HYDRODYNAMICAL RESEARCHES OF GASES IN MOTION

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Contents

Introduction

1. From the dynamical point of view, the so-called fluids are distinct from the so-called solids by the following fact: The stress in solid happens to appear when it is merely deformed, whereas, as for fluid, the stress except the so-called static pressure comes out only when the speed of deformation appears.

As well known, this dynamical character is not proper to a certain substance, if it may be called for the time a solid or fluid, but a temporary character which appears and disappears as the scale of time and space is changed with which the dynamical phenomena occur. Namely, the name of solid or fluid is not one which is connected with material itself but one connected with phenomena. Thus in our present consideration it is necessary to bear in mind that we should not miss our steps on the scale of time and space with which the material is fully considered as fluid.

The classification of dynamical phenomena from this point of view is still too rough. According to the character of the matter considered, many other conditions are to be put on its motion with the scale of time and space, thus the domain of the dynamics called hydrodynamics is consequently complicated and vast. Accordingly the phenomena which are called equally hydroynamical are not solved by a mere single method. For instance, the application of the theory of complex function, the conformal representation of regular function, has been a very effective method and special importance has been attached to it. But it is the case only when the objects of research are restricted within a narrow domain.

It seems that there are two kinds of attitudes of research. From one point of view we take the dynamical phenomena abstracted without taking into accout real matter, and from the other point of view we take a specially fixed matter in which the phenomena occur complicatedly. The example of the former is the ideal fluid* where we do not consider any really existing matter. We take the matter imagined from a particular hydrodynamical state of real one. Therefore, as to a proper substance, the scale of time and space is restricted where it is considered to be the ideal fluid. To research thoroughly the phenomena of a proper matter, it is necessary to define many kinds of abstracted fluids and to consider fully each of them. Each one is simple and we can not explain the existing phenomena merely by the consideration of this abstracted one. The image of real fluid exists on the envelope of the images of all kinds of abstracted fluids. Without understanding the operation to make the envelope in each case, we may be perplexed with various kinds of paradoxes. From the other point of view, we take a proper substance and observe it in the various scales of motion when many kinds of factors appear and disappear with scale of time and space. In this case we may confront the same confusion as appeared in the fluid mechanics in the former times.

A matter comes out wearing various kinds of masks and taking the features of various kinds of abstracted fluids with the scale of time and space. We must solve the problems by seeing through these facts and making use of the simplicity of each abstracted fluid. That is to make the classification of the dynamical phenomena and of the methods of research by means of the idea of dynamical similarity.

In the following chapters, we shall first of all make the classification and then work out the more or less concrete problems.

I. Hydrodynamical Similarity

2. As well known, many authors have discussed the similarity of flow by means of the method of dimensional analysis.[1] Here the present author discusses the problem from more of less general point of view that, when two flows or more are needed to be geometrically similar, three kinds of equations, i.e., the equation of continuity (mass conservation), Navier-Stokes' equation (momentum conservation) and the equations of energy (energy conservation) put some limitations on the quantities which appear in the phenomena. It may be important to pick up thoroughly the elements which decide the field of flow and to understand their meanings.

Although most results thus derived are not new nor original, but we can get a general view and the benefit for the further researches of complicated phenomena.

3. When there are two fields of flow, there will be more than one meaning in

* We define the ideal fluid as that which has no viscosity and is of constant density and temperature.

[1] L. Prandtl und O. Tietjens, *Hydro- und Aeromechanik* (Julius Springer, Berlin, 1931), Bd, II, S. 6. W. F. Durand, *Aerodynamic Theory* (Julius Springer, Berlin, 1935), Vol. VI, p. 250. H. Lamb, *Hydrodynamics*, 6th edition (Combridge, University Press, 1932), p. 682. H. O. Croft, *Thermodynamics, Fluid Flow and Heat Transmission* (McGraw-Hill Co., New York, 1938), p. 16. A. Busemann, *Handbuch der Experimental Physik* (1931), IV. Teil 1, S. 360.

saying that these flows are similar. Here we take its meaning as follows: Taking the orthogonal coördinate system x, y, z, we represents the velocity components by u, v, w. First of all, we put

$$\left.\begin{array}{lll} x=l_0 x', & y=l_0 y', & z=l_0 z', \\ u=q_0 u', & v=q_0 v', & w=q_0 w', \end{array}\right\} \qquad \cdots\cdots\cdots (\text{I. } 1)$$

where l_0 and q_0 are proper to each field of flow representing their scale, and x', y', z', u', v', w' are the common non-dimensional quantities. We take these as the representation of the geometrical and kinematical similarity of two fields of flow.

Now let us put the limitation on the pressure p, temperature T, and density ρ, the variables of thermodynamical state to fulfill the conditions of similarity. Sometimes we put

$$p=(\rho q^2/2)p' \qquad \cdots\cdots\cdots (\text{I. } 2)$$

as the conditions of similarity in pressure. We have the reason to do so: In the field of flow where the density is nearly constant and further the action of viscosity can be neglected, we can say that there is the linear relation between pressure p and $1/2\cdot\rho q^2$ from Navier-Stokes' Eq.

For the present we put

$$p=p_0 p', \quad T=T_0 T', \quad \rho=\rho_0 \rho', \qquad \cdots\cdots\cdots (\text{I. } 3)$$

where T represents absolute temperature. Thus, taking R the gas constant and m gram molecule, the equation of state of gas is

$$p\rho^{-1}=RT/m.$$

Considering (I. 3) we get

$$(p_0 \rho_0^{-1} T_0^{-1} m R^{-1}) p' \rho'^{-1} = T'.$$

Under the condition that $\rho_0 \rho_0^{-1} T_0^{-1} m R^{-1}$ is constant for all the fields, p', T', ρ' are able to be common to all. Under the condition that the two gases to be compared are of the same kind and $p_0 \rho_0^{-1} T_0^{-1} m R^{-1}$ is common to both, the equation of state does not put any other obligation upon us to complete the similarity. Namely the equation of state does not add any condition to make p', T' and ρ' common to the two fields.

Now the coefficient of viscosity μ and the thermal conductivity λ are in general functions of temperature:

$$\left.\begin{array}{l} \mu=f(T), \\ \lambda=k\mu=kf(T). \end{array}\right\} \qquad \cdots\cdots\cdots (\text{I. } 4)$$

The form of function f and the constant k are proper to each substance. But as the requirement upon T has already been given in (I. 3), any other form of similarity is not to be put on here further. On the contrary we are able to use $f(T)$ for T as a variable of state.

4. The law of mass conservation, in the present case, is

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0.$$

By means of (I, 1) and (I. 3), this is rewritten as follows:

$$\frac{\rho_0 q_0}{l_0}\left\{\frac{\partial\rho'}{\partial t'}+u'\frac{\partial\rho'}{\partial x'}+v'\frac{\partial\rho'}{\partial y'}+w'\frac{\partial\rho'}{\partial z'}+\rho'\left(\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'}+\frac{\partial w'}{\partial z'}\right)\right\}=0,$$

where we put

$$t=(l_0/q_0)t'. \qquad \cdots\cdots\cdots\text{(I. 5)}$$

Namely

$$\frac{\partial\rho'}{\partial t'}+\frac{\partial(\rho'u')}{\partial x'}+\frac{\partial(\rho'v')}{\partial y'}+\frac{\partial(\rho'w')}{\partial z'}=0.$$

Thus the mass conservation law does not put any limitation on the present similarity.*

5. The equation of motion (Navier-Stokes' Eq.) is

$$\left.\begin{aligned}\rho\frac{du}{dt}&=-\frac{\partial p}{\partial x}-\frac{2}{3}\frac{\partial}{\partial x}\left\{\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right\}+2\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)\\&\quad+\frac{\partial}{\partial y}\left\{\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\}+\frac{\partial}{\partial z}\left\{\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial y}\right)\right\},\\\rho\frac{dv}{dt}&=-\frac{\partial p}{\partial y}-\cdots\cdots\cdots\cdots,\\\rho\frac{dw}{dt}&=-\frac{\partial p}{\partial z}-\cdots\cdots\cdots\cdots,\end{aligned}\right\}\cdots\cdots\cdots\text{(I. 6)}$$

where

$$\frac{d}{dt}\equiv\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}.$$

Into this substituting (I. 1), (I. 3) and (I. 5), we get

$$\begin{aligned}\frac{\rho_0 q_0^2}{l_0^2}\rho'\frac{du'}{dt'}&=-\frac{p_0}{l_0}\frac{\partial p'}{\partial x'}-\frac{2}{3}\frac{q_0}{l_0^2}\mu\left\{\frac{\partial}{\partial x'}\left(\frac{\partial u'}{\partial x'}+\cdots\right)\right\}\\&\quad-\frac{2}{3}\frac{q_0 T_0}{l_0^2}\frac{d\mu}{dT}\frac{\partial T'}{\partial x'}\left(\frac{\partial u'}{\partial x'}+\cdots\right)+\cdots,\ \text{etc.}\end{aligned}$$

Dividing the both hand sides of the equation by $\rho_0 q_0^2/l_0$ we get

$$\rho'\frac{du'}{dt'}=-\frac{p_0}{\rho_0 q_0^2}\frac{\partial p'}{\partial x'}+\frac{\mu}{l_0\rho_0 q_0}(\,'\,)+\frac{T_0}{l_0\rho_0 q_0}\frac{d\mu}{dT}(\,''\,),$$

where $(\,'\,)$ and $(\,''\,)$ are non-dimensional terms. To conserve the similarity, the equation of motion put the limitation on the quantities that $p_0/(\rho_0 q_0^2)$, $\mu_0/(l_0\rho_0 q_0)$ and $T_0/(l_0\rho_0 q_0)\cdot(d\mu/dT)_0$ must be common respectively. Here we must assume that the form of the function f in Eq. (I. 4) is universal for all the substances considered to make μ as μ_0.

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* When we assume that the flow is adiabatic, i.e., $p\rho^{-\gamma}=p_0\rho_0^{-\gamma}$ (γ=[specific heat at constant pressure]/[specific heat at constant volume]), we get for steady flow

$$(u^2+v^2+w^2)/2=\gamma/(\gamma-1)\cdot(p_0/\rho_0)\{1-(p/p_0)^{(\gamma-1)/\gamma}\} \qquad \cdots\cdots\cdots\text{i)}$$

from the equation of motion. Further considering $p\rho^{-1}=RT/m$, p, ρ and T become the functions of q. Thus the equation of continuity is

$$\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'}+\frac{\partial w'}{\partial z'}=\frac{1}{2}\frac{q_0^2}{a^2}\frac{dq'^2}{dt'}=\frac{1}{2}\frac{q_0^2}{a_0^2}\frac{dq'^2}{dt'}\rho'^{(-\gamma+1)}, \qquad \cdots\cdots\cdots\text{ii)}$$

where

$$a^2=\gamma p/\rho,\quad a_0^2=\gamma p_0/\rho_0,$$
$$q^2=u^2+v^2+w^2.$$

In this case a new limitation is put on the functions of state. Namely q_0^2/a_0^2 must be common. q_0^2/a_0^2 is represented by M^2 and M is the so-called Mach's number. On the other hand, we shall see in §6 that the ratio of intrinsic energy to energy of motion is automatically common from this condition.

In hydrodynamics, it is customary with us to call

$$p_0/(\rho_0 q_0^2) \qquad \cdots\cdots\cdots (\text{I. 7})$$

Euler's number, and the reciprocal of

$$\mu/(l_0 \rho_0 q_0) \equiv 1/R_e \qquad \cdots\cdots\cdots (\text{I. 8})$$

Reynolds' number.*

As mentioned above, provided that we take (I. 2) as the condition of similarity of pressure, the coincidence of Euler's number is automatically fulfilled. Thus the similarity of flow from the point of view of momentum is satisfied by making Reynolds' number common. In the following analysis we see that Reynolds' number represents the order of the ratio of shearing stress to inertia force.

6. Finally let us deal with the problem from the point of view of energy. The conservation of law of energy is expressed as follows:

$$\left.\begin{aligned}&\rho\frac{d}{dt}\left\{c_vT+\frac{1}{2}(u^2+v^2+w^2)\right\}\\&\quad=-\left\{\frac{\partial}{\partial x}(up_{xx}+vp_{xy}+wp_{xz})+\frac{\partial}{\partial y}(up_{xy}+v_{yy}+wp_{yz})\right.\\&\quad\left.+\frac{\partial}{\partial z}(up_{xz}+vp_{yz}+wp_{zz})\right\}\\&\quad+\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right).\end{aligned}\right\}\cdots\cdots (\text{I. 9})$$

Here p_{xx}, p_{xy}, $\cdots$ etc. have such meanings as

$$p_{xx}=p-2\mu\frac{\partial u}{\partial x}+\frac{3}{2}\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right),$$

$$p_{xy}=-\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),$$

$$\cdots,$$

$$\cdots.$$

Substituting (I. 1), (I. 3), (I. 5) etc. into the present equation, we see that the conditions of similarity are fulfilled when there exist the proper proportions between the following quantities;

$\rho_0 c_v T_0 q_0/l_0$	intrinsic energy,	i)
$\rho_0 q_0^3/l_0$	energy of motion,	ii)
$\rho_0 q_0/l_0$	work done by pressure,	iii)
$q_0^2\mu/l_0^2$, $[q_0^2 T_0/(l_0\rho_0 q_0)\cdot d\mu/dT]$	heat produced by stress,	iv)
$T_0\lambda/l_0^2$, $[T_0^2/l_0^2\cdot d\lambda/dT]$	heat conducted,	v)

each being that which is convected or produced in unit time in unit volume.

a) Dividing ii) by i), we get

$$q_0^2/(c_vT)=Rq_0^2/(c_v m p_0\rho_0^{-1})=(\gamma-1)\gamma M^2, \qquad \cdots\cdots\cdots (\text{I. 10})$$

where

$$M^2=q_0^2/a_0^2,\quad a_0^2=(c_p/c_v)p_0\rho_0^{-1},\quad \gamma=c_p/c_v.$$

M is called Mach's number. Although it is defined originally in connection with the

* Taking $\mu=kT^n$, we get the relation; $\dfrac{T}{l_0 q_0\rho_0}\dfrac{d\mu}{dT}\propto\dfrac{\mu}{l_0 q_0\rho_0}$.

propagations of waves, Mach's number has to be taken in the form of M^2 from the mechanical point of view (cf. the foot note of p. 102).

b) Next we make (iii) ÷ (i);

$$(p_0 q_0/l_0).\ l_0/(\rho_0 c_v T_0 q_0) = p_0/(\rho_0 c_v T) = (c_p - c_v)/c_v. \quad \cdots\cdots\cdots (\text{I. } 11)$$

This is fixed according to the freedom of the molecular motion of the gases considered.

c) The ratio of (iv) to (i) is

$$(q_0^2 \mu/l_0^2).\ l_0/(\rho_0 c_v T_0 q_0) = \gamma(\gamma - 1) M^2/R_e. \quad \cdots\cdots\cdots (\text{I. } 12)$$

This ratio is not independent of the above mentioned numbers as we can take only two independent numbers from the three M, R_e, M^2/R_e.

d) The ratio of (v) to (i) is

$$(T_0 \lambda/l_0^2).\ l_0/(\rho_0 c_v T_0 q_0) = \lambda/(l_0 \rho_0 c_v q_0). \quad \cdots\cdots\cdots (\text{I. } 13)$$

The reciprocal of this is called Peclet's number.

7. To make up the similarity which we proposed at the beginning, all the ratios above mentioned should be common to the two fields of flows considered. As there are so many kinds of them as five to be considered, we have scarecely ever perfectly similar two flows. But taking our domain restrictedly narrow, we see that only a few quantities of them are more significant as compared with the others. For instance, in the case of a drop of fog hanging in the air $1/R_e$ and Euler's number are larger as compared, for example, with the case of a ball in base-ball play. In this case, thus, Stokes and Oseen succeeded in their methods.

After all it is important to classify the problems from the point of view of similarity and to understand the hydorodynamical character of each class. Thus the method of research will be naturally fixed.

In this case it is supposed that the problems will be altered according to where we place the basis of similarity and to what we claim* similarity.

II. Vorticity and Entropy in the High-Speed Flow of Gas
(On the Two-Dimensional Subsonic Flow)

8. In the flow where the so-called Reynolds' number is *large*, it is supposed that the diffusion of vorticity which exists a priori as a continuous layer in the surface in contact with solid body is restricted within the so-called boundary layer. On the other hand, we have the law of Helmholtz that the flow having no vorticity at the beginning will not have it for ever. Thus we are used to expect no vorticity in the flow out of the boundary layer.

Granting it to be reasonable to neglect the vorticity caused by the boundary layer, it is doubtful whether we could assume that the law of Helmholtz is right held also

* In (I. 1) it is natural and reasonable to put

$$x = l_0 x', \quad y = \delta_0 y', \quad l_0 \gg \delta_0,$$

when we deal with the problem of boundary layer flow. See the foot note of § 4.

in the high-speed flow, for example, of air where viscosity is varied with the variation of density.

In the present chapter, we limit our discussion to the case where the terms of M^2 are effective and those of M^4, M^6 are to be neglected. Therefore the application of the discussion is valid, strictly speaking, only in the so-called subsonic flow. Further we take the assumption of adiabatic flow. Essentially speaking, the vorticity thus produced is resulted directly from the lack of uniformity of density and viscosity in the equation of motion. The lack of the adiabatic character in energy plays only a complimentary role in the present problem.

9. Taking an orthogonal coordinate system x, y, we express respectively by u, v the velocity components in the directions of x, y. Further, to representing the character and the state of the gas, we adopt the following notations:

μ: coefficient of viscosity,
ρ: density,
ν: μ/ρ,
p: pressure,
c_p: specific heat at constant pressure,
γ: (specific heat at constant pressure)/(specific heat at constant volume),
λ: conductivity of heat,
T: absolute temperature,
t: time.

