j^0 + \frac{1}{\sqrt{q_k^{02}}}(q_j' - n_j^0 n_l^0 q_l') = n_j^0 + n_j'$$

$\cdots\cdots\cdots$ (III, 3): outward normal to the boundary of strained state.

The boundary conditions for initial state are:

$$n_j^0 A_{ij}^0(y) = F_i^0(u, v) \qquad \cdots\cdots\cdots (III, 4)$$

with external surface traction F_i^0. This surface traction must of course be in equilibrium with body force $\rho^0 X_i^0$ in the whole. The conditions for strained state are:

$$n_j A_{ij}(t, x) = F_i(t, u, v),$$

and from these 2 systems of relations we get, using (II, 3) and (III, 3):

$$F_i(t, u, v) - F_i^0(u, v) = (n_j' + n_k^0 \omega_{jk}) A_{ij}^0 + n_j^0 \omega_{ki} A_{kj}^0 + n_j^0 A_{ij}'.$$

The 1st terms of these formulae can be rewritten as follows:

$$\frac{\partial \beta_j}{\partial u}=\frac{\partial \xi_j \partial f_k}{\partial x_k \partial u}=(\sigma_{jk}-\omega_{jk})\frac{\partial f_k}{\partial u},$$

$$q_1'=(\sigma_{2k}-\omega_{2k})\left(\frac{\partial f_k}{\partial u}\frac{\partial f_3}{\partial v}-\frac{\partial f_k}{\partial v}\frac{\partial f_3}{\partial u}\right)+(\sigma_{3k}-\omega_{3k})\left(\frac{\partial f_k}{\partial v}\frac{\partial f_2}{\partial u}-\frac{\partial f_k}{\partial u}\frac{\partial f_2}{\partial v}\right)$$

$$=-\sigma_{k1}q_k^0+\sigma_{kk}q_1^0+\omega_{k1}q_k^0,$$

$$\frac{q_j'}{\sqrt{q_k^{02}}}=-\sigma_{kj}n_k^0+\sigma_{kk}n_j^0+\omega_{kj}n_k^0,$$

$$n_j'+\omega_{jk}n_k^0=-\sigma_{kj}n_k^0+\sigma_{kl}n_k^0 n_l^0 n_j^0.$$

Hence the boundary conditions take the form:

$$F_i-F_i^0=-\sigma_{kj}n_k^0 A_{ij}^0+\sigma_{kl}n_k^0 n_l^0 n_j^0 A_{ij}^0+\omega_{ki}n_j^0 A_{kj}^0+n^0 A'$$
$$=-\sigma_{kj}n_k^0 A_{ij}^0+\sigma_{kl}n_k^0 n_l^0 F_i^0+\omega_{ki}F_k^0+n_j^0 A_{ij}'. \qquad \cdots\cdots\cdots (\mathrm{III}, 5)$$

The equations (4, 8) in Biot[1] are of the form:

$$F_i-F_i^0=-\sigma_{kj}n_k^0 A_{ij}^0+\sigma_{kk}n_j^0 A_{ij}^0+\omega_{ki}n_j^0 A_{kj}^0+n_j^0 A_{ij}',$$

and the 2nd terms are different from those of (III, 5).

When in unstrained state neither body force nor surface traction act on the solid, (III, 1) and (III, 4) become:

$$\left.\begin{aligned}&\frac{\partial A_{ij}^0}{\partial x_j}=0,\\ &n_j^0 A_{ij}^0=0,\end{aligned}\right\}\cdots\cdots\cdots (\mathrm{III}, 6)$$

and when in strained state also no external force acts, we have from (III, 2) and (III, 5):

$$\left.\begin{aligned}&\rho^0\frac{\partial^2\xi_i}{\partial t^2}=-\sigma_{jk}\frac{\sigma A_{ij}^0}{\partial x_k}+\frac{\partial \omega_{jk}}{\partial x_k}A_{ij}^0+\frac{\partial \omega_{ji}}{\partial x_k}A_{kj}^0+\frac{\partial A_{ij}'}{\partial x_j},\\ &0=-\sigma_{kj}n_k^0 A_{ij}^0+n_j^0 A_{ij}',\end{aligned}\right\}\cdots\cdots\cdots (\mathrm{III}, 7)$$

and in the static problem under surface traction we have:

$$\left.\begin{aligned}&0=-\sigma_{jk}\frac{\partial A_{ij}^0}{\partial x_k}+\frac{\partial \omega_{jk}}{\partial x_k}A_{ij}^0+\frac{\partial \omega_{ji}}{\partial x_k}A_{kj}^0+\frac{\partial A_{ij}'}{\partial x_j},\\ &F_i=-\sigma_{jk}n_k^0 A_{ij}^0+n_j^0 A_{ij}',\end{aligned}\right\}\cdots\cdots\cdots (\mathrm{III}, 8)$$

with

$$A_{ij}'=\lambda\sigma_{kk}\delta_{ij}+2\mu\sigma_{ij}$$

in (III, 7) and (III, 8) for isotropic body.