Thus the equations of motion are

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}-\frac{2}{3}\frac{\partial}{\partial x}\left\{\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right\}+2\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left\{\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\}, \qquad \cdots\cdots\cdots \text{(II. 1)}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}-\frac{2}{3}\frac{\partial}{\partial y}\left\{\mu\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right\}+2\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial x}\left\{\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\}. \qquad \cdots\cdots\cdots \text{(II. 2)}$$

The equation of continuity is

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0. \qquad \cdots\cdots\cdots \text{(II. 3)}$$

As the equation of energy we take

$$p\rho^{-\gamma}=\text{const} \qquad \cdots\cdots\cdots \text{(II. 4)}$$

according to the reason mentioned in the introductory sentences of the present chapter. The equations of state of the present gas are

$$p\rho^{-1}T^{-1}=\text{const}, \qquad \cdots\cdots\cdots \text{(II. 5)}$$

$$\mu T^{-n}=\text{const}. \qquad \cdots\cdots\cdots \text{(II. 6)}$$

In the last equation, assuming that the gas is air, n is 1/2 approximatly. In the following discussion we take

$$n=1/2. \qquad \cdots\cdots\cdots \text{(II. 7)}$$

10. As is usual with us, partially differentiating (II. 1) and (II. 2) respectively with respect to y and x and substituting (II. 2) from (II. 1) of each hand side, we get the following result:

$$\frac{\partial\omega}{\partial t}+u\frac{\partial\omega}{\partial x}+v\frac{\partial\omega}{\partial y}=\frac{1}{2\rho^2}\frac{\partial\rho}{\partial y}\Big[-\frac{2}{3}\frac{\partial}{\partial x}\Big\{\mu\Big(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\Big)\Big\}+2\frac{\partial}{\partial x}\Big(\mu\frac{\partial u}{\partial x}\Big)+\frac{\partial}{\partial y}\Big\{\mu\Big(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\Big)\Big\}\Big]$$
$$-\frac{1}{2\rho^2}\frac{\partial\rho}{\partial x}\Big[-\frac{2}{3}\frac{\partial}{\partial y}\Big\{\mu\Big(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\Big)\Big\}+2\frac{\partial}{\partial y}\Big(\mu\frac{\partial v}{\partial y}\Big)+\frac{\partial}{\partial x}\Big\{\mu\Big(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\Big)\Big\}\Big]$$
$$-\Big(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\Big)\omega-\frac{1}{\rho}\frac{\partial^2}{\partial x\partial y}\Big\{\mu\Big(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\Big)\Big\}-\frac{1}{2\rho}\frac{\partial^2}{\partial y^2}\Big\{\mu\Big(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\Big)\Big\}$$
$$+\frac{1}{2\rho}\frac{\partial^2}{\partial x^2}\Big\{\mu\Big(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\Big)\Big\}, \qquad \cdots\cdots\cdots(\text{II. }8)$$

where

$$\omega\equiv\frac{1}{2}\Big(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\Big), \qquad \cdots\cdots\cdots(\text{II. }9)$$

and the relations of (II. 1), (II. 2), (II. 4) are used.

Now let us assume that $\mu=0$ for the time $t<0$. Then from (II. 8) we get

$$\frac{\partial\omega}{\partial t}+u\frac{\partial\omega}{\partial x}+v\frac{\partial\omega}{\partial y}=-\Big(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\Big)\omega.$$

During $t<0$, if $\omega=0$ at the beginning, ω is always equal to zero everywhere.* In this case, accordingly, a steady flow under the conditions of continuity and irrotationality can be realized. Now let us assume that μ appears suddenly at the time $t=0$ as shown in (II. 6). After this time, ω can not be zero for the result of existence of viscosity. Namely the steady flow becomes unsteady according to (II. 8). The unsteadiness continues untill the flow becomes steady anew under the condition that μ and ω also are not zero. As, assuming that ω thus appearing in the flow is small, it does not give rise to a large alteration of the field of flow and the variation of T and ρ,** we can adopt the values of μ and ρ fixed corresponding to the innitial steady flow throughout the unsteady period.*** According to this consideration, putting μ equal to zero in (II. 1) and (II. 2), and assuming that $\omega=0$, as the first approximation,

$$\frac{1}{2}q^2+\int\frac{dp}{\rho}=\text{const} \qquad \cdots\cdots\cdots(\text{II. }10)$$

all over the field. Using (II. 4) one can integrate this equation and get

$$\frac{p}{\rho}=\frac{p_0}{\rho_0}+\frac{1}{2}\frac{\gamma-1}{\gamma}(U^2-q^2). \qquad \cdots\cdots\cdots(\text{II. }11)$$

Here $q^2=u^2+v^2$, and U is the value of q at the point where $p=p_0$, and $\rho=\rho_0$. Eliminating p from (II. 11) and (II. 4) we get

$$\frac{\rho}{\rho_0}=\Big\{1+\frac{1}{2}\frac{\gamma-1}{a_0^2}(U^2-q^2)\Big\}^{1/(\gamma-1)},$$

* As is deduced from (II. 8), this is realized even when μ is not zero, if μ and ρ are constant everywhere.

** Thus we can neglect the terms of M^4, M^6, . . .

*** μ and ρ are functions of location but not of time.

where $a_0^2=\gamma p_0/\rho_0$. Putting

$$M^2=U^2/a_0^2, \quad q=Uq',$$

and neglecting the terms of M^4, M^6, etc., we get

$$\frac{\rho}{\rho_0}=1+\frac{1}{2}M^2(1-q'^2). \quad \cdots\cdots\cdots(\text{II. }12)$$

Similarly

$$\frac{p}{p_0}=1+\frac{\gamma}{2}M^2(1-q'^2). \quad \cdots\cdots\cdots(\text{II. }13)$$

From (II, 12), as for the variation of ρ, we get

$$\frac{\delta\rho}{\rho_0}=-M^2q'\,\delta q'. \quad \cdots\cdots\cdots(\text{II. }14)$$

As for μ, from (II. 6),

$$\mu=\mu_0\left(\frac{T}{T_0}\right)^{1/2}=\mu_0\left\{1+\frac{1}{4}(\gamma-1)M^2(1-q'^2)\right\}.$$

Thus, (II. 12) being taken into consideration, we get

$$\frac{\mu}{\rho}=\nu_0\left\{1+\frac{1}{4}(\gamma-3)M^2(1-q'^2)\right\} \quad \cdots\cdots\cdots(\text{II. }15)$$

and as the variation of μ, making use of (II. 5) and (II. 6),

$$\frac{\delta\mu}{\mu}=\frac{1}{2}\left(\frac{\delta p}{p}-\frac{\delta\rho}{\rho}\right)=\frac{1}{2}(\gamma-1)\frac{\delta\rho}{\rho}. \quad \cdots\cdots\cdots(\text{II. }16)$$

By means of (II. 14) and (II. 15), we get

$$\frac{\delta\mu}{\rho}=-\frac{1}{2}(\gamma-1)\nu_0M^2q'\,\delta q'. \quad \cdots\cdots\cdots(\text{II. }17)$$

From (II. 16),

$$\frac{\delta^2\mu}{\mu}-\left(\frac{\delta\mu}{\mu}\right)^2=\frac{1}{2}(\gamma-1)\left\{\frac{\delta^2\rho}{\rho}-\left(\frac{\delta\rho}{\rho}\right)^2\right\}.$$

Considering (II. 15) and (II. 16), we get

$$\frac{\delta^2\mu}{\rho}=-\frac{1}{2}(\gamma-1)\nu_0M^2(q'\,\delta^2q'+\delta q'\,\delta q'). \quad \cdots\cdots\cdots(\text{II. }18)$$

Now let us adopt the following notation:

$$\frac{\partial u}{\partial x}\equiv\delta+\bar{\delta},\quad \frac{\partial u}{\partial y}\equiv-\omega+\bar{\omega},\quad \frac{\partial v}{\partial x}\equiv\omega+\bar{\omega},\quad \frac{\partial v}{\partial y}\equiv\delta-\bar{\delta}, \quad \cdots\cdots\cdots(\text{II. }19)$$

$$\left.\begin{array}{l} q\equiv Uq',\quad u\equiv Uu',\quad v\equiv Uv',\quad x\equiv l\xi,\quad y\equiv l\eta,\quad t\equiv l/U\cdot t', \\ \omega\equiv U/l\cdot\omega',\quad \delta\equiv U/l\cdot\delta',\quad \bar{\omega}\equiv U/l\cdot\bar{\omega}',\quad \bar{\delta}\equiv U/l\cdot\bar{\delta}'. \end{array}\right\} \cdots\cdots(\text{II. }20)$$

Thus (II. 8) becomes

$$\begin{aligned}\frac{\partial\omega'}{\partial t'}=&-u'\frac{\partial\omega'}{\partial\xi}-v'\frac{\partial\omega'}{\partial\eta}-2\delta'\omega'+\left\{-\frac{1}{2}\frac{\partial q'^2}{\partial\eta}\left(\frac{4}{3}\frac{\partial\delta'}{\partial\xi}-\frac{\partial\omega'}{\partial\eta}\right)+\frac{1}{2}\frac{\partial q'^2}{\partial\xi}\left(\frac{4}{3}\frac{\partial\delta'}{\partial\eta}+\frac{\partial\omega'}{\partial\xi}\right)\right\}\frac{M^2}{R_e}\\ &+\left\{\frac{1}{2}\frac{\partial q'^2}{\partial\eta}\left(\frac{\partial\delta'}{\partial\xi}-\frac{\partial\omega'}{\partial\eta}\right)-\frac{1}{2}\frac{\partial q'^2}{\partial\xi}\left(\frac{\partial\omega'}{\partial\xi}+\frac{\partial\delta'}{\partial\eta}\right)+\frac{1}{2}\frac{\partial^2q'^2}{\partial\xi\partial\eta}\bar{\delta}'+\frac{1}{4}\left(\frac{\partial^2q'^2}{\partial\eta^2}-\frac{\partial^2q'^2}{\partial\xi^2}\right)\bar{\omega}'\right\}(\gamma-1)\frac{M^2}{R_e}\\ &+\left\{1+\frac{1}{4}(1-q'^2)(\gamma-3)M^2\right\}\left(\frac{\partial^2\omega'}{\partial\xi^2}+\frac{\partial^2\omega'}{\partial\eta^2}\right)\frac{1}{R_e},\end{aligned} \quad \cdots\cdots\cdots(\text{II. }21)$$

where $R_e=lU/\nu_0$.

As assumed above, we make everywhere $\omega'=0$ when $t'=0$. Consequently, after the time interval $\Delta t'$, i.e., when $t'=\Delta t'$, ω' is equal to $\omega'_{t'=\Delta t'}$, where, integrating (II, 21),

$$\omega'_{t'=\Delta t'}=\Big[-\frac{2}{3}\frac{\partial q'^2}{\partial\eta}\frac{\partial\delta'}{\partial\xi}+\frac{2}{3}\frac{\partial q'^2}{\partial\xi}\frac{\partial\delta'}{\partial\eta}$$
$$+(\gamma-1)\Big\{\frac{1}{2}\frac{\partial q'^2}{\partial\eta}\frac{\partial\delta'}{\partial\xi}-\frac{1}{2}\frac{\partial q'^2}{\partial\xi}\frac{\partial\delta'}{\partial\eta}+\frac{1}{2}\frac{\partial^2 q'^2}{\partial\xi\partial\eta}\bar{\delta}'+\frac{1}{4}\Big(\frac{\partial^2 q'^2}{\partial\eta^2}-\frac{\partial^2 q'^2}{\partial\xi^2}\Big)\bar{\omega}'\Big\}\Big]\frac{M^2}{R_e}\Delta t'.$$

Inserting the result into the right hand side of (II. 21), we get the value of ω' at the time $t'=2\Delta t'$ by integration. After these successive integrations, we shall get the result converging finitely for the value of t' sufficiently large. This assumption is required reasonablly from the fact that the steady high-speed flow really exists and that the vorticity in that flow is usually trivial. The result thus obtained has apparently M^2/R_e as the coefficient. In addition to this fact we see that from (II. 3), (II. 14)

$$\delta'\equiv\frac{\partial u'}{\partial\xi}+\frac{\partial v'}{\partial\eta}=\Big(u'q'\frac{\partial q'}{\partial\xi}+v'q'\frac{\partial q'}{\partial\eta}+q'\frac{\partial q'}{\partial t'}\Big)M^2.$$

Getting, thus, the general view of ω' and δ', we see that in (II. 21) there remain the terms which are of the order of M^4 or M^2/R_e^2. Neglecting these terms anew, we get the equation of steady flow for t' sufficiently large;

$$u'\frac{\partial\omega'}{\partial\xi}+v'\frac{\partial\omega'}{\partial\eta}=\frac{1}{2}(\gamma-1)\frac{M^2}{R_e}\Big\{\bar{\delta}'\frac{\partial^2 q'^2}{\partial\xi\partial\eta}-\frac{1}{2}\bar{\omega}'\Big(\frac{\partial^2 q'^2}{\partial\xi^2}-\frac{\partial^2 q'^2}{\partial\eta^2}\Big)\Big\}. \quad\cdots\cdots\text{(II. 22)}$$

As stated repeatedly, from the point of view of neglecting the terms of M^4, M^2/R_e^2, etc., u', v', q', $\bar{\delta}'$, $\bar{\omega}'$ are independent of ω'. That is to say (II. 22) is linear of ω'. All the quantities except ω' are got under the conditions of irrotational motion and continuity. (II. 22) is rewritten by (II. 20);

$$u\frac{\partial\omega}{\partial x}+v\frac{\partial\omega}{\partial y}=\frac{1}{2}(\gamma-1)\frac{\nu_0}{a_0}\Big\{\bar{\delta}\frac{\partial^2 q^2}{\partial x\partial y}-\frac{1}{2}\bar{\omega}\Big(\frac{\partial^2 q^2}{\partial x^2}-\frac{\partial^2 q^2}{\partial y^2}\Big)\Big\}. \quad\cdots\cdots\cdots\text{(II. 23)}$$

Although the state represented by the suffix 0 is to be selected at will, it is reasonable to take the state corresponding to the mean velocity so as to make $1-q'^2$ in (II. 12) as small as possible throughout the field.

11. *Calculation of vorticity.* As the equation (II. 23) is linear with regard to ω, the integration of it is easy. The characteristic equations are

$$\frac{dx}{u}=\frac{dy}{v}=\frac{d\omega}{\frac{1}{2}(\gamma-1)\frac{\nu_0}{a_0^2}\Big\{\bar{\delta}\frac{\partial^2 q^2}{\partial x\partial y}-\frac{1}{2}\bar{\omega}\Big(\frac{\partial^2 q^2}{\partial x^2}-\frac{\partial^2 q^2}{\partial y^2}\Big)\Big\}}, \quad\cdots\cdots\cdots\text{(II. 24)}$$

or

$$\frac{dx}{dy}=\frac{u}{v}, \quad\cdots\cdots\cdots\text{(II. 25)}$$

$$\frac{d\omega}{dx}=\frac{\frac{1}{2}(\gamma-1)\frac{\nu_0}{a_0^2}\Big\{\bar{\delta}\frac{\partial^2 q^2}{\partial x\partial y}-\frac{1}{2}\bar{\omega}\Big(\frac{\partial^2 q^2}{\partial x^2}-\frac{\partial^2 q^2}{\partial y^2}\Big)}{u}. \quad\cdots\cdots\cdots\text{(II. 26)}$$

Here (II. 25) represents the stream lines under the condition of irrotational motion. Referring to this equation, ω is calculated from (II. 26) where the right hand side is also determined from the condition of irrotationality. As shown in the foot note of page 106, when the speed is negligibly small as compared with the sonic velocity, an irrotational flow in infinitely distant place goes down irrotationally for ever following (II. 26).

12. *Application.* For example, let us apply (II. 24) to the case of the flow around a circular cylinder. Taking the center of the circle as the origin (Fig. 1) we make the coordinate system, the x-axis being parallel to the main flow, the y-axis perpendicular to it. Let the radius be r_0, the velocity of main flow be U, then

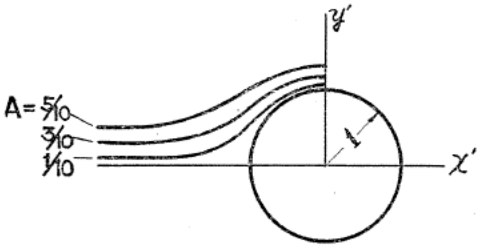

Fig. 1

$$\frac{d\omega'}{dx'}=12(\gamma-1)\frac{\nu_0}{Ur_0}\frac{U^2}{a_0^2}A\frac{(-r'^2-1)}{r'^4(r'^2-1)\{r'^4+(r'^2-2x'^2)\}}, \quad \cdots\cdots\cdots(\text{II. }27)$$

$$x'^2=r'^2-A\frac{r'^4}{(r'^2-1)^2}, \quad \cdots\cdots\cdots(\text{II. }28)$$

where

$$\omega=\omega'\frac{U}{r_0},$$

$r=r'r_0$ (the distance of a point from the origin),

$x=x'r_0$.

A being the number which indicates a stream line, the relation between A and the point on the y-axis where the flow passes is as follows;

Table 1

A	1/10	2/10	3/10	4/10	5/10
y'	1.052	1.105	1.161	1.220	1.277

The integration of (II. 27) is worked out graphycally. The mutually corresponding values of x' and y' are obtained from (II. 28). Concerning each given value of A, $d\omega'/dx'$ is calculated at each point from (II. 27). The result is indicated in Fig. 2.

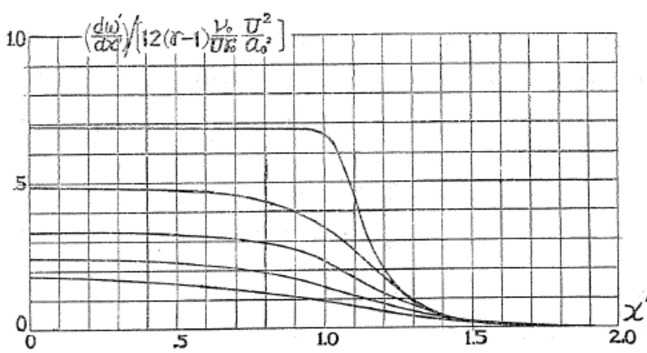

Fig. 2

Of each stream line (A is constant) we integrate from $x=\infty$ to $x=0$ graphycally, and get the result as follows;

Table 2.