From now we treat the terms with A_{ij}^0 or X_j^0 in (III, 2), (III, 5), (III, 7) and (III, 8) as correction terms. It seems that this simplification does not change the character of the solution. Let $\xi_{i,0}$ be solutions of (III, 2) with (III, 5) in the case with

$$A_{ij}^0=0,\ X_j^0=0,\ F_j^0=0$$

so that, putting

$$\sigma_{ij,0}=\frac{1}{2}\left(\frac{\partial \xi_{j,0}}{\partial x_i}+\frac{\partial \xi_{i,0}}{\partial x_j}\right),$$

$$\omega_{ij,0}=\frac{1}{2}\left(\frac{\partial \xi_{j,0}}{\partial x_i}-\frac{\partial \xi_{i,0}}{\partial x_j}\right),$$

$$A_{ij,0}'=\lambda\sigma_{kk,0}\delta_{ij}+2\mu\sigma_{ij,0},$$

the equations hold:

$$\left.\begin{aligned}&\rho^0\frac{\partial^2\xi_{i,0}}{\partial t^2}=\frac{\partial A'_{ij,0}}{\partial x_j}+\rho^0 X'_i,\\&F_i=n^0_j A'_{ij,0}.\end{aligned}\right\}\cdots\cdots\cdots(\mathrm{III},9)$$

Further put:

$$\xi_i=\xi_{i,0}+\zeta_i,\ \sigma_{ij}=\sigma_{ij,0}+\tau_{ij},\ A'_{ij}=A'_{ij,0}+B_{ij},$$

then we obtain from (III, 2) and (III, 5) the equations and conditions:

$$\left.\begin{aligned}&\rho^0\frac{\partial^2\zeta_i}{\partial t^2}=\rho^0 K_i(t,x)+\frac{\partial B_{ij}}{\partial x_j},\\&n^0_j B_{ij}=S_i(t,u,v)\end{aligned}\right\}\cdots\cdots\cdots(\mathrm{III},10)$$

with

$$\left.\begin{aligned}&\rho^0 K_i=\rho^0(-\sigma_{kk,0}X^0_i+\omega_{ij,0}X^0_j)-\sigma_{jk,0}\frac{\partial A^0_{ij}}{\partial x_k}+\frac{\omega_{jk,0}}{\partial x_k}A^0_{ij}+\frac{\partial\omega_{ji,0}}{\partial x_k}A^0_{kj},\\&S_i=-F^0_i+\sigma_{kj,0}n^0_k A^0_{ij}-\sigma_{kl,0}n^0_k n^0_l F^0_i-\omega_{ki,0}F^0_k,\end{aligned}\right\}(\mathrm{III},11)$$

and from (III, 7):

$$\rho_0\frac{\partial^2\zeta_i}{\partial t^2}=\rho^0 K'_i+\frac{\partial B_{ij}}{\partial x_j},\qquad n^0_j B_{ij}=S'_i\qquad\cdots\cdots\cdots(\mathrm{III},12)$$

with

$$\left.\begin{aligned}&\rho^0 K'_i=-\sigma_{jk,0}\frac{\partial A^0_{ij}}{\partial x_k}+\frac{\partial\omega_{jk,0}}{\partial x_k}A^0_{ij}+\frac{\partial\omega_{ji,0}}{\partial x_k}A^0_{kj},\\&S'_i=\sigma_{kj,0}n^0_k A^0_{ij},\end{aligned}\right\}\cdots\cdots\cdots(\mathrm{III},13)$$

and finally from (III, 8):

$$0=\rho^0 K'_i+\frac{\partial B_{ij}}{\partial x_j},\qquad n^0_j B_{ij}=S'_i\qquad\cdots\cdots\cdots(\mathrm{III},14)$$

with the same expressions of K'_i and S'_i as in (III, 13).

IV. Rotating Body

When the elastic body rotates about x_3-axis with constant angular velocity ω, we assume that x_1, x_2 and x_3 are of the rotating system. Then the equations for the equilibrium state take the form:

$$\left.\begin{aligned}&\omega^2\rho^0(x)x_i+\rho^0(x)X^0_i(x)+\frac{\partial A^0_{ij}(x)}{\partial x_j}=0,\qquad(i=1,2)\\&\rho^0(x)X_3(x)+\frac{\partial A^0_{3j}(x)}{\partial x_j}=0,\end{aligned}\right\}\cdots\cdots\cdots(\mathrm{IV},1)$$

which correspond to (III, 1), while those of motion are:

$$\omega^2\rho(t,x)x_i+\rho(t,x)X_i(t,x)+\frac{\partial A_{ij}(t,x)}{\partial x_j}=\rho^0(x)\Big(\frac{\partial^2\xi_i(t,x)}{\partial t^2}\mp 2\omega\frac{\partial\xi_{i}}{\partial t}\Big),$$
$$(i=1,2)$$

$$\rho(t,x)X_3(t,x)+\frac{\partial A_{3j}(t,x)}{\partial x_j}=\rho^0(x)\frac{\partial^2\xi_3(t,x)}{\partial t^2}.$$

From these eqnations and (II, 4) we have:

$$\left.\begin{aligned}
\rho^0\left(\frac{\partial^2\xi_i}{\partial t^2}\mp 2\omega\frac{\partial\xi_i}{\partial t}\right) &= \rho^0(X_i' - \sigma_{kk}X_i^0 + \omega_{ij}X_j^0) - \sigma_{jk}\frac{\partial A_i^0}{\partial x_k} \\
&\quad + \frac{\partial\omega_{jk}}{\partial x_k}A_{ij}^0 + \frac{\partial\omega_{ji}}{\partial x_k}A_{kj}^0 + \frac{\partial A_{ij}'}{\partial x_j} + \omega^2\rho^0\{\xi_i - \sigma_{kk}x_i - \sum_{k=1}^{2}\omega_{ki}x_k\}, \\
&\qquad\qquad (i=1,2) \\
\rho^0\frac{\partial^2\xi_3}{\partial t^2} &= \rho^0(X_3' - \sigma_{kk}X_3^0 + \omega_{3j}X_j^0) - \sigma_{jk}\frac{\partial A_{3j}^0}{\partial x_k} \\
&\quad + \frac{\partial\omega_{jk}}{\partial x_k}A_{3j}^0 + \frac{\partial\omega_{j3}}{\partial x_k}A_{kj}^0 + \frac{\partial A_{3j}'}{\partial x_j} - \omega^2\rho^0\omega_{k3}x_k.
\end{aligned}\right\}\cdots(\mathrm{IV},2)$$

The boundary conditions are the same as in the case III.

PART II

1) Hydrostatic Pressure

$$A_{ij}^0 = A\,\delta_{ij},\qquad A:\ \text{constant}.$$

In this case we have from (III, 1) and (III, 4):

$$X_i^0 = 0,\qquad F_i^0 = n_i^0 A,$$

$$\frac{\partial A_{ij}^0}{\partial x_k} = 0,\qquad \frac{\partial\omega_{jk}}{\partial x_k}A_{ij}^0 = \frac{\partial\omega_{ik}}{\partial x_k}A,\qquad \frac{\partial\omega_{ji}}{\partial x_k}A_{kj}^0 = \frac{\partial\omega_{ki}}{\partial x_k}A,$$

and (III, 2) become:

$$\rho^0\frac{\partial^2\xi_i}{\partial t^2} = \rho^0 X_i' + \frac{\partial A_{ij}'}{\partial x_j},$$

and the influence of initial stress disappears, while from (III, 5)

$$n_j^0 A_{ij}' = F_i - n_i^0 A + \sigma_{ki}n_k^0 A - \sigma_{kl}n_k^0 n_l^0 n_i^0 A - \omega_{ki}n_k^0 A.$$

2) Uniform Compression

The equations (III, 8) have, under the assumption that λ and μ are constant, the solution for the uniform compression:

$$\begin{aligned}
&\xi_j = \alpha x_j,\qquad \alpha:\ \text{constant},\\
&\sigma_{ij} = \alpha\,\delta_{ij},\quad \omega_{ij} = 0,\\
&-\sigma_{jk}\frac{\partial A_{ij}^0}{\partial x_k} = -\alpha\frac{\partial A_{ij}^0}{\partial x_j} = 0,\quad A_{ij}' = (3\lambda + 2\mu)\alpha\,\delta_{ij},\\
&-\sigma_{kj}n_k^0 A_{ij}^0 = -\alpha n_j^0 A_{ij}^0 = 0,\\
&F_i = n_j^0 A_{ij}' = (3\lambda + 2\mu)\alpha n_i^0.
\end{aligned}$$

In this case the influence of initial stress vanishes also on the boundary conditions.

3) Plane Waves in an Infinite Solid with Constant A_{ij}^0.[6]

We can take the system x_1, x_2, x_3 such that $A_{ij}^0 = 0$, $(i \neq j)$ and take another coordinate system y_1, y_2, y_3. In this system let η_j be displacment components and

$$\eta_j = f_j\left(t - \frac{y_3}{c}\right)$$

be the plane wave progressing in the y_3-positive direction.

Then (III, 7) take the form:

$$\left.\begin{aligned}\rho\frac{\partial^2\eta_3}{\partial t^2}&=(\lambda+2\mu)\frac{\partial^2\eta_3}{\partial y_3^2}-\sum_{j=1,2}B_{3j}^0\frac{\partial^2\eta_j}{\partial y_3^2},\\ \rho\frac{\partial^2\eta_1}{\partial t^2}&=\mu\frac{\partial^2\eta_1}{\partial y_3^2}-\frac{1}{2}(B_{11}^0-B_{33}^0)\frac{\partial^2\eta_1}{\partial y_3^2}-\frac{1}{2}B_{12}^0\frac{\partial^2\eta_2}{\partial y_3^2},\\ \rho\frac{\partial^2\eta_2}{\partial t^2}&=\mu\frac{\partial^2\eta_2}{\partial y_3^2}-\frac{1}{2}B_{12}^0\frac{\partial^2\eta_1}{\partial y_3^2}-\frac{1}{2}(B_{22}^0-B_{33}^0)\frac{\partial^2\eta_2}{\partial y_3^2},\end{aligned}\right\}\cdots\cdots\cdots(3,1)$$