A	1/10	2/10	3/10	4/10	5/10
$\omega' \Big/ \left[12(\gamma-1)\frac{\nu_0}{Ur_0}\frac{U^2}{a_0^2}\right]$	0.8005	0.5469	0.3711	0.2578	0.1758

In the case of air, taking $\gamma=1.4$

$$\nu_0=1.3\times10^{-1} \text{ dyne sec cm/g}$$

$$a_0=33000 \text{ cm/sec}, \quad r_0=1 \text{ cm}, \quad U=20000 \text{ cm/sec},$$

the following is calculated:

$$\frac{\nu_0}{Ur_0}=6.5\times10^{-6}, \quad \frac{U}{a_0}=0.65.$$

Thus ω', being of the order $12(\gamma-1)\frac{\nu_0 U}{r_0 a_0^2}\fallingdotseq 10^{-5}$, may be said trivial in this case.

13. Although ω' is trivial in our daily experience, yet it will be so large that we can not neglect in the case of shock line* where the mathematical analysis in view of perfectly irrotational flow loses physical meaning. This will be the reason why there exists really the field nearly satisfying the boundary condition with which the mathematical analysis fails.[1]

These are also to be said about entropy; namely the adiabatic condition is to be corrected in the same way as the irrotational condition. In the case of oblique shock wave, Crocco discussed more or less kinematically the same phenomena as the present.[2]

III. Graphycal Solution of the Two-Dimensional Euler's Equation

14. The two-dimensional Euler's equation in orthogonal coordinate system x, y is

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x}, \qquad \cdots\cdots\cdots (\text{III. }1)$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y}, \qquad \cdots\cdots\cdots (\text{III. }2)$$

where u, v represent respectively the velocity components in the direction of x, y and ρ density, p pressure. The equation of continuity with regard to steady state is

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0. \qquad \cdots\cdots\cdots (\text{III. }3)$$

In addition to these, the equation of energy is

$$p\rho^{-\gamma}=\text{const.} \qquad \cdots\cdots\cdots (\text{III .}4)$$

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* Cf. §16 and Chapter IV of the present paper.

1) T. Koga, J. Soc. Aero. Sic. Japan, **71**, 241 (1941).

2) L. Crocco, Zeitschrift f. angew. Math. u. Mech., Bd. 17, Heft 1, 1 (1937).

Now, eliminating p from (III. 1) and (III. 2), and applying (III. 4), one can take*

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0. \qquad \cdots\cdots\cdots (\text{III. } 5)$$

Thus one can integrate (III. 1) and (III. 2) applying the conditions (III. 4) and (III. 5), the result obtained is

$$u^2+v^2\equiv q^2=\frac{2\gamma}{\gamma-1}\frac{p_0}{\rho_0}\left\{1-\left(\frac{p}{p_0}\right)^{(\gamma-1)/\gamma}\right\}, \qquad \cdots\cdots\cdots (\text{III. } 6)$$

where the suffix 0 expresses the state where $q=0$, i.e., the stagnation point.

As shown in Fig. 3, we take a flow uniform at the infinitely distant place which, flowing down, gets disturbed making up our problems to be solved. At infinitely distant place, the velocity is uniformly q_m, the pressure p_m, the density ρ_m, and there we imagine stream lines, the distance of each being s_m. These quantities change respectively to q, p, ρ, s, as the flow goes down adiabatically.

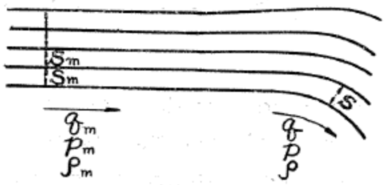

Fig. 3

Let the radius of curvature of a stream line be r and the origine of the coordinate system be on the stream line, x being in the direction of the tangential line and y normal to x and directed oppositely to the center of curvature, then (III. 2) becomes

$$\frac{\rho q^2}{r}=\frac{\partial p}{\partial y}, \qquad \cdots\cdots\cdots (\text{III. } 7)$$

and (III. 3)

$$\rho q s=\rho_m q_m s_m \qquad \cdots\cdots\cdots (\text{III. } 8)$$

and (III. 4)

$$p\rho^{-\gamma}=p_m\rho_m^{-\gamma}. \qquad \cdots\cdots\cdots (\text{III. } 9)$$

The flow is determined by means of (III. 6), (III. 7), (III. 8) and (III. 9). When the flow is of low-speed and the variation of state is not large throughout the field, we can take ρ=const., and in place of (III. 8)

$$qs=q_m s_m, \qquad \cdots\cdots\cdots (\text{III. } 10)$$

and of (III. 7)

$$\frac{q}{r}=-\frac{\partial q}{\partial y}. \qquad \cdots\cdots\cdots (\text{III. } 11)$$

Further for (III. 6) we can put

$$p+\tfrac{1}{2}\rho q^2=p_0. \qquad \cdots\cdots\cdots (\text{III. } 12)$$

15. *Graphycal method.*

i) *The case of the boundary condition of the first type where the form of a standard stream line and the state on it are given.* As the variable of state which is given on the standard stream line, any one of three (pressure, density and velocity) may well be adopted. One being known, the other two are decided by

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* Cf. Chapter II.

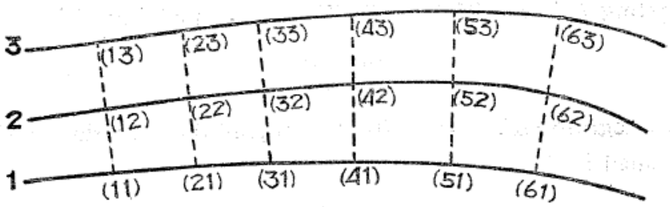

Fig. 4

means of (III. 6) and (III. 9). Let us suppose that the stream line 1 in Fig. 4 and the pressure distribution on it are given beforehand. On this line we take properly the points 11, 21, 31 etc. The states at these points are denoted by the suffixes 11, 21, 31 etc. Firstly we write down the pressures

$$p_{11} \quad p_{21} \quad p_{31} \quad p_{41} \quad p_{51} \quad p_{61} \quad p_{71} \cdots .$$

Then we measure the radius of curvature of the stream line at each point. When the center of curvature is on the side opposite to the direction to which we intend to proceed drawing the succesive stream lines, we take the radius as positive. Namely

$$r_{11} \quad r_{21} \quad r_{31} \quad r_{41} \quad r_{51} \quad r_{61} \cdots .$$

From (III. 6) we can decide q, thus we can arrange them

$$q_{11} \quad q_{21} \quad q_{31} \quad q_{41} \quad q_{51} \quad q_{61} \cdots .$$

From (III. 9), similarly,

$$\rho_{11} \quad \rho_{21} \quad \rho_{31} \quad \rho_{41} \quad \rho_{51} \quad \rho_{61} \cdots ,$$

and from (III. 8)

$$s_{11} \quad s_{21} \quad s_{31} \quad s_{41} \quad s_{51} \quad s_{61} \cdots .$$

The s's mean the distances between the stream line firstly given and that to be drawn secondly. Consequently we make the normal line of length of s at each point on the stream line 1, and then connect the terminals of these lines with each neighbouring one. Thus we get the second stream line, i.e., the stream line 2. The state on the second stream line is decided as follows: From (III. 7) we know

$$\left(\frac{\partial p}{\partial y}\right)_{11} \quad \left(\frac{\partial p}{\partial y}\right)_{21} \quad \left(\frac{\partial p}{\partial y}\right)_{31} \quad \left(\frac{\partial p}{\partial y}\right)_{41} \quad \left(\frac{\partial p}{\partial y}\right)_{51} \cdots .$$

The pressure differences Δp between two neighbouring stream lines are

$$\Delta p_{11} = s_{11} \times \left(\frac{\partial p}{\partial y}\right)_{11}, \quad \Delta p_{21} = s_{21} \times \left(\frac{\partial p}{\partial y}\right)_{21}, \quad \Delta p_{31} = s_{31} \times \left(\frac{\partial p}{\partial y}\right)_{31}, \cdots .$$

The state on the line 2 is generally denoted by the suffixes 12, 22, 32, 42, $\cdots$;

$$p_{12} = p_{11} + \Delta p_{11}, \quad p_{22} = p_{21} + \Delta p_{21}, \quad p_{32} = p_{31} + \Delta p_{31}, \cdots .$$

Continuing the above stated operation succesively, we can proceed to draw all the stream lines.

As these operations are more or less complicated, we adopt in fact the following plan: Let M_m be the Mach's number of the standard state, then

$$\frac{s}{s_m}=\frac{\left(\frac{\gamma-1}{2}M_m^2+1\right)^{1/(1-\gamma)}\sqrt{1-\left(\frac{\gamma-1}{2}M_m^2+1\right)^{-1}}}{\left(\frac{p}{p_0}\right)^{1/\gamma}\sqrt{1-\left(\frac{p}{p_0}\right)^{(\gamma-1)/\gamma}}}.$$

This, being a function of p/p_0, is represented by A, and its numerical values are prepared beforehand in a table (Table 3). Next

$$\Delta\left(\frac{p}{p_0}\right)=B\times\frac{1}{r/s_m},$$

$$B=\frac{2\gamma}{\gamma-1}\sqrt{1-\left(\frac{p}{p_0}\right)^{(\gamma-1)/\gamma}}\sqrt{1-\left(\frac{\gamma-1}{2}M_m^2+1\right)^{-1}}\left(\frac{\gamma-1}{2}M^2+1\right)^{1/(1-\gamma)}.$$

This B is also a function of p/p_0 and similarly prepared in Table 3. Next we

Table 3

p/p_0	A	B	q/a_0	q/a	p/p_0	A	B	q/a_0	q/a	p/p_0	A	B	q/a_0	q/a
1.00	∞	**0**	**0**	**0**	**0.60**	**1.012**	**0.668**	**0.822**	**0.885**	**0.20**	**1.343**	**1.100**	**1.354**	**1.707**
0.99	4.874	0.097	0.120	0.120	0.59	1.009	0.678	0.835	0.901	0.19	1.376	1.114	1.371	1.741
0.98	3.465	0.138	0.169	0.170	0.58	1.006	0.688	0.847	0.916	0.18	1.412	1.128	1.388	1.777
0.97	2.845	0.169	0.208	0.209	0.57	1.004	0.698	0.860	0.932	0.17	1.453	1.142	1.405	1.814
0.96	2.477	0.195	0.240	0.242	0.56	1.002	0.708	0.872	0.948	0.16	1.497	1.157	1.424	1.854
0.95	2.228	0.219	0.269	0.271	0.55	1.001	0.718	0.884	0.964	0.15	1.547	1.172	1.442	1.896
0.94	2.046	0.240	0.296	0.298	0.54	1.000	0.728	0.896	0.980	0.14	1.604	1.188	1.462	1.940
0.93	1.905	0.260	0.320	0.323	0.53	1.000	0.738	0.909	0.996	0.13	1.668	1.204	1.482	1.988
0.92	1.792	0.278	0.343	0.347	0.52	1.000	0.749	0.921	1.012	0.12	1.741	1.221	1.503	2.040
0.91	1.700	0.296	0.364	0.369	0.51	1.001	0.758	0.933	1.028	0.11	1.825	1.239	1.525	2.096
0.90	**1.622**	**0.312**	**0.384**	**0.390**	**0.50**	**1.002**	**0.768**	**0.946**	**1.045**	**0.10**	**1.924**	**1.258**	**1.548**	**2.157**
0.89	1.556	0.328	0.404	0.410	0.49	1.003	0.778	0.958	1.062	0.095	1.981	1.267	1.560	2.190
0.88	1.499	0.343	0.423	0.430	0.48	1.005	0.789	0.971	1.079	0.09	2.042	1.278	1.572	2.224
0.87	1.449	0.358	0.441	0.450	0.47	1.007	0.799	0.983	1.096	0.085	2.110	1.288	1.585	2.261
0.86	1.405	0.372	0.458	0.468	0.46	1.010	0.809	0.995	1.113	0.08	2.184	1.299	1.598	2.300
0.85	1.366	0.386	0.475	0.486	0.45	1.013	0.819	1.008	1.130	0.075	2.268	1.310	1.612	2.341
0.84	1.331	0.400	0.492	0.504	0.44	1.017	0.829	1.020	1.148	0.07	2.361	1.321	1.626	2.386
0.83	1.300	0.413	0.508	0.522	0.43	1.021	0.839	1.033	1.166	0.065	2.466	1.333	1.641	2.433
0.82	1.271	0.426	0.524	0.539	0.42	1.026	0.849	1.045	1.185	0.06	2.587	1.346	1.656	2.484
0.81	1.246	0.438	0.539	0.556	0.41	1.031	0.860	1.058	1.203	0.055	2.725	1.359	1.673	2.541
0.80	**1.222**	**0.451**	**0.555**	**0.573**	**0.40**	**1.037**	**0.870**	**1.071**	**1.222**	0.05	2.886	1.373	1.690	2.602
0.79	1.201	0.463	0.570	0.589	0.39	1.044	0.880	1.083	1.241	0.045	3.078	1.388	1.708	2.671
0.78	1.181	0.475	0.584	0.606	0.38	1.051	0.891	1.096	1.260	0.04	3.309	1.404	1.728	2.748
0.77	1.164	0.487	0.599	0.622	0.37	1.058	0.901	1.109	1.280	0.035	3.595	1.421	1.749	2.836
0.76	1.147	0.498	0.613	0.637	0.36	1.066	0.912	1.122	1.300	0.03	3.960	1.440	1.772	2.938
0.75	1.132	0.509	0.627	0.653	0.35	1.075	0.923	1.136	1.321	0.025	4.443	1.461	1.798	3.060
0.74	1.118	0.521	0.641	0.669	0.34	1.085	0.934	1.149	1.342	0.02	5.125	1.485	1.827	3.211
0.73	1.106	0.532	0.654	0.685	0.33	1.096	0.944	1.162	1.364	0.015	6.172	1.513	1.862	3.413
0.72	1.094	0.543	0.668	0.701	0.32	1.107	0.955	1.176	1.386	0.01	8.053	1.548	1.905	3.700
0.71	1.083	0.554	0.681	0.716	0.31	1.119	0.967	1.189	1.408	0.005	12.774	1.598	1.966	4.220
0.70	**1.073**	**0.564**	**0.695**	**0.731**	**0.30**	**1.132**	**0.978**	**1.203**	**1.431**	**0**	∞	**1.806**	**2.222**	∞
0.69	1.064	0.575	0.708	0.747	0.29	1.147	0.989	1.217	1.455	**0.527**	**1.000**	**0.741**	**0.913**	**1.000**
0.68	1.056	0.586	0.721	0.762	0.28	1.162	1.001	1.232	1.480					
0.67	1.048	0.596	0.734	0.777	0.27	1.179	1.013	1.246	1.505					
0.66	1.041	0.607	0.747	0.793	0.26	1.197	1.024	1.261	1.531					
0.65	1.035	0.617	0.759	0.808	0.25	1.216	1.036	1.275	1.558					
0.64	1.029	0.627	0.772	0.823	0.24	1.238	1.049	1.290	1.585					
0.63	1.024	0.638	0.785	0.839	0.23	1.261	1.061	1.306	1.614					
0.62	1.019	0.648	0.797	0.854	0.22	1.286	1.074	1.322	1.644					
0.61	1.015	0.658	0.810	0.870	0.21	1.313	1.087	1.338	1.675					
0.60	**1.012**	**0.668**	**0.822**	**0.885**	**0.20**	**1.343**	**1.100**	**1.354**	**1.707**					

prepare a table for complimentary calculation as Table 4.

In this graphycal method, the most difficult operation is to measure the curvatures

Table 4

	Point on stream line					
	1	2	3	4	5	6
$(p/p_0)_1$						
$(r/s_m)_1$						
A_1						
B_1						
$\Delta(p/p_0)_1$						
$(p/p_0)_2$						
$(r/s_m)_2$						
A_2						
B_2						

of line. For convenience' sake, it would be beneficial to prepare a rule.*

For the special case where the density is constant, and the velocity distribution is given on the first stream line, it is simpler and easier to deal with the problem:

$$q_{11} \quad q_{21} \quad q_{31} \quad q_{41} \quad q_{51} \cdots.$$

The radii of curvature are also measured:

$$r_{11} \quad r_{21} \quad r_{31} \quad r_{41} \quad r_{51} \cdots.$$

From (III. 10)

$$s_{11} \quad s_{21} \quad s_{31} \quad s_{41} \quad s_{51} \cdots.$$

are got. The differnces of the velocity Δq between the neighbouring stream lines are

$$\Delta q_{11} = s_{11} \times \left(\frac{\partial q}{\partial y}\right)_{11}, \quad \Delta q_{21} = s_{21} \times \left(\frac{\partial q}{\partial y}\right)_{21}, \quad \Delta q_{31} = s_{31} \times \left(\frac{\partial q}{\partial y}\right)_{41}, \quad \cdots$$

refering to (III. 11).

Thus

$$q_{12} = q_{11} + \Delta q_{11}, \quad q_{23} = q_{21} + \Delta q_{21}, \quad q_{32} = q_{31} + \Delta q_{31}, \quad q_{42} = q_{41} + \Delta q_{41}, \cdots.$$

In this case we get

$$\frac{s}{s_m} = \frac{q_m}{q}, \quad \Delta\left(\frac{q}{q_m}\right) = -\frac{q}{q_m} \times \frac{s}{r} = -\frac{s_m}{r}.$$

Thus the drawing of the successive stream line can be worked out. Here also we use a table for complimentary calculation as Table 5.

* In fact the author prepared one temporarily, but it may be not the best and the illustration of it is here omitted.