where B_{ij}^0 are the stress components in the y-coordinate system and ρ is used for ρ^0 for simplicity. When we put

$$B_{12}^0=0$$

by taking the axes y_1 and y_2 suitably, (3, 1) become:

$$\left.\begin{aligned}\rho\frac{\partial^2\eta_3}{\partial t^2}&=(\lambda+2\mu)\frac{\partial^2\eta_3}{\partial y_3^2}-\sum_{j=1,2}B_{3j}^0\frac{\partial^2\eta_j}{\partial y_3^2},\\ \rho\frac{\partial^2\eta_1}{\partial t^2}&=\mu_1\frac{\partial^2\eta_1}{\partial y_3^2},\\ \rho\frac{\partial^2\eta_2}{\partial t^2}&=\mu_2\frac{\partial^2\eta_2}{\partial y_3^2},\end{aligned}\right\}\cdots\cdots\cdots(3,2)$$

with

$$\begin{aligned}\mu_1&=\mu-\frac{1}{2}(B_{11}^0-B_{33}^0),\\ \mu_2&=\mu-\frac{1}{2}(B_{22}^0-B_{33}^0).\end{aligned}\qquad\cdots\cdots\cdots(3,3)$$

Hence (3, 2) can be satisfied by the 3 waves:

$$\eta_1=f_1\left(t-\frac{y_3}{c_1}\right),\quad \eta_2=0,\quad \eta_3=\frac{B_{13}^0}{\lambda+2\mu-\mu_1}\eta_1;$$

$$\eta_1=0,\quad \eta_2=f_2\left(t-\frac{y_3}{c_2}\right),\quad \eta_3=\frac{B_{23}^0}{\lambda+2\mu-\mu_2}\eta_2;$$

$$\eta_1=\eta_2=0,\quad \eta_3=f_3\left(t-\frac{y_3}{c_0}\right);$$

with velocities $c_1=\sqrt{\frac{\mu_1}{\rho}}$, $c_2=\sqrt{\frac{\mu_2}{\rho}}$ and $c_0=\sqrt{\frac{\lambda+2\mu}{\rho}}$ respectively. Here the planes of oscillation of the 1st and 2nd waves are perpendicular to each other.

The conditon

$$B_{12}^0=0$$

can be satisfied by taking ϕ according to the equation:

$$\begin{aligned}\{(A_{11}^0-A_{33}^0)pq+(A_{22}^0-A_{33}^0)p'q'\}\cos 2\phi\\ =\tfrac{1}{2}\{(A_{11}^0-A_{33}^0)(q^2-p^2)+(A_{22}-A_{33})(q'^2-p'^2)\}\sin 2\phi.\end{aligned}$$

	x_1	x_2	x_3
y_1	$p\cos\phi-q\sin\phi$	$p'\cos\phi-q'\sin\phi$	$-\sin\theta\cos\phi$
y_2	$-p\sin\phi-q\cos\phi$	$-p'\sin\phi-q'\cos\phi$	$\sin\theta\sin\phi$
y_3	$-q'\sin\theta$	$q\sin\theta$	$\cos\theta$

$$p=\cos\varphi\cos\theta,\qquad p'=\sin\varphi\cos\theta,\qquad q=\sin\varphi,\qquad q'=-\cos\varphi.$$

4) Reflection of Plane Waves by a Free Plane Surface

We take the axis x_3 to be the outward normal to the plane surface of the elastic medium and the axes x_1, x_2 on the surface such that the normal of the incident wave lies in the x_1x_3-plane, then from the surface conditions we have:

$$A_{i3}^0 = 0, \qquad (i = 1, 2, 3).$$

Putting

$$\eta_j = \Re E_j e^{i\nu(t-(y_3/c))} = \Re E_j e^{i\nu(t-(x_1 \sin\theta + x_3 \cos\theta)/c)}$$

into (3, 1), we have

$$(\mu_1 - \rho c^2) E_1 - \tfrac{1}{2} B_{12}^0 E_2 = 0,$$
$$-\tfrac{1}{2} B_{12}^0 E_1 + (\mu_2 - \rho c^2) E_2 = 0,$$
$$-B_{13}^0 E_1 - B_{23}^0 E_2 + (\lambda + 2\mu - \rho c^2) E_3 = 0$$

with (3, 3).

From these equations we obtain 3 waves:

	x_1	x_2	x_3
y_1	$\cos\theta$	0	$-\sin\theta$
y_2	0	1	0
y_3	$\sin\theta$	0	$\cos\theta$

$$E_1 \neq 0, \quad E_2 = \frac{-B_{12}^0 E_1}{\mu_1 - \mu_2 + M},$$

$$E_3 = \frac{2E_1}{2(\lambda + 2\mu) - (\mu_1 + \mu_2) - M}\left\{B_{13}^0 - \frac{B_{12}^0 B_{23}^0}{\mu_1 - \mu_2 + M}\right\},$$

with velocity

$$c_1 = \sqrt{\frac{\mu_1 + \mu_2 + M}{2\rho}};$$

$$E_1 = \frac{B_{12}^0 E_2}{\mu_1 + \mu_2 + M}, \quad E_2 \neq 0, \quad E_3 = \frac{2E_2}{2(\lambda + 2\mu) - (\mu_1 + \mu_2) + M}\left\{B_{23}^0 + \frac{B_{12}^0 B_{13}^0}{\mu_1 + \mu_2 + M}\right\},$$

with velocity

$$c_2 = \sqrt{\frac{\mu_1 + \mu_2 - M}{2\rho}};$$

and

$$E_1 = E_2 = 0, \quad E_3 \neq 0 \quad \text{with velocity} \quad c_0 = \sqrt{\frac{\lambda + 2\mu}{\rho}},$$

where

$$M = \sqrt{(\mu_1 - \mu_2)^2 + B_{12}^{0\,2}}.$$

Let the incident wave be

$E_j = \beta_j E$ with velocity c,

progressing in the direction denoted by θ, and let the reflected waves be

$$E_j^{(m)} = \beta_j^{(m)} E^{(m)}$$

with velocities c_m progressing in the directions $\theta^{(m)}$ respectively. $(m = 0, 1, 2)$

At the free surface $x_3 = 0$ the boundary conditions (III, 7) become:

$$\frac{E}{c}\{\beta_1 \cos 2\theta + \beta_3 \sin 2\theta\} + \sum_{m=0}^{2} \frac{E^{(m)}}{c_m}\{\beta_1^{(m)} \cos 2\theta^{(m)} + \beta_3^{(m)} \sin 2\theta^{(m)}\} = 0,$$

$$\frac{\beta_2 E}{c}\cos\theta + \sum_{m=0}^{2} \frac{\beta_2^{(m)} E^{(m)}}{c_m}\cos\theta^{(m)} = 0,$$

$$\lambda\frac{\beta_3 E}{c}+2\mu\frac{E\cos\theta}{c}\{-\beta_1\sin\theta+\beta_3\cos\theta\}$$

$$+\sum_{m=0}^{2}\{\lambda\frac{\beta_3^{(m)}E^{(m)}}{c_m}+2\mu\frac{E^{(m)}\cos\theta^{(m)}}{c_m}(-\beta_1^{(m)}\sin\theta^{(m)}+\beta_3^{(m)}\cos\theta^{(m)})\}=0.$$

with $\frac{\sin\theta}{c}=\frac{\sin\theta^{(m)}}{c_m}$. $\quad (m=0, 1, 2)$

In general these equations determine $E^{(m)}$ and $\theta^{(m)}$ as functions of E and θ.

5) Reflection and Refraction of Plane Waves by a Discontinuous Plane

Take the x_1x_2-plane as the plane of discontinuous initial stress and let the constant initial stress be

A_{ij}^0 in the region: $x_3<0, \quad -\infty<x_1, \quad x_2<+\infty$,

B_{ij}^0 in the region: $x_3>0, \quad -\infty<x_1, \quad x_2<+\infty$,

then the conditions

$$A_{i3}^0=B_{i3}^0 \qquad (i=1, 2, 3) \qquad \cdots\cdots\cdots(5, 1)$$

must be satisfied.

We can take the axes x_1 and x_2, so that the condition

$$A_{12}^0=0$$

is satisfied.

Taking the incident wave progressing in the x_3-positive direction in the region $x_3<0$:

$$\xi_j=\Re D_j e^{i\nu(t-(x_3/c))},$$

we obtain 3 waves:

$$D_1\neq 0, \quad D_2=0, \quad D_3=\frac{A_{13}^0}{\lambda+2\mu-\mu_1}D_1 \quad \text{with } c_1=\sqrt{\frac{\mu_1}{\rho}};$$

$$D_1=0, \quad D_2\neq 0, \quad D_3=\frac{A_{23}^0}{\lambda+2\mu-\mu_2}D_2 \quad \text{with } c_2=\sqrt{\frac{\mu_2}{\rho}}; \qquad \cdots(5, 2)$$

and

$$D_1=D_3=0, \quad D_3\neq 0 \qquad \text{with } c_0=\sqrt{\frac{\lambda+2\mu}{\rho}}.$$