Table 5

	Point on stream line					
	1	2	3	4	5	6
$(q/q_m)_1$						
$(r/s_m)_1$						
$(s/s_m)_1$						
$\Delta(q/q_m)_1$						
$(q/q_m)_2$						
$(r/s_m)_2$						

ii) *The case of the boundary condition of the second type where a standard equipotential curve and the state on it are given.* Let us assume that we know the pressure distribution on an equipotential line and consequently the corresponding distances for the common distance s_m between mutually neighbouring points on the line by means of (III. 6), (III. 8) and (III. 9):

$$s_{11} \quad s_{12} \quad s_{13} \quad s_{14} \quad s_{15} \quad s_{16} \cdots$$

At each point the p's are already known:

$$p_{11} \quad p_{12} \quad p_{13} \quad p_{14} \quad p_{15} \quad p_{16} \cdots .$$

Then we culculate the curvature of each stream line at each point by means of (III. 7):

$$r_{11} \quad r_{12} \quad r_{13} \quad r_{14} \quad r_{15} \quad r_{16} \cdots .$$

Thus we can draw elements of stream lines and the second potential line perpendicular to them. Now we can measure the s's on the second equipotential line:

$$s_{21} \quad s_{22} \quad s_{23} \quad s_{24} \quad s_{25} \quad s_{26} \cdots .$$

Table 6

	Point on equipotential line					
	1	2	3	4	5	6
$(p/p_0)_1$						
$A_1 \equiv (s/s_m)_1$						
$\Delta(p/p_0)_1$						
B_1						
$(r/s_m)_1$						
$A_2 \equiv (s/s_m)_2$						
$(p/p_0)_2$						

Then the q's on them are obtained from (III. 6), (III. 8) and (III. 9) and consequently the p's, and so on. In the same way we continue to draw the successive equipotential lines. Here we can use beneficially Table 6 for the complimentary calculation.

When the density is constant, we work out the problem similarly to the case i) by Table 7.

Table 7

	Ponit on equipotential line					
	1	2	3	4	5	6
$(q/q_m)_1$						
$(s/s_m)_1$						
$\Delta(q/q_m)_1$						
$(r/s_m)_1$						
$(s/s_m)_2$						
$(q/q_m)_2$						

16. *Examples.* The method in the present chapter is available equally to all kinds of problems, i.e., to the case of field of ellipsoidal, parabolic, or hyperbolic type. This is the strong point of the present method. On the other hand, the boundary conditions permitted in this method are more or less restricted, and this is the weak point together with the fact that there is the unavoidable limit of accuracy of the obtained result being caused by the limited accuracy of measurement of curvature of a line.

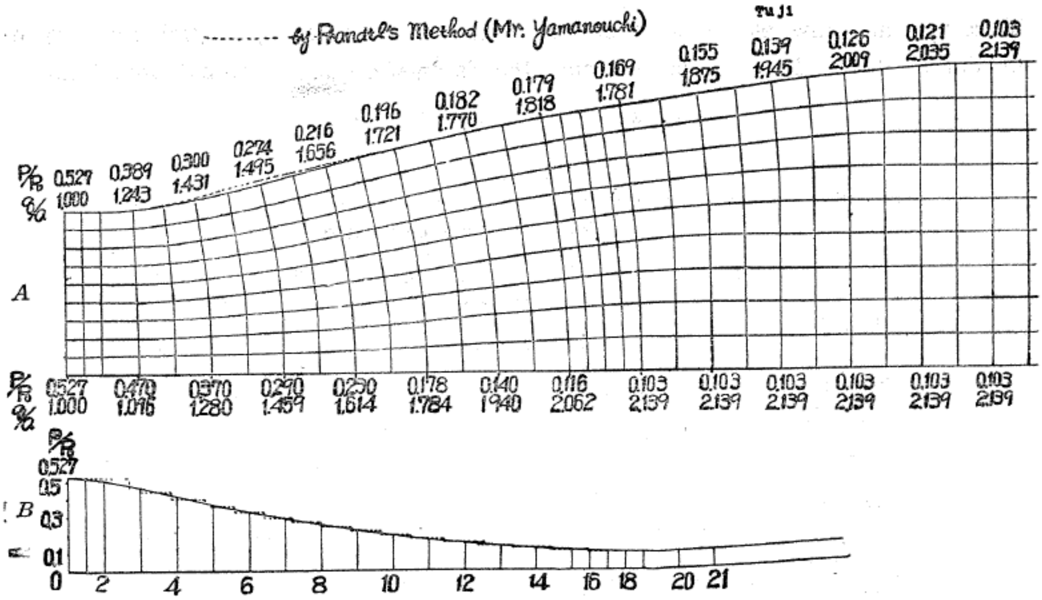

Fig. 5

i) *The flows in a tube.* Let us examine flows in a two-dimensional tube, the form of one of the two walls and the pressure distribution on it being given. We can also decide the forms of the two walls, the pressure distribution on the central line (taking symmetrically the two walls, this line is a central line) being given at will. In Fig. 5 there is shown one of the examples of this kind ($\gamma=1.4$). The result* by means of the method of Prandtl[1] and others is also written in the same figure with the dotted line. The difference between the full line (our result) and the dotted line is scarcely distinguishable. In case of the Prandtl's method, the pressure distribution is given in the form of steps as shown with the dotted line in *B* of the figure.

ii) *Flow around a circular cylinder.* Taking the density to be constant, the problem of the flow around a circular cylinder is easily solved by means of the analytical method. Therefore it will be interesting to make the comparison between the result of mathematical analysis and what we obtain by our method. Firstly, from this point of view, we worked out analytically to get the pressure distribution on

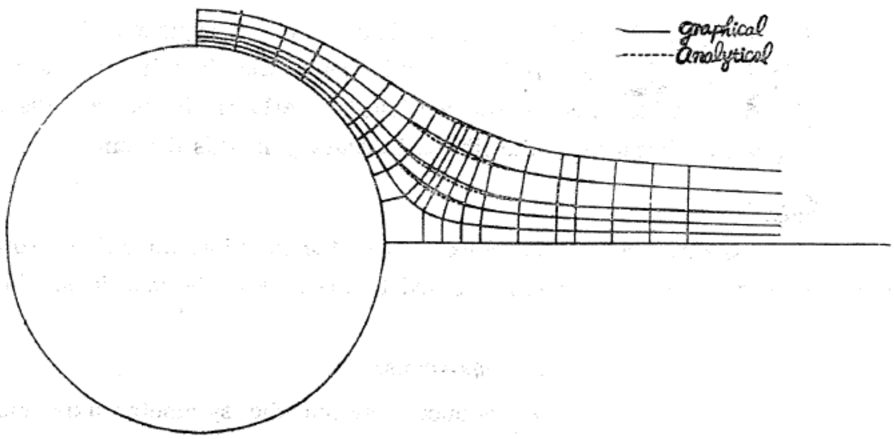

Fig. 6

the stream line which passes through the stagnation points. Secondly, taking this line as the basis, we proceed to draw outwardly the successive stream lines. Lastly, these lines are compared with those which are obtained quite analytically. Near the stagnation point, as the radius of curvature becomes small suddenly, we must have more or less knack in this operation. Speaking truely, as the basis we should not take such a line from the first. At any rate we have got the result as shown in Fig. 6. The dotted lines are of mathematical analysis.

iii) *Shock line.* On a shock line,** the acceleration is infinitely large. In Fig. 7, we are given firstly the stream line indicated by 0 and the pressure distribution on it. The pressure distribution shows that the flow along this stream line is thoroughly

* This is obtained by Mr. M. Yamanouchi.

** See the next chapter of the present article.

1) A. Busemann, *Handbuch ber Experimental Physik* (1931), IV, Teil 1, S. 428.

subsonic ($\gamma=1.4$). Firstly the graphycal operation is proceeded upwards and we get the result that the flow becomes slower. Secondly, downwards, the flow then gets accelerated and up to supersonic one. At last, we arrive at the points where the

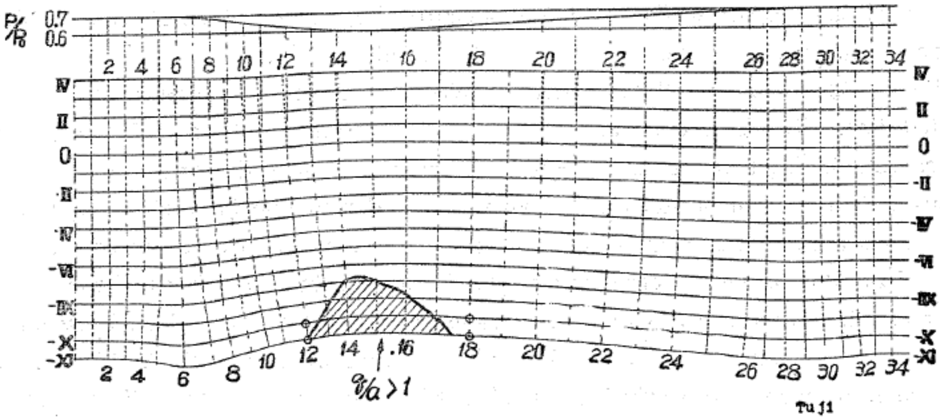

Fig. 7

stream line has a trip (indicated by ∘ in the figure) and consequently the acceleration is infinitly large. Before arriving at this point, the flow has already been supersonic (the domain indicated by oblique lines). Perhaps, the present discusion may be too inquisitive, because of the limitted accuracy in this domain.

17. *Appendix.*

a. Extension to the case of 3-dimensions. When the flow has an axis of rotatory symmetry, we can extend the present method to the case of 3-dimensions. In this case we take

$$2\pi R\rho qs=\text{const}$$

instead of (III. 8). Here R is the distance between the symmetry axis and the point considered.

b. Application to problems of elasticity etc. By the present method, we can solve the problem of torsion of infinitely long rod of uniform section of any form, i.e., the problem of de Saint-Venant.

In addition to this, it is available for the cases of stable temperature field, electric field, magnetic field etc.

IV. Characters of Flow of the Speed near That of Sound

18. As for the phenomena of shock wave in flow in a tube, we have more or less knowledges[1] of the condition which gives rise to them. But we are scarecely

1) W. F. Durannd, *Aerodynamic Theory* (Julius Springer, Berlin, 1935), III. pp. 222–229. F. Ringleb, Z. für angewandte Math. u. Mech., Bd. 20, Heft 4, S. 185 (1940). Th. von Karman, J. of the Aeronautical Sciences, Vol. 8, No. 9, 337 (1941). L. Prandtl, Physikalische Z., VIII, 23 (1907). A. Busemann, *Handbuch der Experimental Physik* (1931), IV, Teil 1, SS. 421–431. T. Koga, J. Soc. Aero. Sci. Japan, **71**, 241 (1941).

able to say about the problems of the same kind concerning the flow around an aerofoil.*

In this section the author deals with the problem as the extension or generalization of the case of a tube.

19. *Comparison between one-dimensional flow and flow in a tube.* The change of velocity of one-dimentional steady flow is represented by du/dx. As there is no change in the direction perpendicular to the flow velocity, there exists essentially no conception of vorticity in one-dimensional flow. In general, du/dx appears only when there are sink or source.

$$\frac{d(\rho u)}{dx}=m, \qquad \cdots\cdots\cdots \text{(IV. 1)}$$

where m represents sink or soruce. Namely

$$\rho\frac{du}{dx}+u\frac{d\rho}{du}\frac{du}{dx}=m, \qquad \cdots\cdots\cdots \text{(IV. 2)}$$

where du/dx is equal to zero when m vanishes. Particularly when $\rho+ud\rho/du=0$, du/dx needs not to be zero. In this case

$$\frac{d\rho}{\rho}+\frac{du}{u}=0. \qquad \cdots\cdots\cdots \text{(IV. 3)}$$

This condition is identical with one where u is equal to the sound velocity of the considered point (the local sound velocity). The reason is as follows: The equation of momentum is

$$\rho u\frac{du}{dx}=-\frac{dp}{dx}. \qquad \cdots\cdots\cdots \text{(IV. 4)}$$

Eliminating du from this equation and (IV. 3), we get

$$u^2=\frac{dp}{d\rho}.$$

Namely, u is equal to the sound velocity. At the point where u is equal to the local sound velocity, there can exist the change of velocity du/dx without sink or source. If, further, there is sink or source, du/dx becomes infinitely large. This is indentical with the shock line of which we will consider in §§ 20, 21. Under this condition, we get

$$\frac{p}{p_0}=\left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)} \qquad \cdots\cdots\cdots \text{(IV. 5)}$$

after a simple calculaltion. Here p_0 is the pressure of the stagnation point and γ is (heat capacity at constant pressure)/(heat capacity at constant volume). Taking $\gamma=1.405$ by air

$$\frac{p}{p_0}=0.527. \qquad \cdots\cdots\cdots \text{(IV. 5)}$$

Secondly, as for the steady flow in a tube,

$$\rho uS=\text{const}, \qquad \cdots\cdots\cdots \text{(IV. 6)}$$

---

* The present chapter was written in the year 1944.

taking S as the sectional area of the tube (a function of x) and changing easily with x. Differentiating (IV. 6), we get

$$\frac{d\rho}{\rho}+\frac{du}{u}+\frac{dS}{S}=0.$$

From (IV. 4) and the equation of state, we can take ρ a function of u only. Thus

$$du\left(\frac{d\rho}{du}\frac{1}{\rho}+\frac{1}{u}\right)+\frac{dS}{S}=0. \qquad \cdots\cdots\cdots(\text{IV. }7)$$

Accordingly, $du=0$ at the sections where $dS=0$. Particularly when (IV. 3) is satisfied, du is finite and not zero only under the condition that $dS=0$. If, even for a time, the part of flow of sound velocity exists at the point where $dS\neq 0$, the flow can not be steady and the unsteadiness continues until the part of flow of sound velocity reaches the point where $dS=0$.

We can compare flows in a tube to one-dimensional flow only when we take the consequence of the change of sectional area along the axis as the sink and source on the side of one-dimentional flow.

Namely, to designate the form of a tube and the flow on a section is identical with indicating the distribution of sink and source in the case of one-dimensional flow. The influence of boundary layer on the wall of a tube, when it is large, can change the flow even at the center into two-dimentional one (or three dimensional), and the flow can scarecely be taken as one-dimensional. The flow in tube, in order to be taken as a one-dimentional flow and to be useful in our consideration, must be such that which is isolated from the complication mentioned above. To avoid the complication, it is useful to consider as follows: We set a great number of little roller bearings specially designed on the wall and rotate them with proper speed suitablly to the local speed of the flow.

20. *Flow in a tube.* Thus we can experience the character of an one-dimensional flow by means of a flow in a suitablly selected tube. It may be better to say that there is a proper one-dimensional flow which can be represented by a flow in a tube. As a matter of cource, there are various kinds of one-dimensional flows representing flows in a tube, because of the fact that the distribution of m (IV. 1) is decided not only by the form of a tube but also by the velocity of flow.

Comparing (IV. 2) and (IV. 7), we see that

$$\frac{dS}{S}=-\frac{m}{(du/dx)}\frac{du}{\rho u}.$$

Namely m is also a function of u and ρ_0, p_0 (the values at the stagnation point). The Bernouill's equation of gas is

$$q^2=\frac{2\gamma}{\gamma-1}\frac{p_0}{\rho_0}\left\{1-\left(\frac{p}{p_0}\right)^{(\gamma-1)/\gamma}\right\}. \qquad \cdots\cdots\cdots(\text{IV. }8)$$

Here q is the velocity, p_0 and ρ_0 are the pressure and the density of the stgnation point, γ is (specific heat at constant pressure)/(specific heat at constant volume), and p is the pressure at the point in question. Denoting the sectional area of a tube S, there is a relation

$$\rho q S = \text{const.}$$

Where $q=0$, S is infinitely large. When $p=0$ and $\rho=0$, q is finite. In this case also $S=\infty$. Namely there are two cases where S is infinitely large: 1) $p=p_0$, $\rho=\rho_0$, $q=0$, 2) $p=0$, $\rho=0$, $q=\{2\gamma/(\gamma-1)\cdot p_0/\rho_0\}^{1/2}$. Let us set up a blower in the tube shown in Fig. 8 *a*, *A*, and make flow in it. After the flow has arrived at the necessary state, we put out the blower. Provided that the pressure at the outlet of the tube of down flow where $S=\infty$, is kept at the same pressure p_0 as that of the opposite end, the inlet, the pressure is distributed along the tube as shown by the curves between 1 and n in Fig. 8 *a*, *B*, each curve

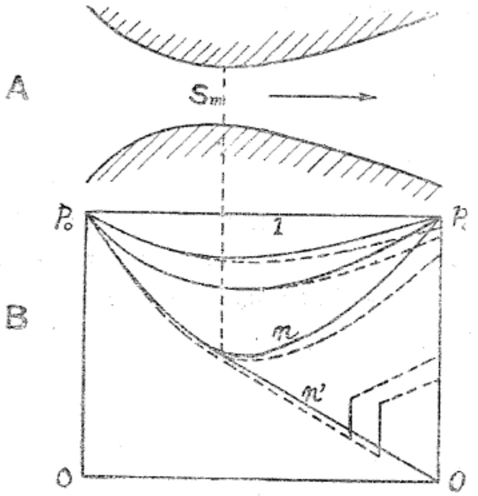

Fig. 8 *a*

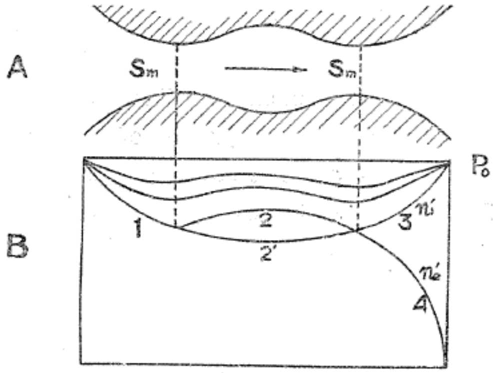

Fig. 8 *b*

corresponding to the sum of the momentum given at the beginning by the blower. The curve n' is attained only when the pressure at the outlet is kept zero. When the pressure at the outlet p_∞ is between p_0 and zero, there appears a shock wave on the way (between S_m and the outlet), the point where the shock wave appears being fixed by the pressure p_∞. Through a shock wave, the pressure rises up unadiabatically and abruptly. In this case, the state of the uperstream part is not changed.* This may be said to be an impotant fact. Next, let us take a tube of the form as shown in Fig. 8 *b*, *A*, where the minimum section S_m is arranged at two places. Thus the pressure distribution is resulted as *B* in Fig. 8 *b*. When p_∞ is zero, the flow can rise up once to supersonic state (passing through the first minimum section) and can go down again to subsonic state adiabatically (passing through the second minimum section), and the pressure distribution is 1-2′-3 indicated in the figure. When the pressure of outlet p_∞ is zero, the pass can be 1-2′-4. In experiment S_m near the inlet must be slightly smaller than S_m near the outlet to compensate the influence of viscosity on the wall.[1]

* There the equation of flow is of the hyperbolic type.

[1] J. Ackeret, *Convegno di Scienze Fisische, Matematiche Naturali* (Volta Congress, 1935), p. 521.

21. *The Ringleb's solution*[1] *about radial flows and its physical meaning.* In a two-dimensional flow, let us take u, v as the components of velocity respectively in the directions x, y axes mutually orthogonal. Then, from the conditions of continuity and irrotationality, we see that there exist two functions, φ and ψ which satisfy the following relations:

$$u=\frac{\partial\varphi}{\partial x},\quad v=\frac{\partial\varphi}{\partial y}, \qquad \cdots\cdots\cdots(\text{IV. 9})$$

$$-\frac{\rho}{\rho_0}v=\frac{\partial\psi}{\partial x},\quad \frac{\rho}{\rho_0}u=\frac{\partial\psi}{\partial y}, \qquad \cdots\cdots\cdots(\text{IV. 10})$$

where ρ denotes the density of gas and one that has the suffix 0 belongs to the state of stagnation point. φ is called the velocity potential, and ψ the stream function.

Into the following relation

$$\left.\begin{aligned} d\varphi&=udx+vdy,\\ d\psi&=\frac{\rho}{\rho_0}(-vdx+udy),\end{aligned}\right\} \qquad \cdots\cdots\cdots(\text{IV. 11})$$

we introduce a polar coordinate system, and q, θ are taken instead of u, v, where q represents the magnitude of velocity vector and θ the inclination with respect to the axis x. The state is expressed also on the hodograph plane as shown in Fig. 9, b. The physical plane is shown in a.

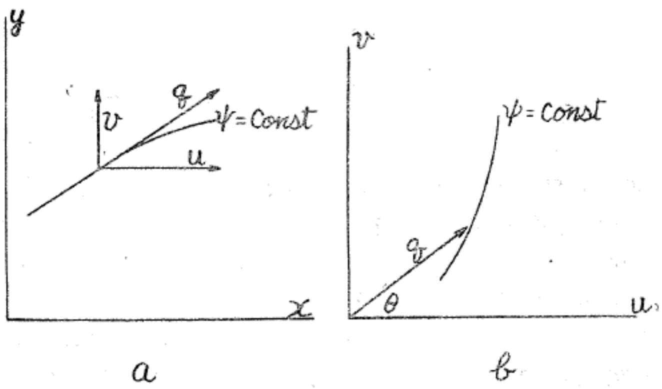

Fig. 9

Now

$$u=q\cos\theta,\quad v=q\sin\theta,$$

consequently, from (IV. 11)

$$\left.\begin{aligned} dx&=\frac{\cos\theta}{q}d\varphi-\frac{\rho_0}{\rho}\frac{\sin\theta}{q}d\psi,\\ dy&=\frac{\sin\theta}{q}d\varphi+\frac{\rho_0}{\rho}\frac{\cos\theta}{q}d\psi.\end{aligned}\right\} \qquad \cdots\cdots\cdots(\text{IV. 12})$$

Here the following relations should be satisfied:

$$\frac{\partial^2 x}{\partial q\partial\theta}=\frac{\partial^2 x}{\partial\theta\partial q},\qquad \frac{\partial^2 y}{\partial q\partial\theta}=\frac{\partial^2 y}{\partial\theta\partial q}.$$

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[1] E. Ringleb, Z. für angewandte Math. und Mech., Bd. 20, Heft 4, 185 (1940).

Consequently

$$\left.\begin{aligned}\frac{\partial\varphi}{\partial q}&=-\frac{\rho_0}{\rho}\left(1-\frac{q^2}{a^2}\right)\frac{1}{q}\frac{\partial\psi}{\partial\theta},\\ \frac{\partial\varphi}{\partial\theta}&=\frac{\rho_0}{\rho}q\frac{\partial\psi}{\partial q},\end{aligned}\right\} \qquad \cdots\cdots\cdots\text{(IV. 13)}$$

considering the relation $\frac{d}{dq}\left(\frac{\rho_0}{\rho}\right)=\frac{\rho_0}{\rho}\frac{q}{a^2}$, where a is the local sound velocity. Differentiating the first of (IV. 13) with regard to θ, and the second with regard to q, we can eliminate φ as follows:

$$q^2\frac{\partial^2\psi}{\partial q^2}+q\left(1+\frac{q^2}{a^2}\right)\frac{\partial\psi}{\partial q}+\left(1-\frac{q^2}{a^2}\right)\frac{\partial^2\psi}{\partial t^2}=0. \qquad \cdots\cdots\cdots\text{(IV. 14)}$$

By means of the Bernoull's Eq. (IV. 8), a is a function of q, thus

$$q^2(1-\alpha q^2)\frac{\partial^2\psi}{\partial q^2}+q(1-\beta q^2)\frac{\partial\psi}{\partial q}+(1-\delta q^2)\frac{\partial^2\psi}{\partial\theta^2}=0, \qquad \cdots\cdots\cdots\text{(IV. 15)}$$

where

$$\alpha=\frac{\gamma-1}{2a_0^2},\qquad \beta=\frac{\gamma-3}{2a_0^2},\qquad \delta=\frac{\gamma+1}{2a_0^2}. \qquad \cdots\cdots\cdots\text{(IV. 16)}$$

Now, as for the case of radial flow, each stream line has its own constant θ, namely, ψ is a function of θ only. Eq. (IV. 15) thus becomes

$$\frac{d^2\psi}{d\theta^2}=0.$$

Consequently

$$\psi=c\theta+c' \qquad \cdots\cdots\cdots\text{(IV. 17)}$$

where c and c' are constant. In addition to this, as

$$\frac{\partial\psi}{\partial q}=0,\qquad \frac{\partial\psi}{\partial\theta}=c,$$

so, from (IV. 13)

$$\frac{\partial\varphi}{\partial q}=-c\frac{\rho_0}{\rho}\left(1-\frac{q^2}{a^2}\right)\frac{1}{q},\qquad \frac{\partial\varphi}{\partial\theta}=0,$$

and from (IV. 12)

$$\left.\begin{aligned}\frac{\partial x}{\partial q}&=-c\frac{\rho_0}{\rho}\left(1-\frac{q^2}{a^2}\right)\frac{\cos\theta}{q^2}, & \frac{\partial x}{\partial\theta}&=-c\frac{\rho_0}{\rho}\frac{\sin\theta}{q},\\ \frac{\partial y}{\partial q}&=-c\frac{\rho_0}{\rho}\left(1-\frac{q^2}{a^2}\right)\frac{\sin\theta}{q^2}, & \frac{\partial y}{\partial\theta}&=c\frac{\rho_0}{\rho}\frac{\cos\theta}{q}.\end{aligned}\right\} \qquad \text{i)}$$

Considering that ρ is a function of q only, the second and fourth equations in i) are to be integrated with regard to θ, and

$$\left.\begin{aligned}x&=c\frac{\rho_0}{\rho}\frac{\cos\theta}{q}+f(q),\\ y&=c\frac{\rho_0}{\rho}\frac{\sin\theta}{q}+g(q).\end{aligned}\right\} \qquad \text{ii)}$$

The equation ii) obtained above is introduced in the first and third equations of i) and

$$f'(q)=0,\qquad g'(q)=0.$$

Thus, taking K_1 and K_2 as constant, we get

$$x=c\frac{\rho_0}{\rho}\frac{\cos\theta}{q}+K_1,\qquad y=c\frac{\rho_0}{\rho}\frac{\sin\theta}{q}+K_2.$$

Here we put

$$K_1=K_2=0,\qquad c=a_0 r_0.$$

This means that the meeting point of all the radial stream lines is situated at the origin of coordinates and the velocity on a certain circle is given. Namely, if $\theta=k$,

$$x=r_0\frac{\rho_0}{\rho}\frac{a_0}{q}\cos k,\qquad y=r_0\frac{\rho_0}{\rho}\frac{a_0}{q}\sin k, \qquad \cdots\cdots\cdots \text{(IV. 18)}$$

and the curve on which the speed is constant is given by

$$x^2+y^2=r_0^2\left(\frac{\rho_0 a_0}{\rho q}\right)^2. \qquad \cdots\cdots\cdots \text{(IV. 19)}$$

Namely this curve is a circle.

Now ρ is a function of q by means of Bernouill's equation and the adiabatic condition:

$$\frac{\rho}{\rho_0}=\left\{1-\frac{(\gamma-1)}{2}\left(\frac{q}{a_0}\right)^2\right\}^{1/(\gamma-1)}.$$

Consequently $(\rho q)/(\rho_0 a_0)$ in (IV. 19) is a function of q/a_0 as shown in Fig. 10 (taking $\gamma=1.4$). $(\rho q)/(\rho_0 a_0)$ has a maximum value where $q=a$. In general the element of a stream line ds is given by

$$(ds)^2=(dx)^2+(dy)^2=a_0^2 r_0^2\left(\frac{\rho_0}{\rho}\right)^2\left(1-\frac{q^2}{a^2}\right)^2\frac{1}{q^4}(dq)^2.$$

Namely

$$\frac{dq}{ds}=\pm\frac{\rho}{\rho_0}\cdot\frac{q^2}{(1-q^2/a^2)\,a_0 r_0}.$$

On the circle where $q=a$,

$$\frac{dq}{ds}=\infty.$$

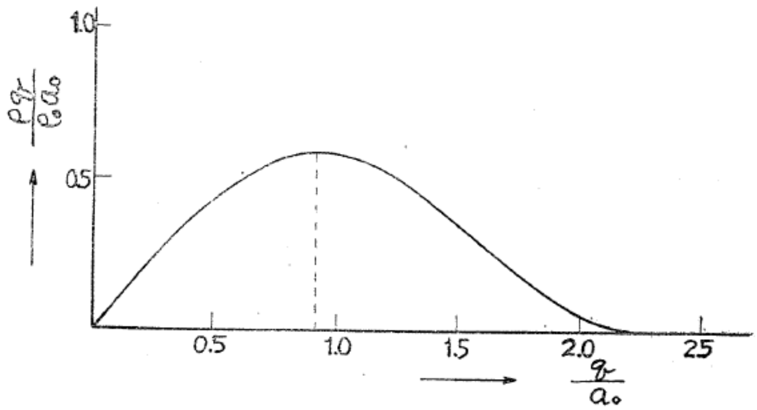

Fig. 10

Now

$$\frac{dq}{ds}=\frac{dq}{dt}\frac{1}{(ds/dt)}=\frac{dq}{dt}\frac{1}{q},$$

where t denotes the time. Namely in our present case

$$\frac{dq}{dt}=\infty.$$

Ringleb calls a curve on which $(dq/dt)=\infty$ a shock line.

When a gas flows centripetally into a point from the surroundings infinitely distant, where its velocity is zero, the nearer the flow comes to the center, the more the flow is accelerated, and at last on a circle the flow arrives at the velocity of sound of that state. The circle is called 'shock line'. On this line the direction of flow becomes reversed and accelerated more over. The maximum velocity is of $2.2a_0$ (taking $\gamma=1.4$) and that is reached on the circle, the radius of which is infinitely large.

22. The above is the outline of Ringleb's discussion. This is not only of mathematical interest but also full of physical meanings: We take a tube which is constructed of two parts A and B, each being a tube radially expanding and connected with each other on the line M as shown in Fig. 11. Let a subsonic flow come to M from the left hand side being accelerated and take the velocity equal to that of sound of that state at M. Now the acceleration must become infinitely large if the form of the tube continues to be radially converging in the same manner as ever. The flow is scarecely accelerated infinitely, when the tube being connected to B stops to converge, and then begin to diverge. Thus the acceleration continues to be finite over the sound velocity passing M. The equation of flow over the sound velocity comes to be that of hyperbolic type. The part B of the flow may be said the reversing flow in Ringleb's discussion.

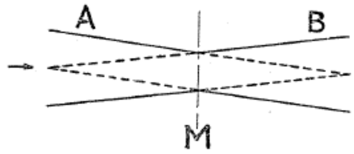

Fig. 11

23. *The theory of Ringleb and Kármán about 'shock line', and the meaning of graphycal solution by means of characteristic curve in 2-dimensional flow.* Taking s the length along a stream line and t the time, the acceleration in the direction of flow is dq/dt:

$$\frac{dq}{dt}=\frac{\partial q}{\partial s}\frac{ds}{dt}=\frac{\partial q}{\partial s}q=\left(\frac{\partial q}{\partial \varphi}\frac{\partial \varphi}{\partial s}+\frac{\partial q}{\partial \psi}\frac{\partial \psi}{\partial s}\right)q=\frac{\partial q}{\partial \varphi}q^2.$$

Accordingly the condition

$$\frac{\partial q}{\partial \varphi}=\infty$$

is equal to the condition

$$\frac{dq}{dt}=\infty. \qquad \cdots\cdots\cdots (\text{IV. } 20)$$

Between q, θ, ψ and φ mentioned in § 21, there are in general the following relations:

$$\left.\begin{aligned}\frac{\partial q}{\partial \psi}=\frac{1}{\varDelta}\frac{\partial \varphi}{\partial \theta}, \quad \frac{\partial \theta}{\partial \psi}=-\frac{1}{\varDelta}\frac{\partial \varphi}{\partial q},\\ \frac{\partial q}{\partial \varphi}=-\frac{1}{\varDelta}\frac{\partial \psi}{\partial \theta}, \quad \frac{\partial \theta}{\partial \varphi}=\frac{1}{\varDelta}\frac{\partial \psi}{\partial q},\end{aligned}\right\} \qquad \cdots\cdots\cdots(\text{IV. }21)$$

where

$$\varDelta\equiv\frac{\partial \varphi}{\partial q}\frac{\partial \psi}{\partial \theta}-\frac{\partial \varphi}{\partial \theta}\frac{\partial \psi}{\partial q}. \qquad \cdots\cdots\cdots(\text{IV. }22)$$

Thus, excepting the case where

$$\frac{\partial \psi}{\partial \theta}=\infty\,, \qquad \cdots\cdots\cdots(\text{IV. }23)$$

$\partial q/\partial \varphi$ is infinitely large only when

$$\varDelta=0\,. \qquad \cdots\cdots\cdots(\text{IV. }24)$$

Now the irrotational condition is

$$\frac{1}{q}\frac{\partial q}{\partial n}-\frac{\partial \theta}{\partial s}=0\,, \qquad \cdots\cdots\cdots(\text{IV. }25)$$

where n means the normal of the stream line, and the equation of continuity is

$$(1-M^2)\frac{1}{q}\frac{\partial q}{\partial s}+\frac{\partial \theta}{\partial n}=0, \qquad \cdots\cdots\cdots(\text{IV. }26)$$

where M denotes q/a. These two conditions are rewritten as follows:

$$\left.\begin{aligned}\frac{\rho}{\rho_0}\frac{1}{q}\frac{\partial q}{\partial \psi}-\frac{\partial \theta}{\partial \varphi}=0\,,\\ (1-M^2)\frac{1}{q}\frac{\partial q}{\partial \varphi}+\frac{\rho}{\rho_0}\frac{\partial \theta}{\partial \psi}=0\,.\end{aligned}\right\} \qquad \cdots\cdots\cdots(\text{IV. }27)$$

Further, considering (IV. 21), we get

$$\left.\begin{aligned}\frac{\rho}{\rho_0}\frac{1}{q}\frac{\partial \varphi}{\partial \theta}-\frac{\partial \psi}{\partial q}=0\,,\\ (1-M^2)\frac{\rho_0}{\rho}\frac{1}{q}\frac{\partial \psi}{\partial \theta}+\frac{\partial \varphi}{\partial q}=0\,.\end{aligned}\right\} \qquad \cdots\cdots\cdots(\text{IV. }28)$$

Accordingly the equation (IV. 24) is written as follows:

$$(1-M^2)\left(\frac{\partial \psi}{\partial \theta}\right)^2+q^2\left(\frac{\partial \psi}{\partial q}\right)^2=0\,. \qquad \cdots\cdots\cdots(\text{IV. }29)$$

This is valid only when $M^2>1$, and (IV. 24) is fulfilled only When $q^2>a^2$. At a point on hodograph plane (see Fig. 9), taking β the angle between the two tangential straight lines, one in contact with the circle q=const and the other with the curve ψ=const, we see that

$$\tan\beta=\frac{(dq)_{\psi=\text{const}}}{q(d\theta)_{\psi=\text{const}}}\,,$$

or, according to (IV. 21)

$$\tan\beta=-\frac{1}{q}\frac{(\partial \psi/\partial \theta)}{(\partial \psi/\partial q)}\,.$$

Thus, from (IV. 29),

$$\tan^2\beta=\frac{1}{M^2-1}. \qquad \cdots\cdots\cdots(\text{IV. }30)$$

This means that β is equal to Mach's angle and the curve $\psi=\text{const}$ is tangential to the characteristic curve at this point. The locus of such a point makes 'shock line.' On this line (on physical plane), the vertical component of velocity is equal to the sound velocity corresponding to that state. Thus we may say that a shock line is the special case of Mach's wave. These are the opinions of Ringleb, Kármán,[1] and others.