Exchanging the axes x_1 and x_2, we see that the 1st and 2nd waves of (5, 2) interchnge to each other. Accordingly we treat the 1st wave as the incident one. To this 1st wave the 2 waves are reflected at the plane surface $x_3=0$ and progress in x_3-negative direction:

$$F_1^{(1)}, \quad F_2^{(1)}=0, \quad F_3^{(1)} \quad \text{with } c_1;$$

$$F_1^{(0)}=F_2^{(0)}=0, \quad F_3^{(0)} \quad \text{with } c_0.$$

Further the 3 refracted waves progressing in the x_3-positive direction shall be denoted by:

$$E_1^{(1)}, \quad E_2^{(1)}, \quad E_3^{(1)} \qquad \text{with} \quad V_1=\sqrt{\frac{\mu_1^*+\mu_2^*+N}{2\rho}};$$

$$E_1^{(2)}, \quad E_2^{(2)}, \quad E_3^{(2)} \qquad \text{with} \quad V_2=\sqrt{\frac{\mu_1^*+\mu_2^*-N}{2\rho}};$$

$$E_1^{(0)}=E_2^{(0)}=0, \quad E_3^{(0)} \quad \text{with} \quad V_0=\sqrt{\frac{\lambda+2\mu}{\rho}},$$

where

$$\mu_1^* = \mu - \tfrac{1}{2}(B_{11}^0 - B_{33}^0),$$
$$\mu_2^* = \mu - \tfrac{1}{2}(B_{22}^0 - B_{33}^0),$$
$$N = \sqrt{(\mu_1^* - \mu_1^*)^2 + B_{12}^{0^2}}.$$

The boundary conditions at $x_3 = 0$ are

$$D_j + \sum_{m=0}^{1} F_j^{(m)} = \sum_{m=0}^{2} E_j^{(m)}, \qquad (j = 1, 2, 3) \qquad \cdots\cdots(5, 3)$$

and

$$\left.\begin{aligned} &(2\mu - A_{11}^0)\left(\frac{D_1}{c} - \sum_{m=0}^{1}\frac{F_1^{(m)}}{c_m}\right) - (2\mu - B_{11}^0)\sum_{m=0}^{2}\frac{E_1^{(m)}}{V_m} = 0, \\ &(2\mu - A_{22}^0)\left(\frac{D_2}{c}\sum_{m=1}^{1} - \frac{F_2^{(m)}}{c_m}\right) - (2\mu - B_{22}^0)\sum_{m=0}^{2}\frac{E_2^{(m)}}{V_m} = 0, \\ &\frac{D_3}{c} - \sum_{m=0}^{1}\frac{F_3^{(m)}}{c_m} - \sum_{m=0}^{2}\frac{E_3^{(m)}}{V_m} = 0. \end{aligned}\right\} \cdots\cdots(5, 4)$$

from (III, 5) and (5, 1).

(5, 3) and (5, 4) determine $E_j^{(m)}$ and $F_j^{(m)}$ completely.

The longitudinal wave with velocity c_0 undergoes no reflection by the surface $x_3 = 0$ and passes through without any change into the region $x_3 > 0$.

6) Surface Wave

Let the elastic body occupy the region: $x_3 < 0$, $-\infty < x_1, x_2 < +\infty$. As in the case 4) we obtain $A_{i3}^0 = 0$ in the whole region $x_3 < 0$. Considering the surface wave

$$\eta_j = \Re E_j e^{\alpha x_3 + i(fx_1 - lt)}, \qquad \Re\alpha > 0,$$

we obtain by (III, 7), A_{ij}^0 being constant,

$$\left\{(b^2 - \frac{A_{11}^0}{2\rho})\alpha^2 - a^2 f^2 + l^2\right\}E_1 + \frac{1}{\rho}(f^2 - \frac{\alpha^2}{2})A_{12}^0 E_2 + if\alpha(a^2 - b^2 + \frac{A_{11}^0}{2\rho})E_3 = 0,$$

$$-\frac{A_{12}^0}{2\rho}\alpha^2 E_1 + \left\{(b^2 - \frac{A_{22}^0}{2\rho})\alpha^2 - \frac{f^2}{2\rho}(A_{11}^0 - A_{22}^0) - b^2 f^2 + l^2\right\}E_2 + if\alpha\frac{A_{12}^0}{2\rho}E_3 = 0,$$

$$if\alpha(a^2 - b^2 - \frac{A_{11}^0}{2\rho})E_1 - if\alpha\frac{A_{12}^0}{2\rho}E_2 + \left\{a^2\alpha^2 - f^2(b^2 + \frac{A_{11}^0}{2\rho}) + l^2\right\}E_3 = 0$$

where $a = \sqrt{\dfrac{\lambda + 2\mu}{\rho}}$ and $b = \sqrt{\dfrac{\mu}{\rho}}$.

From these equations we obtain 3 waves

$$E_j^{(m)} = \beta_j^{(m)} E^{(m)}, \text{ with definite } \alpha^{(m)}. \quad (m = 1, 2, 3)$$

Accordingly the surface wave takes the form:

$$\eta_j = \Re\sum_{m=1}^{3}\beta_j^{(m)} E^{(m)} e^{\alpha^{(m)}x_3 + i(fx_1 - lt)}.$$

Boundary conditions (III, 7) at $x_3 = 0$ become

$$\sum_{m=1}^{3}(if\beta_3^{(m)} + \alpha^{(m)}\beta_1^{(m)})E^{(m)} = 0,$$
$$\sum_{m=1}^{3}\alpha^{(m)}\beta_2^{(m)}E^{(m)} = 0,$$
$$\sum_{m=1}^{3}\{\lambda if\beta_1^{(m)} + (\lambda + 2\mu)\alpha^{(m)}\beta_3^{(m)}\}E^{(m)} = 0.$$

The elimination of $E^{(m)}$ from these equations completely determines the velocity $c=\frac{l}{f}$ of the surface wave.

7) Compression of a Spherical Shell

We assume that the distribution of initial stress is of spherical symmetry and further consider the radial displacement.

Let

$$x_i^2 = r^2,$$
$$A_{ij}^0 = f\delta_{ij} + g x_i x_j,$$
$$\xi_i = \eta x_i$$

where f, g and η are functions of r, the region of the shell extending in $a' \leqq r \leqq a$.

From the equations (III, 6) of unstrained state, we obtain the relation:

$$f' + r^2 g' + 4rg = 0.$$

From (III, 7) we obtain

$$\eta'' + \left(\frac{4}{r} - Q'(r)\right)\eta' = 0,$$

where $Q(r)=\frac{f+r^2 g}{\lambda+2\mu}$, with $Q(a)=Q(a')=0$, which represent the boundary conditions (III, 6).

This equation can be solved by

$$\eta = c\int_a^r \frac{e^Q}{r^4}\,dr + c_1,$$

the 2 constants c and c_1 being determined by the boundary conditions (III, 8).

8) Compression of a Cylindrical Body with Arbitrary Cross Section

Let the cylinder occupy the region $-l < x_3 < +l$ and assume:

$$\xi_1 = bx_1, \quad \xi_2 = bx_2, \quad \xi_3 = cx_3, \quad b \neq c,$$

then we calculate:

$$A'_{ij} = \begin{cases} 2(\lambda+\mu)b + \lambda c, & i=j=1 \text{ or } 2 \\ (\lambda+2\mu)c + 2\lambda b, & i=j=3 \\ 0, & i \neq j \end{cases}$$

$$\frac{\partial A'_{ij}}{\partial x_j} = 0.$$

$$-\sigma_{jk}\frac{\partial A_{ij}^0}{\partial x_k} = (b-c)\frac{\partial A_{i3}^0}{\partial x_3}.$$

Hence from (III, 8) we have: $\frac{\partial A_{i3}^0}{\partial x_3} = 0.$

Considering the conditions (III, 6) i.e.

$$A_{i3}^0 = 0 \quad \text{at} \quad x_3 = \pm l,$$

we have

$$A_{i3}^0 = 0, \qquad (i=1,2,3)$$

in the whole internal region of the cylinder.

Hence (III, 6) become

$$\sum_{j=1}^{2} \frac{\partial A_{ij}^0}{\partial x_j} = 0, \qquad (i=1,2)$$

and

$$\sum_{j=1}^{2} A_{ij}^0 n_j^0 = 0, \qquad (i=1,2)$$

on the lateral surface. Hence A_{11}^0, A_{22}^0, A_{12}^0 can be any functions of x_1, x_2, x_3 with these 2 systems of equations and conditions.

The values of b and c can be determined from the pressures on the lateral and basic surfaces.

9) Compression of a Circular Tube

Let

$$r = \sqrt{x_1^2 + x_2^2},$$

and the tube occupy the region:

$$-l < x_3 < +l, \qquad a' < r < a.$$

Further we assume that the distribution of initial stress is given by:

$$\left.\begin{aligned} A_{ij}^0 &= p\delta_{ij} + q x_i x_j, \qquad (i, j = 1, 2) \\ A_{i3}^0 &= 0, \qquad\qquad (i = 1, 2, 3) \end{aligned}\right\} \cdots\cdots\cdots (9, 1)$$

with p and q functions of r only.

(III, 6) take the form:

$$Q = \frac{p + r^2 q}{\lambda + 2\mu}, \qquad Q' = -\frac{rq}{\lambda + 2\mu}, \qquad Q(a) = Q(a') = 0.$$

Putting

$$\left.\begin{aligned} \xi_i &= \eta x_i = \eta(r) x_i, \qquad (i = 1, 2) \\ \xi_3 &= \zeta x_3, \qquad \zeta = \text{const.} \end{aligned}\right\} \cdots\cdots\cdots (9, 2)$$

into (III, 8), we obtain

$$\eta'' + \left(\frac{3}{r} - Q'\right)\eta' = 0.$$

From this equation η is found to be

$$\eta = c_1 + c\int_a^r \frac{e^Q}{r^3}\, dr.$$

The boundary conditions (III, 8) take the form:

$$\begin{aligned} 2(\lambda+\mu)\eta + (\lambda+2\mu) r\eta' + \lambda\zeta &= -P, \qquad (r = a) \\ \text{''} \qquad\qquad \text{''} \qquad\qquad &= -P', \qquad (r = a') \\ \lambda(2\eta + r\eta') + (\lambda+2\mu)\zeta \qquad &= F(r). \qquad (x_3 = \pm l) \end{aligned}$$

with lateral pressures P and P' and tension $F(r)$. The circumstances that $F(r)$ must be taken suitably come from the simple assumption of ξ_1, ξ_2 and ξ_3 as the functions of x_3.