24. *On the case where* $\frac{\partial\psi}{\partial\theta}=\infty$. So far, we have not discussed the case where

$$\frac{\partial\psi}{\partial\theta}=\infty. \qquad \cdots\cdots\cdots(\text{IV. }23)$$

Now let us consider this case. As well known from (IV. 28), $(\partial\varphi)/(\partial q)$ is of the same order as of $(\partial\psi)/(\partial\theta)$, and $(\partial\varphi)/(\partial\theta)$ as of $(\partial\psi)/(\partial q)$. The conditions under which $(\partial q)/(\partial\varphi)$ is finite must be (cf. (IV. 21))

$$\Delta=O\left(\frac{\partial\psi}{\partial\theta}\right)$$

or

$$\frac{\rho_0}{\rho}\frac{1}{q}(1-M^2)\left(\frac{\partial\psi}{\partial\theta}\right)^2+\frac{\rho_0}{\rho}q\left(\frac{\partial\psi}{\partial q}\right)^2=O\left(\frac{\partial\psi}{\partial\theta}\right)$$

from (IV. 22) and (IV. 28). Here O denotes 'Order'. Accordingly we see that

$$\frac{\partial\psi}{\partial q}=O\left(\frac{\partial\psi}{\partial\theta}\right),$$

In the same way as in the case of (IV. 30)

$$\tan^2\beta=\frac{\frac{1}{q^2}\left(\frac{\partial\psi}{\partial\theta}\right)^2}{\left(\frac{\partial\psi}{\partial q}\right)^2}=\frac{\frac{1}{q^2}\left(\frac{\partial\psi}{\partial\theta}\right)^2}{\frac{(M^2-1)}{q^2}\left(\frac{\partial\psi}{\partial\theta}\right)^2+\frac{1}{q^2}\frac{\rho}{\rho_0}O\left(\frac{\partial\psi}{\partial\theta}\right)}$$

$$=\frac{1}{M^2-1}. \qquad \cdots\cdots\cdots(\text{IV. }30')$$

The result is the same as (IV. 30), but two more conditions must be added:

$$\left.\begin{aligned}\frac{\partial\psi}{\partial q}&=\infty,\\ \frac{\partial\psi}{\partial\theta}&=\infty.\end{aligned}\right\} \qquad \cdots\cdots\cdots(\text{IV. }31)$$

Namely, on hobograph plane, many steam lines are piled up on one line. In general, under the condition that $\Delta\leqq O\left(\frac{\partial\psi}{\partial\theta}\right)$, β is equal to Mach's angle. When Δ is of the lower order than $\frac{\partial\psi}{\partial\theta}$, the acceleration is infinitely large (cf. (IV. 21)). On the other hand

[1] Th. von Kármân, J. of the Aeronautical Science, Vol. 8, 337 (1941).

when $\Delta = O\left(\frac{\partial\psi}{\partial\theta}\right)$, the acceralation is finite. In both cases the stream line is tangential to the characteristic curve on hodograph plane.

25. As well known there is a method of graphycal solution of supersonic flow where we make use of the characteristic curves on hodograph plane.[1] By means of the discussions mentioned above we can understand the meanings of this method: At every point in the supersonic field (physical plane), there are two branches of characteristic curves, and the field is considered to be covered with a net, each mesh of which is infinitely small, and strings which make meshes of the net are characteristic curves. In the graphycal method we make an approximated image of real field, and there we take each mesh as of finite scale. In the inner part of each mesh there appears no acceleration. It appears only when the flow passes over a string, and is infinitely large. The uniform state of the inner part of a mesh is represented by one point on the hodograph plane (for instance the point 1 in Fig. 12). When the flow passes over the strings (of infintely samll breadth) on physical plane, the change of the flow is finite, and the state goes over to the point 2 in Fig. 12 (on hodograph plane). In the present figure the lines which connect the point 1, 2, 3, 4, $\cdots$ successively are those obtained in graphical solution and the dotted line is that which corresponds to a real stream line on physical plane. The zig-zag line corresponds to the stream line in the field which has the net of meshes of finite scale. When the flow moves from the point 1 to 2 along the characteristic curve which connects the two points, the acceleration vector $\overrightarrow{12}$ in hodograph plane is vertical to a characteristic curve on physical plane (Mach's wave or shock line), and the curve (ψ = const) coincides temporarily to a characteristic curve in hodograph plane, namely the acceleration is infinitely large (cf. Eq. (IV. 30)). Thus when we move with the flow from one point to the neighbouring point on hodograph plane, the change of the flow on physical plane is fulfilled at a point on a line (shock line) and the acceleration is infinitely large. But when we take the distances of each pair of neighbouring points in Fig. 12 infinitely small, the shock line comes to Mach's line. From the fact that the solution thus obtained agrees with the experimental results and further there appear waves of finite amplitude in real flow by means of small irregularity on the wall surface, without so much disturbing the whole flow, we can suspect that the shock lines in mathematical solution do not always mean the entire impossibility of existence of the real flow corresponding to the present solution containing shock lines. There may perhaps appear a kind of flexibility by means of the viscosity and heat

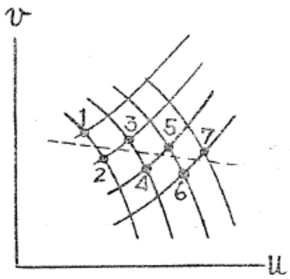

Fig. 12

[1] A. Busemann, *Handbuch der Experimental Physik* (1931), IV, Teil 1, S. 428.

conductivity. Perhaps there may appear the same kind of phenomena as shown in Chapter II (namely the small deviation from the perfectly ideal fluid).

26. *An opinion.* Thus we arrive at an opinion: As well known, a locus of the points at which stream lines in hodograph plane are tangential to characteristic curves makes in general a shock line. But there still can be the cases where the so-called shock line (Mach's line) really exists as a characteristic curve on physical plane, if many stream lines are piled up on one line on hodograph plane ((cf. IV. 31)). The graphical solution mentioned in §25, making use of the characteristic curves on hodograph plane is nothing but the operation joining many meshes which are small but of finite scale and cut off along the shock lines in the present meaning. The possibility of joining them from the dynamical point of view is expected from the fact that the smaller we make the meshes, the nearer the shock lines of the present case comes to Mach's lines. Thus we may say that it is possible to realise in its almost original meaning a shock line in mathematics of the absolute perfect gas. And further more, even in the cases where the shock lines do not appear really, we are using them in deciding the field of flow by making them exist temporarily in the graphical solution. On the other hand, when the given boundary condition does not allow the meshes to be joined smoothly, there appear in the gas multifarious characters as of an existing substance and the flow must obey complicated dynamical and thermodynamical qualifications. Here, I suppose, exist the causes of the multifariousness of shock phenomena in general. This may be remarkable especially when there should be any subsonic part in the boundary condition where any characteristic curve (Mach's wave) does not really exist and we can not cut the flow along it. And it is supposed that, if the joining of flow in such a domain is done smoothly, the shock lines in mathematical potential theory in general would not so much disturb the existence of the corresponding flow in nature.* Concerning the flow locally supersonic about a two-dimensional aerofoil, the case which satisfies the above conditions may correspond to the flow along the line 1–2′–3 in Fig. 8*b*. Further to say, this should be marked especially where the supersonic flow returns again to the subsonic state, because of the fact that the decelerating flow is unstable near the surface of solid body (owing to viscosity).

27. *Moderate decelaration.* The acceleration of one-dimensional flow appears only when there is a sink or source. Particularly where the velocity is the same as that of sound, there exists finite acceralation without any sink or source, whereas the acceleration is infinitely large with sink or source. This is the circumstance about one-dimensional flow as shown in §19. As for the two-dimensional cases, it is supposed that there must be more or less different circumstances to make the acceleration moderate at the point where the velocity is the same as that of sound. Let us assume that the Euler's equation is available, the field steady, the variation moder-

*) This presumption seems to be different from that of Kármán (see the paper quoted at p. 118).

ate and further the condition of irrotation and continuity adoptable.

Let a point in the field be the coordinate origin and the tangential line to the stream line at that point be the x axis. Further taking appropriately the y axis we make a set of orthogonal axes. Taking u_1 the speed of flow at the origin, the velocity components u and v near the origin are expressed as follows:

$$\left.\begin{aligned} u&=u_1+u',\\ v&=v'. \end{aligned}\right\} \qquad \cdots\cdots\cdots (\text{IV. }32)$$

From the assumption set first, u'/u_1 and v'/v_1 are of small value and the terms of higher than the second power can be neglected. The pressure p and the density ρ are also given as follows:

$$\left.\begin{aligned} p&=p_1+p',\\ \rho&=\rho_1+\rho'. \end{aligned}\right\} \qquad \cdots\cdots\cdots (\text{IV. }33)$$

Let the pressure be p_0 and the density ρ_0 at the stagnation point, and then

$$u^2+v^2=\frac{2\gamma}{\gamma-1}\left(\frac{p_0}{\rho_0}-\frac{p}{\rho}\right),$$

when γ is (heat capacity at constant pressure)/(heat capacity at constant volume). Inserting the values of (IV. 32) and (IV. 33) into the above equation and neglecting the terms of higher power of small quantities, we get

$$u'=-\frac{\gamma}{\gamma-1}\frac{1}{u_1}\frac{p_1}{\rho_1}\left(\frac{p'}{p_1}-\frac{\rho'}{\rho_1}\right). \qquad \cdots\cdots\cdots (\text{IV. }34)$$

In the same way, the adiabatic condition is written as follows:

$$\frac{p'}{p_1}-\gamma\frac{\rho'}{\rho_1}=0 \qquad \cdots\cdots\cdots (\text{IV. }35)$$

and the irrotational condition:

$$\frac{\partial u'}{\partial y}=\frac{\partial v'}{\partial x}, \qquad \cdots\cdots\cdots (\text{IV. }36)$$

and further the equation of continuity:

$$\frac{\partial u'}{\partial x}\left(1-\frac{u_1^2}{a_1^2}\right)+\frac{\partial v'}{\partial y}=0, \qquad \cdots\cdots\cdots (\text{IV. }37)$$

where

$$a_1^2=\gamma\frac{p_1}{\rho_1}, \qquad \cdots\cdots\cdots (\text{IV. }38)$$

a_1 is the velocity of sound at the origin.

The expression of the acceleration of sonic state is as follows: Let us take the condition $a_1=u_1$ and then, remembering the premise that the variation should be moderate, we get from (IV. 37)

$$\frac{\partial v'}{\partial y}=0. \qquad \cdots\cdots\cdots (\text{IV. }39)$$

Near the origin u' and v' are expressed as follows:

$$\left.\begin{aligned} u'&=\sum\sum\alpha_{ij}x^iy^j,\\ v'&=\sum\sum\beta_{i'j'}x^{i'}y^{j'}, \end{aligned}\right\} \qquad \cdots\cdots\cdots (\text{IV. }40)$$

where i and j annexed to x and y are positive integers and indicate the powers of x and y, whereas i and j annexed to α and β are the suffixes. As, u' and v' are zero at the origin,

$$\alpha_{00}=\beta_{00}=0. \qquad \cdots\cdots\cdots(\text{IV. }41)$$

Inserting (IV. 40) into (IV. 36), and using of the method undetermined multipliers, we get

$$j\,\alpha_{ij}=i'\,\beta_{i'j'},$$

taking

$$i=i'-1,$$
$$j=1+j'.$$

Namely

$$j\,\alpha_{ij}=(i+1)\beta_{(i+1)(j-1)}. \qquad \cdots\cdots\cdots(\text{IV. }42)$$

Further, form (IV. 39)

$$j\,\beta_{ij}=0. \qquad \cdots\cdots\cdots(\text{IV. }43)$$

According to these conditions,

$$\left.\begin{aligned} u&=a_1+\alpha_{10}x+\alpha_{20}x^2+\alpha_{30}x^3+\cdots \\ &\qquad +\alpha_{01}y+\alpha_{11}xy+\alpha_{21}x^2y+\alpha_{31}x^3y+\cdots, \\ v&=\alpha_{01}x+\tfrac{1}{2}\alpha_{11}x^2+\tfrac{1}{3}\alpha_{21}x^3+\cdots. \end{aligned}\right\} \cdots\cdots\cdots(\text{IV. }44)$$

There remain no other terms.

In the neighbourhood of the present origin, we express the locus of the points where the velocity is the same as that of sound in the following way:

$$x=A_1y+A_2y^2+A_3y^3+\cdots. \qquad \cdots\cdots\cdots(\text{IV. }45)$$

On this curve

$$a_1^2=(a_1+u')^2+v'^2.$$

Neglecting the terms of the second power

$$2a_1u'=0$$

or

$$u'=0.$$

From (IV. 44) and (IV. 45) we get

$$\begin{aligned} &\alpha_{10}(A_1y+A_2y^2+A_3y^3+\cdots) \\ &+\alpha_{20}(A_1y+A_2y^2+A_3y^3+\cdots)^2 \\ &+\alpha_{30}(A_1y+A_2y^2+A_3y^3+\cdots)^3 \\ &+\cdots\cdots\cdots\cdots\cdots\cdots \\ &+\alpha_{01}y+\alpha_{11}y^2+\alpha_{21}y^3+\cdots\cdots=0. \end{aligned}$$

By the method of undetermined multipliers

$$\left.\begin{aligned} &A_1\alpha_{10}+\alpha_{01}=0, \\ &A_2\alpha_{10}+A_1^2\alpha_{20}+\alpha_{11}=0, \\ &A_3\alpha_{10}+2A_1A_2\alpha_{20}+A_1^3\alpha_{30}+\alpha_{21}=0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots. \end{aligned}\right\} \cdots\cdots\cdots(\text{IV. }46)$$

Now let us express a stream line near the origin as follows:

$$y=y_0+B_1x+B_2x^2+B_3x^3+\cdots, \qquad \cdots\cdots\cdots(\text{IV. }47)$$

where B_1, B_2, B_3, $\cdots$ etc. are respectively functions of y_0. Especially when we take

$$y_0=0,$$

we put the corresponding values of B as B_{10}, B_{20}, B_{30}, etc. Along the curve (IV. 47)

$$\frac{dy}{dx}(a_1+u')=v'.$$

Namely from (IV. 44)

$$\begin{aligned}(B_1+2B_2x+3B_3x^2+4B_4x^3+\cdots)\times\\ \times(a_1+\alpha_{10}x+\alpha_{20}x^2+\alpha_{30}x^3+\cdots\\ +\alpha_{01}y_0+\alpha_{01}B_1x+\alpha_{01}B_2x^2+\alpha_{01}B_3x^3+\cdots\\ +\alpha_{11}y_0x+\alpha_{11}B_1x^2+\alpha_{11}B_2x^3+\cdots\\ +\alpha_{21}y_0x^2+\alpha_{21}B_1x^3+\cdots\\ +\alpha_{31}y_0x^3+\cdots\\ +\cdots)\\ =\alpha_{01}x+\tfrac{1}{2}\alpha_{11}x^2+\tfrac{1}{3}\alpha_{21}x^3+\cdots\cdots\cdots.\end{aligned}$$

By the method of undetermined multipliers

$$\left.\begin{aligned}&B_1(a_1+\alpha_{01}y_0)=0,\\ &2B_2(a_1+\alpha_{01}y_0)=\alpha_{01},\\ &3B_3(a_1+\alpha_{01}y_0)+2B_2(\alpha_{10}+\alpha_{11}y_0)=\frac{1}{2}\alpha_{11},\\ &4B_4(a_1+\alpha_{01}y_0)+3B_3(\alpha_{10}+\alpha_{11}y_0)\\ &\qquad+2B_2(\alpha_{20}+\alpha_{01}B_2+\alpha_{21}y_0)=\frac{1}{3}\alpha_{21},\\ &\cdots\cdots\cdots\cdots\cdots\cdots.\end{aligned}\right\}\quad\cdots\cdots\text{(IV. 48)}$$

Now according to the assumption that the acceleration is moderate $(\alpha_{01}\neq\infty)$ we must put

$$B_{10}=0.$$

By means of only the relations (IV. 46) and (IV. 47), the coefficients are not perfectly determined. But when the sonic state is present on the wall, we can avoid the sharp acceleration near the point taking care in selecting the form of the wall. Putting the origin on the wall, and inserting the result (IV. 48) into (IV. 47) we get

$$y=B_{20}x^2+B_{30}x^3+\cdots. \qquad\cdots\cdots\text{(IV. 49)}$$

In the equation (IV. 45) A_1 can be of any value between zero and infinity. In this case there is some fear of α_{10} and α_{01} in (IV. 46) being infinitely large. The fear disappears only when we take α_{01} equal to zero:

$$\alpha_{01}=0. \qquad\cdots\cdots\text{(IV. 50)}$$

How can we assure this condition by the form of the wall only? According to (IV. 48), the fact that

$$B_{20}=0$$

is available. In the present case, α_{01} being independent of y_0,

$$B_2=B_{20}=0. \qquad\cdots\cdots\text{(IV. 51)}$$

Consequently

$$B_3=\frac{\alpha_{11}}{6a_1}. \qquad\cdots\cdots\text{(IV. 52)}$$

To make sure of it, putting also B_{30} equal to zero,

$$\left.\begin{aligned} &\alpha_{11}=0, \\ &B_4=\frac{\alpha_{21}}{12a_1}. \end{aligned}\right\} \qquad \cdots\cdots\cdots (\text{IV. }52)$$

B_5, B_6 etc. depend on y_0.

Under these circumstances, we understand from (IV. 46) and (IV. 50) that the finite value (not zero) of α_{10} can exist only when the condition

$$A_1 \to 0$$

is satisfied. Further if we adopt the condition (IV. 52), then

$$A_2 \to 0$$

must be satisfied. It is, however, clear that the above mentioned can not decide all the field by itself without the other conditions which determine the outer parts, i.e., the places where y_0 is large. Namely there does not exist any given condition which keeps the flow be in the sonic state at the origin. It is only a qualification to be annexed, in order to make the acceleration moderate and finite in the sonic state.

In addition to the present discussion, it is not the necessary condition to make acceleration finite that we take $B_2=0$ or $\alpha_{01}=0$. It is only a part of sufficient conditions for the present aim. In fact, it is able to set the sonic state at the point taken as the origin for the present only when the other conditions are given adequately. But, by means of the below mentioned experience, it is understood that our aim is arrived at without so large difficulty by making a domain flat more or less widely on the surface of an aerodoil. Further this consideration should be given at the decelerating part as stated in §26.

28. *Experiments in shallow stream.** There is found the similar character in shallow water stream to that of high speed air flow in two-dimensions. Of the water

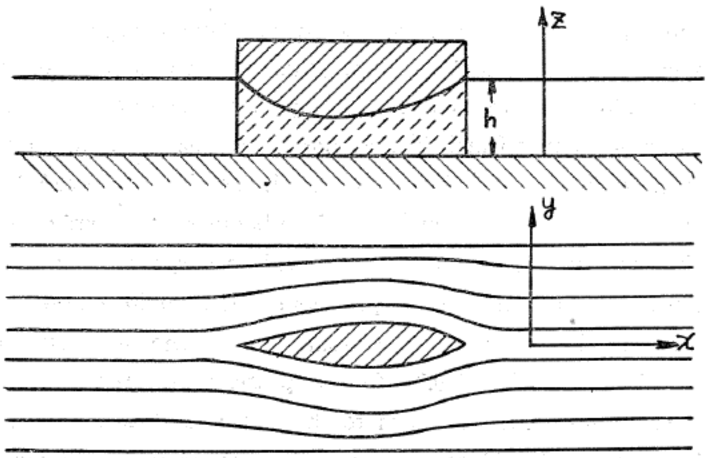

Fig. 13.

* Cf. the next chapter.

stream as shown in Fig. 13, let p denote the pressure, h the depth, z the height from the bottom, and ρ_w the water density, and then we define $\bar{p}$ as follows:

$$\bar{p}=\int_0^h p\,dz=\int_0^h \rho_w g(h-z)\,dz=\frac{\rho_w g}{2}h^2$$

or

$$\bar{p}(\rho_w h)^{-2}=\frac{g}{2\rho_w}=\text{const}, \qquad \cdots\cdots\cdots(\text{IV. }54)$$

where g is the acceleration of gravity. The equation of motion of such a stream is, assuming the change of state along the vertical line to be small,

$$\rho_w h\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial \bar{p}}{\partial x}, \qquad \cdots\cdots\cdots(\text{IV. }55)$$

$$\rho_w h\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial \bar{p}}{\partial y}, \qquad \cdots\cdots\cdots(\text{IV. }56)$$

where u and v are respectively the velocity components in the directions x and y which are taken horizontally. The equation of continuity is

$$\frac{\partial(\rho_w h u)}{\partial x}+\frac{\partial(\rho_w h v)}{\partial y}=0. \qquad \cdots\cdots\cdots(\text{IV. }57)$$

Comparing these (IV. 54), (IV. 55), (IV. 56), (IV. 57) with (III. 4), (III. 1), (III. 2), (III. 3), we see that they are of the same type as those in chapter III, if we take

$$\left.\begin{array}{l}\rho_w h \text{ as } \rho,\\ \bar{p} \text{ as } p,\\ 2 \text{ as } \gamma.\end{array}\right\} \qquad \cdots\cdots\cdots(\text{IV. }58)$$

In the present case the sound velocity a is given as follows:

$$a^2=2\frac{\bar{p}}{\rho_w h}=gh. \qquad \cdots\cdots\cdots(\text{IV. }59)$$

According to these principles, the result of the discussion in last section is tested.

In Fig. 14 the sight of a water trough is given. This trough is about one meter wide and the inclination to the horizontal plane is variable to make the depth and the velocity of shallow stream change. In Fig. 15 the two aerofoils to be tested are shown, the lower one being Clark Y and the upper the new, the thickness of them being 8% of cord-length. Fig. 16 is a photograph showing the aerofoil called Clark Y in test, the depth of water on the upper surface being read directly dy means of two sets of straight lines drawn with ink on the surface perpendicular to each other and with the distance of 1 cm. In these experiments the domain where the water line is under 4 cm corresponds to that of super sonic state. The main stream corresponds to the air flow of Mach's No. of 0.7. In the present case the stream once having gone over the sonic state returns down to the subsonic state not smoothly but suddenly with shock wave (the white wave in the photograph). On the other hand in Fig. 17, we see an example of suitable aerofoil (the new shown in Fig. 15) in the same condition as the above experiment. The stream once having gone up to the

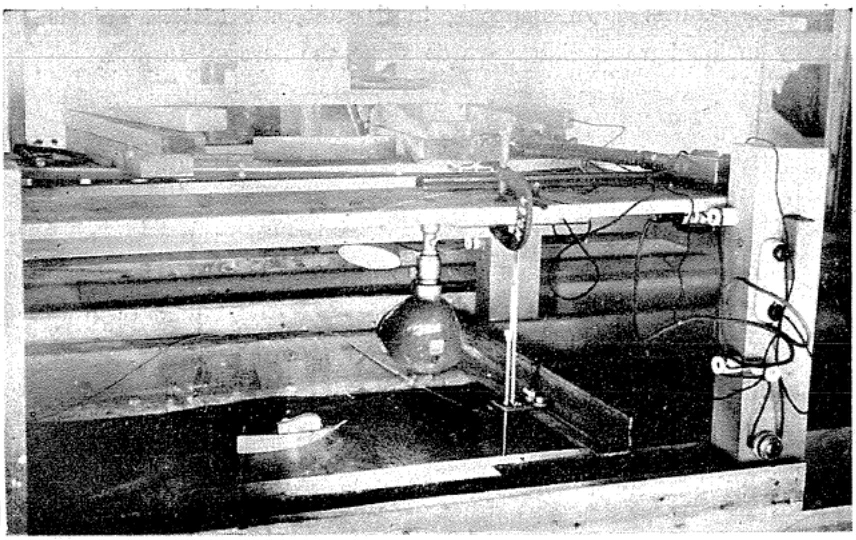

Fig. 14.

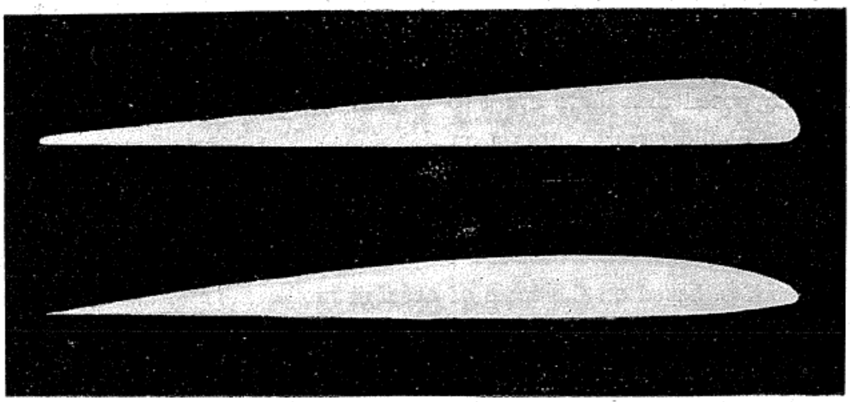

Fig. 15.

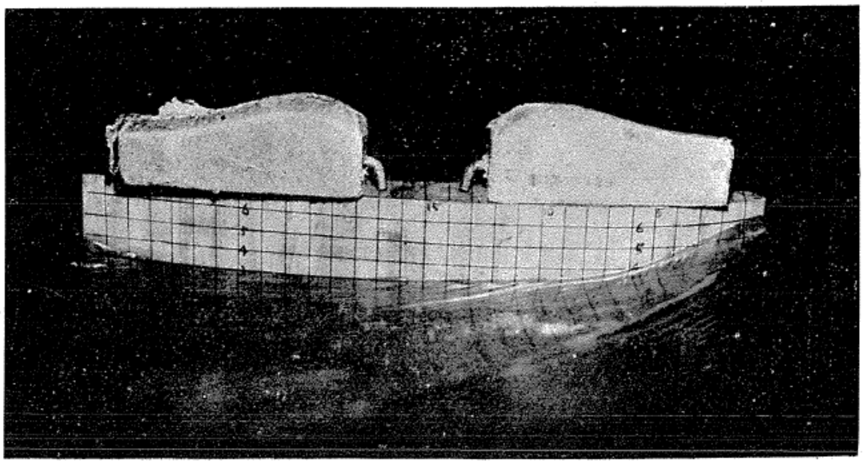

Fig. 16.

supersonic state returns down smoothly to the subsonic state. This is shown by the smoothly inclined water line upwards along the aerofoil surface.

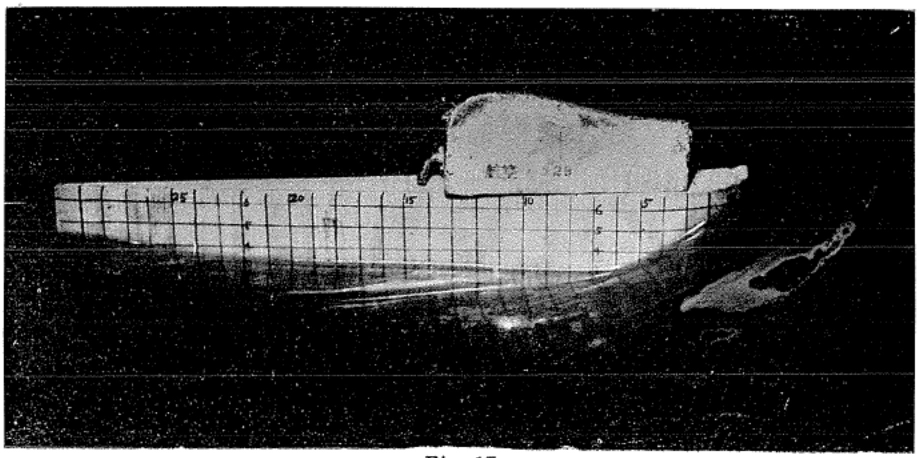

Fig. 17.

V. Two-Dimensional Character of Shallow Water Stream *
(The influence of the vertical component of velocity)

29. It is supposed that there are two origins which disturb the two-dimensional character of a shallow water stream on flat floor. The one is the existence of the vertical component of velocity and the other is the influence of viscosity near the bottom upon the horizontal component of velocity. Therefore the coincidence between any two experiments which are expected from the coincidence of Mach's No. and Reynolds' No. based on the scale of aerofoil model fails. In the case where the model is moved in the water which stays still relatively to the floor, the influence of viscosity near the bottom will be neglected and here let us discuss the other problem.

The steady water flow is expressed by the following set of equations where ρ is the density of water, p the pressure, g the acceleration of gravity, x and y the rectangular coordinates on horizontal plane, z the vertical coordinate, and u, v and w are respectively the velocity components in the directions of x, y and z:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}, \qquad \cdots\cdots\cdots (\text{V. }1)$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}, \qquad \cdots\cdots\cdots (\text{V. }2)$$

$$\rho\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}-\rho g, \qquad \cdots\cdots\cdots (\text{V. }3)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \qquad \cdots\cdots\cdots (\text{V. }4)$$

One can get the irrotational condition ** from these equations, and then can integrate (V. 1), (V. 2), (V. 3) obtaining

* Cf. § 28.

** Cf. § 10.

$$\frac{1}{2}\rho q^2+p+\rho gz=A. \qquad \cdots\cdots\cdots(\text{V. }5)$$

Here A is constant, q is the value of velocity:

$$q^2=u^2+v^2+w^2. \qquad \cdots\cdots\cdots(\text{V. }6)$$

30. *Vertical component of velocity* (i). Denoting the water depth by h, the vertical component of velocity on the water surface by w_s, the horizontal component on the surface by q_{sxy}, we see that

$$\begin{aligned} w_s &= \operatorname{grad} h \cdot q_{sxy} \\ &= u_s\frac{\partial h}{\partial x}+v_s\frac{\partial h}{\partial y}. \end{aligned} \qquad \cdots\cdots\cdots(\text{V. }7)$$

As w disappears on the flat floor, w_s is considered as the maximum value of w along a vertical line through a given point on the x-y plane. This is only a supposition, but, as mentioned later, in the discussion of dimensional analysis, only the extent of value or its change is important and w_s is supposed to be sufficient to represent the extent of w which appears in the present problem.

Denoting the value of general flow with U, the representation of the present boundary scale with l (for instance the cord length of aerofoil), and the depth of genneral flow with h_0, we put as follows:

$$\begin{aligned} u&=Uu', \\ v&=Uv', \\ x&=lx', \\ y&=ly', \\ z&=h_0z', \\ h&=h_0h'. \end{aligned}$$

These expressions are introduced into (V. 3), where

$$\rho u\frac{\partial w}{\partial x} \text{ is of the order of } \rho U^2h_0\frac{1}{l^2}u'\frac{\partial}{\partial x'}\left[u'\frac{\partial h'}{\partial x'}+v'\frac{\partial h'}{\partial y'}\right],$$

$$\rho v\frac{\partial w}{\partial y} \text{ is of the order of } \rho U^2h_0\frac{1}{l^2}v'\frac{\partial}{\partial y'}\left[u'\frac{\partial h'}{\partial x'}+v'\frac{\partial h'}{\partial y'}\right],$$

$$\rho w\frac{\partial w}{\partial z} \text{ is of the order of } \rho U^2h_0\frac{1}{l^2}\left[\left(u'\frac{\partial h'}{\partial x'}+v'\frac{\partial h'}{\partial y'}\right)^2\frac{1}{h'}\right].$$

Assuming that the change of flow with respect to space is moderate, the dimensionless terms of the right hand side in the bracket are of the order of unity. Taking M* as

$$M^2=\frac{U^2}{gh_0}, \qquad \cdots\cdots\cdots(\text{V. }8)$$

(V. 3) becmes

$$M^2\frac{h_0^2}{l^2}f=-\frac{1}{\rho g}\frac{\partial p}{\partial z}-1, \qquad \cdots\cdots\cdots(\text{V. }9)$$

where f is a dimensionless value of the order of unity.

When $M^2h_0^2/l^2$ is sufficiently small comparing with unity, we get from (V. 9)

$$p=-\rho gz+\rho gh+p_s, \qquad \cdots\cdots\cdots(\text{V. }10)$$

* M is called Mach's No. after (IV. 59).

where p_s is the pressure on the water surface, namely the atmospheric pressure. Introducing (V. 10) into (V. 5)

$$\frac{1}{2}\rho q^2+\rho gh+p_s=A. \qquad \cdots\cdots\cdots (V. 11)$$

Namely q is independent of z, provided that the left hand side of (V. 9) is neglected comparing with unity.

From (V. 11), we get

$$\frac{\partial h}{\partial x}=-\frac{1}{g}q\frac{\partial q}{\partial x}, \qquad \cdots\cdots\cdots (V. 12)$$

$$\frac{\partial h}{\partial y}=-\frac{1}{g}q\frac{\partial q}{\partial y}. \qquad \cdots\cdots\cdots (V. 13)$$

In these epuations there remains the error of the order of neglecting $M^2h_0^2/l^2$ compaing with unity.

31. *Vertical component of velocity* (ii). By means of the result of the discussion so far advanced, we examine closely each term of (V. 1) where we put

$$q=Uq',$$

and

$$q'^2=u'^2+v'^2+w'^2.$$

Now

$$\rho u\frac{\partial u}{\partial x}=\rho U^2\frac{1}{l}u'\frac{\partial u'}{\partial x'},$$

$$\rho v\frac{\partial u}{\partial y}=\rho U^2\frac{1}{l}v'\frac{\partial u'}{\partial y'}.$$

From the condition of irrotaional motion, we get

$$\rho w\frac{\partial u}{\partial z}=\rho w\frac{\partial w}{\partial x},$$

and further, referring to (V. 7), (V. 12), (V. 13), $\rho w\frac{\partial w}{\partial x}$ is of the order of

$$\rho U^6\frac{1}{l^3}\frac{1}{g^2}\left(u'q'\frac{\partial q'}{\partial x'}+v'q'\frac{\partial q'}{\partial y'}\right)\frac{\partial}{\partial x'}\left(u'q'\frac{\partial q'}{\partial x'}+v'q'\frac{\partial q'}{\partial y'}\right).$$

Namely (V. 1) becomes

$$\rho U^2\frac{1}{l}\left(u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}+M^4\frac{h_0^2}{l^2}f_1\right)=-\frac{\partial p}{\partial x}. \qquad \cdots\cdots\cdots (V. 14)$$

where f_1, being composed of q' and $\frac{\partial q'}{\partial x'}$ etc., is of the same order as $u'\frac{\partial u'}{\partial x'}$, i.e., unity.

In the same way, (V. 2) becomes

$$\rho U^2\frac{1}{l}\left(u'\frac{\partial u'}{\partial x'}+v'\frac{\partial v'}{\partial y'}+M^4\frac{h_0^2}{l^2}f_2\right)=-\frac{\partial p}{\partial y}. \qquad \cdots\cdots\cdots (V. 15)$$

The terms of (V. 3) are evaluated as follows: $\rho u\frac{\partial w}{\partial x}$ is of the order of $\rho U^4\frac{1}{l^2}\frac{1}{g}f_{31}$, and $\rho v\frac{\partial w}{\partial y}$ is of the order of $\rho U^4\frac{1}{l^2}\frac{1}{g}f_{32}$. Comparing with (V. 4) we see that

$$\rho w\frac{\partial w}{\partial z}=\rho w\left(-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right),$$

and the right hand side is of the order of $\rho U^4\frac{1}{l^2}\frac{1}{g}f_{33}$. Here f_{31}, f_{32} and f_{33} are of the same order as of f_1 and f_2. Assuming that the change of flow with respect to space is easy, the f's are of the order of unity. Thus(V. 3) becomes

$$\left.\begin{aligned} M^4\frac{h_0^2}{l^2}f_3 &= -\frac{1}{\rho g}\frac{\partial p}{\partial z}-1, \\ f_3 &= f_{31}+f_{32}+f_{33}. \end{aligned}\right\} \qquad \cdots\cdots\cdots(\text{V. }16)$$

The difference between (V. 9) and (V. 16) is reduced to that we use here (V. 12) and (V. 13). In § 30 we considered only the change of h', whereas here we consider the changes of velocity components instead of h' to make harmonized with the other part of discussion. There is no essential difference from the point of view of neglecting h_0/l. Further, in the later discussion of similarity of flow there will occur no interruption, as the coinçidence of M will be presupposed.