10) Torsion of a Circular Cylinder or Tube

Take

$$\begin{aligned} \xi_1 &= -\tau x_2 x_3, \\ \xi_2 &= +\tau x_1 x_3, \\ \xi_3 &= 0, \end{aligned}$$

where τ is constant, then (III, 8) are satisfied when A^0_{ij} fulfill the conditions:

$$A^0_{i3} = 0, \qquad (i = 1, 2, 3)$$

$$\frac{\partial A^0_{ij}}{\partial x_3} = 0. \qquad (i, j = 1, 2)$$

Then (III, 6) require:

$$\sum_{j=1}^{2} \frac{\partial A^0_{ij}}{\partial x_j} = 0, \qquad (i = 1, 2)$$

and

$$\sum_{j=1}^{2} A^0_{ij} n^0_j = 0, \qquad (i = 1, 2)$$

on the lateral surfaces.

The boundary conditions (III, 8) for lateral surfaces are satisfied by $F_j = 0$ and those for $x_3 = \pm l$ become

$$F_1 = \tau\left(\frac{1}{2} A^0_{11} - \mu\right) x_2 - \frac{\tau}{2} A^0_{12} x_1,$$

$$F_2 = -\tau\left(\frac{1}{2} A^0_{22} - \mu\right) x_1 + \frac{\tau}{2} A^0_{12} x_2,$$

$$F_3 = 0.$$

11) Radial Vibration of a Sphere

Let $x_i^2 = r^2$, and the sphere occupy the region: $r < a$.

Taking

$$A^0_{ij} = f(r)\,\delta_{ij} + g(r)\,x_i x_j,$$

$$\xi_i = \Re\,\eta(r) \cdot x_i e^{i\nu t},$$

the equations of motion (III, 7) become

$$\eta'' + \left\{\frac{4}{r} - Q'\right\}\eta' + \frac{\rho^0 \nu^2}{\lambda + 2\mu}\eta = 0$$

with the same Q as in 7).

The boundary conditions (III, 7), i.e.

$$(3\lambda + 2\mu)\eta + (\lambda + 2\mu) r \eta' = 0, \quad \text{at} \quad r = a,$$

determines the frequencies ν of the radial vibration of the sphere.

12) Torsional Vibration of a Circular Cylinder

Let the cylinder occupy the region: $0 < x_3 < l$, $r = \sqrt{x_1^2 + x_2^2} < a$, and assume for A^0_{ij} the expressions:

$$A^0_{ij} = p(r)\delta_{ij} + q(r) x_i x_j, \qquad (i, j = 1, 2)$$

$$A^0_{i3} = 0. \qquad (i = 1, 2, 3)$$

(III, 6) are satisfied by

$$Q = \frac{p + r^2 q}{\lambda + 2\mu}, \qquad Q' = -\frac{rq}{\lambda + 2\mu}, \qquad Q(a) = 0.$$

Putting

$$\xi_1 = -\eta(r) \cdot x_2 \cos \gamma x_3 \cos \nu t,$$

$$\xi_2 = +\eta(r) \cdot x_1 \cos \gamma x_3 \cos \nu t,$$

$$\xi_3 = 0.$$

into (III, 7), we obtain

$$\eta'' + \left(3 + \frac{S'}{S}\right)\eta' + \frac{2\rho^0 \nu^2 \gamma^2 (p - 2\mu)}{S}\eta = 0$$

with $S(r) = 2\mu + r^2 q$.

The boundary conditions (III, 7) at $r = a$ become

$$\eta' = 0$$

and those at $x_3 = 0$ and l are seen to become

$$\sin \gamma\, l = 0;$$

$$\gamma = \frac{n\pi}{l},$$

with n: positive integers.

13) Rotation of a Circular Cylinder about its Axis

Let the infinite cylinder of radius a rotate about its axis with angular velocity ω and we assume the same expressions (9, 1) and (9, 2) for A^0_{ij} and ξ_j. Then from (IV, 1) we obtain:

$$Q(r) = \frac{p + r^2 q}{\lambda + 2\mu}, \qquad Q' = -\frac{1}{\lambda + 2\mu}(\omega^2 \rho r + rq)$$

with the boundary condition $Q(a) = 0$,
and from (IV, 2):

$$\eta'' + \left\{\frac{3}{r} - \left(Q' + \frac{\omega^2 \rho}{\lambda + 2\mu} r\right)\right\}\eta' - \frac{\omega^2 \rho}{\lambda + 2\mu}\zeta = 0.$$

The integral of this equation which is finite at $r = 0$ is:

$$\eta = c + \frac{\omega^2 \rho}{\lambda + 2\mu}\zeta \int_0^r \frac{e^R}{r^3}\left\{\int_0^r r^3 e^{-R} dr\right\} dr$$

with

$$R(r) = Q + \frac{\omega^2 \rho}{2(\lambda + 2\mu)} r^2.$$

As the boundary condition for the deformed state we put the constant pressure P on the lateral surface. Then the constant c is to be determined by

$$-P = (\lambda + 2\mu) r\eta' + 2(\lambda + \mu)\eta + \lambda\zeta \quad \text{at} \quad r = a.$$

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