Further in (V. 11)

$$\begin{aligned} \frac{1}{2}\rho u^2 &= \frac{1}{2}\rho U^2 u'^2, \\ \frac{1}{2}\rho v^2 &= \frac{1}{2}\rho U^2 v'^2, \\ \frac{1}{2}\rho w^2 &= \frac{1}{2}\rho U^6\frac{1}{l^2}\frac{1}{g^2}f_4, \\ f_4 &= \left(u'q'\frac{\partial q'}{\partial x'}+v'q'\frac{\partial q'}{\partial y'}\right)^2. \end{aligned}$$

Thus the equation is rewritten:

$$\frac{1}{2}\rho U^2\left(u'^2+v'^2+U^4\frac{1}{l^2g^2}f_4\right)+\rho g h_0 h'+p_s=A,$$

or

$$\frac{1}{2}M^2\left(u'^2+v'^2+M^4\frac{h_0^2}{l^2}f_4\right)+h'+\frac{1}{\rho g h_0}p_s=A'. \qquad \cdots\cdots\cdots(\text{V. }17)$$

Namely, as far as we take $M^4h_0^2/l^2$ to be minute comparing with unity, q^2 does not contain w^2. And further, as known from (V. 11), q^2 does not depend on z. But, concerning u or v separately, we know nothing. Now let us examine

$$\frac{\frac{\partial u}{\partial z}h}{u}$$

for the sake of later discussion. From the irrotational condition

$$\frac{\partial u}{\partial z}=\frac{\partial w}{\partial x},$$

and we get

$$\frac{\frac{\partial u}{\partial z}h}{u}=\frac{\frac{\partial w}{\partial x}h}{u}.$$

In the same way as the above discussion, $\frac{\partial w}{\partial x}$ is of the order of $U^3\frac{1}{l^2}\frac{1}{g}$. Accordingly

$$\frac{\frac{\partial u}{\partial z}h}{u}=O\left(U^2\frac{1}{l^2}\frac{1}{g}h_0\right)=O\left(M^2\frac{h_0^2}{l^2}\right). \qquad \cdots\cdots\cdots(\text{V. }18)$$

31. *Ideal state.* As an ideal state, we put

$$\frac{h_0}{l}=0, \qquad \cdots\cdots\cdots(\text{V. }19)$$

in (V. 14), (V. 15), (V. 16). Then (V. 1), (V. 2), (V. 3) and (V. 4) become respectively

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}, \qquad \cdots\cdots\cdots(\text{V. }20)$$

$$\rho\left(n\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}, \qquad \cdots\cdots\cdots(\text{V. }21)$$

$$0=-\frac{\partial p}{\partial z}-\rho g, \qquad \cdots\cdots\cdots(\text{V. }22)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \qquad \cdots\cdots\cdots(\text{V. }23)$$

Integrating (V. 22) with regard to z, we get

$$p=-\rho gz+\rho gh+p_s. \qquad \cdots\cdots\cdots(\text{V. }24)$$

And, integrating (V. 20) with regard to z, the left hand side is

$$\int_0^h \rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)dz$$

$$=\rho\int_0^h\left[\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)_{z=0}+\left\{\frac{\partial}{\partial z}\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)\right\}_{z=0}\times z+\cdots\right]dz$$

$$=\rho h\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)_{z=0}+\rho\frac{1}{2}h^2\left\{\frac{\partial}{\partial z}\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)\right\}_{z=0}+\cdots$$

$$=\rho h\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)_{z=0}\left[1+\frac{1}{2}h\frac{\left\{\frac{\partial}{\partial z}\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)\right\}_{z=0}}{\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)_{z=0}}+\cdots\right].$$

The second term in the brachet [ ] is of the order of h_0/l from the discussion in §30, and so, taking h_0/l to be minute as compared with unity,

$$\int_0^h \rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)dz=\rho h\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right),$$

which is taken independent of z.* As for the right hand side of (V. 20),

$$-\int_0^h\frac{\partial p}{\partial x}dz=-\frac{\partial}{\partial x}\int_0^h p\,dz+p_s\frac{\partial h}{\partial x},$$

and, referring to (V. 24),

$$-\int_0^h\frac{\partial p}{\partial x}dz=-\frac{\partial}{\partial x}\left(\frac{1}{2}\rho h^2 g+p_s h\right)+p_s\frac{\partial h}{\partial x}$$

$$=-\frac{\partial}{\partial x}\left(\frac{1}{2}\rho gh^2\right).$$

Denoting $\frac{1}{2}\rho gh^2$ with $\bar{p}$, we get

$$\rho h\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial}{\partial x}\bar{p}. \qquad \cdots\cdots\cdots(\text{V. }25)$$

Similarly

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* $\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)$ is expanded in Taylor's way taking any point on the z-axis as the basis.

$$\rho h\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial}{\partial y}\bar{p}. \qquad \cdots\cdots\cdots(V.\ 26)$$

Integrating (V. 23),

$$\int_0^h\frac{\partial u}{\partial x}dz=\frac{\partial}{\partial x}\int_0^h udz-u_s\frac{\partial h}{\partial x}$$
$$=\frac{\partial}{\partial x}(uh)-u_s\frac{\partial h}{\partial x}^*$$

and similarly,

$$\int_0^h\frac{\partial v}{\partial y}dz=\frac{\partial}{\partial y}(vh)-v_s\frac{\partial h}{\partial y},$$
$$\int_0^h\frac{\partial w}{\partial z}dz=w_s.$$

Referring to (V. 7), we get

$$\frac{\partial}{\partial x}(uh)+\frac{\partial}{\partial y}(vh)=0$$

or

$$\frac{\partial}{\partial x}(\rho hu)+\frac{\partial}{\partial y}(\rho hv)=0, \qquad \cdots\cdots\cdots(V.\ 27)$$

and here

$$\bar{p}(\rho h)^{-2}=\frac{1}{2}\frac{g}{\rho}=\text{const.} \qquad \cdots\cdots\cdots(V.\ 28)$$

33. *Shallow water flow in substitution for two-dimensional gas flow.* Under the condition that the boundary forms are in geometrical similarity, any two experiments are able to have the same gas-dynamical meaning only when

$$\frac{h_0}{l},$$

and further

$$M^2$$

are common to them. (As a matter of course, the Reynolds' number of the two water flows also must be in coincidence.) Further in comparing with air flow (or gas flow), M^2 $(=U^2/gh_0)$ of water flow should be in coincidence with that of gas flow and

$$\frac{h_0}{l},$$

must be minute satisfactorily.

There remains a special circumstance to be mentioned that

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* See the foot note of page 140.

$$u=u_{z=0}+\left(\frac{\partial u}{\partial z}\right)_{z=0}z+\cdots.$$
$$\int_0^h udz=u_{z=0}h+\frac{1}{2}h^2\left(\frac{\partial u}{\partial z}\right)_{z=0}+\cdots$$
$$=u_{z=0}h\left(1+\frac{1}{2}\frac{\frac{\partial u}{\partial z}h}{u}+\cdots\right)_{z=0}.$$

Referring to (V. 18),

$$\int_0^h udz=uh.$$

$$a^2=2\frac{\bar{p}}{\rho h}=gh \quad \text{or} \quad \bar{p}(\rho h)^{-2}=\frac{1}{2}\frac{g}{\rho}=\text{const},$$

and consequently what corresponds to the ratio (specific heat at constant pressure)/(specific heat at constant volume) is of the fixed value 2.

Under the condition that M^2 is of the given value, many experiments of models of geometrically similar form and of various scales of h_0 and l, can offer us the sufficient materials by means of which we are able to work out the extrapolation and to estimate the state where h_0/l is zero. Thus we may say that the shallow water flow can be used perfectly for air flow.

VI. Flow of a Mixed Gas Chemically Unstable.

34. The subject of so-called gas dynamics is the flow of gas chemically stable standing on the assumption that

$$p\rho^{-\gamma}=\text{const.}$$

On the other hand, as for a gas at high temperature, for instance the gas flowing out of the burning room of a rocket, the assumption above mentioned is not taken suitablly, as there occur more or less chemical dissociations and combinations corresponding to state of every moment. As there is a finite velocity in chemical reaction, we can not find any exact equilibrium in flow accelerating or decelerating in space and time. But if the rate of change of flow is easy comaring with velocity of chemical reaction, we can assume approximately the state is always in equilibrium. Namely the real state is an intermediate state between perfectly stable state and the chemical equilibrium. The ratio of reaction velocity to change of flow velocity is zero in the former case, and infinitely large in the latter case. The real state exists between these two limits, and the degree of deviation from them may be discovered comparatively easily by means of our knowledge of reaction velocity. Here we discuss the latter case, and in detail only the case where one kind of chemical reaction occurs. The case of more than two kinds are discussed briefly in the last section.

35. *Momentum in gas flow.* In the steady gas flow, momentum is determined as follows:

$$dp+\rho q dq=0. \qquad \cdots\cdots\cdots(\text{VI. 1})$$

This relation is the same as in the case where the gas is stable. Here p is the pressure of miexd gas, ρ the density of all, and q the speed in a section of stream tube.

36. *Energy in gas flow.* The point that the present problem differs from the usual gas dynamics is to be found in the energy relation. In general, denoting the heat given from the outer part per unit mass of gas by dQ,

$$c_v dT+dU_{T,\rho}-\frac{p}{\rho^2}d\rho=dQ, \qquad \cdots\cdots\cdots(\text{VI. 2})$$

where c_v is the specific heat at constant volume of the mixed gas, U is the internal energy and $dU_{T,\rho}$ represents the absorbed heat of reaction at temperature T and the density ρ.

In the case of usual dynamics

$$dU_{T,\rho}=0,$$
$$dQ=0,$$

and consequently

$$c_v dT-\frac{p}{\rho^2}d\rho=0. \qquad \cdots\cdots\cdots(\text{VI. }3)$$

This is combined with (VI. 1) resulting that

$$d\left(\frac{1}{2}q^2+\frac{\gamma}{\gamma-1}\frac{p}{\rho}\right)=0, \qquad \cdots\cdots\cdots(\text{VI. }4)$$

where c_p denotes the specific heat at constant pressure and γ is c_p/c_v. Thus (VI. 3) is integrated resulting that

$$p\rho^{-\gamma}=\text{const.}$$

This is the relation in the usual adiabatic phenomena, and (VI. 4) is nothing but the Bernouill's equation.

On the other hand, in the present problem,

$$dU_{T,\rho}\neq 0,$$
$$dQ=0$$

of the terms of (VI. 2). $dU_{T,\rho}$ can be calculated by means of the law of mass reaction. When the system in all is in an adiabatic process, this term and the term

$$-\left(c_v dT-\frac{p}{\rho^2}d\rho\right)$$

are cancelled with each other. Namely

$$-\left(c_v dT-\frac{p}{\rho^2}d\rho\right)=dQ_v, \qquad \cdots\cdots\cdots(\text{VI. }5)$$
$$dQ_v=dU_{T,\rho}.$$

Rewriting (VI. 1) as

$$\frac{dp}{\rho}+q\,dq=0,$$

and subtracting from (VI. 5) at each hand side respectively, we get

$$dQ_v=-c_v dT+\frac{p}{\rho^2}d\rho-\frac{dp}{\rho}-q\,dq.$$

Now referring to

$$p\rho^{-1}=RcT,$$
$$Rc=c_p-c_v,$$
$$\gamma=\frac{c_p}{c_v},$$

we get

$$dQ_v=-\frac{\gamma}{\gamma-1}d\left(\frac{p}{\rho}\right)-d\left(\frac{1}{2}q^2\right)+\frac{1}{\gamma-1}\frac{p}{\rho}d(\log c) \qquad \cdots\cdots\cdots(\text{VI. }6)$$

or

$$d\left(\frac{\gamma}{\gamma-1}\frac{p}{\rho}+\frac{1}{2}q^2\right)=-dQ_v+\frac{1}{\gamma-1}\frac{p}{\rho}d(\log c)+d\left(\frac{\gamma}{\gamma-1}\right)\frac{p}{\rho}.$$
$$\cdots\cdots\cdots(\text{VI. }6')$$

The equation (VI. 6) or (VI. 6′) is that which is to be called the Bernouill's Eq. in this case. The right hand side of (VI. 6′) represents the effect of chemical reaction.

37. *Equilibrium condition.* In the case of a mixed gas in equilibrium at every moment, when chemical reaction occurrs between each element, the equilibrium condition is nothing but the content of law of mass reaction. Let

$$p_1,\quad p_2,\quad p_3, \cdots, p_n$$

denote the partial pressures, then

$$p=\sum p_\kappa. \qquad \cdots\cdots\cdots (\text{VI. } 7)$$

Further let

$$m_1,\quad m_2,\quad m_2, \cdots, m_n$$

represent respectively the gram molecule of element and

$$M_1,\quad M_2,\quad M_3, \cdots, M_n$$

respectively the mass of each element contained in unit mass of the mixed gas. c, c' and c'' are defined as follows:

$$\left.\begin{aligned} &c_\kappa=\frac{M_\kappa}{m_\kappa}, \quad c=\sum c_\kappa \\ &c_\kappa'=\rho c_\kappa, \quad c'=\sum c_\kappa' \\ &c_\kappa''=\frac{c_\kappa}{\sum c_\kappa}. \end{aligned}\right\} \qquad \cdots\cdots\cdots (\text{VI. } 8)$$

Thus

$$\left.\begin{aligned} c_v&=\sum c_{v\kappa} M_\kappa, \\ c_p&=\sum c_{p\kappa} M_\kappa, \end{aligned}\right\} \qquad \cdots\cdots\cdots (\text{VI. } 9)$$

$$p\rho^{-1}=RcT=\sum\frac{M_\kappa}{m_\kappa}RT, \qquad \cdots\cdots\cdots (\text{VI. } 10)$$

$$Rc=c_p-c_v=\sum\frac{R}{m_\kappa}M_\kappa. \qquad \cdots\cdots\cdots (\text{VI. } 11)$$

Now let

$$\nu_1,\quad \nu_2,\quad \nu_3, \cdots, \nu_n, \qquad \sum\nu_\kappa=\nu$$

denote respectively the change of mol number of element gas which appears in the considered reaction (the positive sign being taken as representing increase). Putting

$$\left.\begin{aligned} \Pi c_\kappa^{\nu_\kappa}&=K_c, \\ \Pi p_\kappa^{\nu_\kappa}&=K_p, \\ \Pi c_\kappa'^{\nu_\kappa}&=K_{c'}, \\ \Pi c_\kappa''^{\nu_\kappa}&=K_{c''}, \end{aligned}\right\} \qquad \cdots\cdots\cdots (\text{VI. } 12)$$

and referring to the above relations, we get the relation

$$K_p=K_c(\rho RT)^\nu=K_{c'}(RT)^\nu=K_{c''}p^\nu. \qquad \cdots\cdots\cdots (\text{VI. } 13)$$

According to the law of mass reaction, K_p is a function of T only.

When the condition of gas is changed, the state removes from one equilibrium to the other. As, from (VI. 13), $K_{c'}$ is a function of T only and

$$\frac{dK_{c'}}{K_{c'}}-\nu\frac{d\rho}{\rho}=\sum\nu_\kappa\frac{dc_\kappa}{c_\kappa}. \qquad \cdots\cdots\cdots (\text{VI. } 14)$$

In the present case,

$$\frac{dc_1}{\nu_1}=\frac{dc_2}{\nu_2}=\cdots=\frac{dc_n}{\nu_n}=\frac{dc}{\nu}. \qquad \cdots\cdots\cdots \text{(VI. 15)}$$

38. *Absorbed heat of reaction at constant temperature and constant volume.* Corresponding to increases of mol numbers $\nu_1, \nu_2, \nu_3, \cdots, \nu_n$, the absorbed heat is

$$RT^2\frac{d(\log K_{c'})}{dT}. \qquad \cdots\cdots\cdots \text{(VI. 16)}$$

Therefore

$$dQ_v=RT^2\frac{d(\log K_{c'})}{dT}\cdot\frac{dc}{\nu}, \qquad \cdots\cdots\cdots \text{(VI. 17)}$$

corresponding to the variation

$$dc_1,\ dc_2,\ dc_3, \cdots, dc_n, dc.$$

Making the combination of (VI. 17) and (VI. 6′) we get Bernouill's Eq. in a wide sense. Namely the equilibrium state is transfered, the right hand side terms in (VI. 6′) being compensated with the energy of motion and the work by pressure.

39. *Approximate solution.* Let us assume the deviation of real flow from the flow corresponding to ideal gas by means of chemical reaction is not *large.* From this point of view, the terms of right hand side of (VI. 6′) are considered small compared with the left. In the calculation of these compensating terms, the following are adopted as the equations of the first approximation:

$$\left.\begin{aligned}&\frac{1}{2}q^2+\frac{\gamma}{\gamma-1}p\rho^{-1}=\text{const},\\ &p\rho^{-\gamma}=\text{const},\\ &p\rho^{-1}=RcT,\\ &\gamma=\text{const}=\gamma_0,\\ &c=\text{const}=c_0.\end{aligned}\right\} \qquad \cdots\cdots\cdots \text{(VI. 18)}$$

This set of equations represents the motion of gas of a certain constant composition;

$$c_{10},\ c_{20},\ c_{30},\ c_{40}, \cdots, c_{n0}. \qquad \cdots\cdots\cdots \text{(VI. 19)}$$

From these equations q, p and ρ are decided respectively as functions of T. From (VI. 15) we get

$$c_\kappa-c_{\kappa 0}=(c_t-c_{t0})\frac{\nu_\kappa}{\nu_t},$$

$$\kappa=1,\ 2, \cdots, n, \qquad \cdots\cdots\cdots \text{(VI. 20)}$$

where c_t is any one of the c's which is not zero. Referring to (VI. 14)

$$\frac{dK_{c'}}{K_{c'}}=\nu\frac{d\rho}{\rho}+\frac{dc}{\nu}\sum\frac{\nu_\kappa^2}{c_\kappa},$$

therefore

$$dc=\frac{\nu}{\sum\frac{\nu_\kappa^2}{c_\kappa}}\left(\frac{dK_{c'}}{dT}\frac{1}{K_{c'}}dT-\nu\frac{1}{\rho}d\rho\right)$$

$$\doteqdot\frac{\nu}{\left(\sum\frac{\nu_\kappa^2}{c_\kappa}\right)_0}\left(\frac{dK_{c'}}{dT}\frac{1}{K_{c'}}dT-\nu\frac{1}{\rho}d\rho\right)_0,$$

where the value to which the suffix 0 is annexed is calculated according to (VI. 18). Namely

$$(dc)_0=\frac{\nu}{\left(\sum\frac{\nu_\kappa^2}{c_\kappa}\right)_0}\left(\frac{dK_{c'}}{dT}\frac{1}{K_{c'}}dT-\nu\frac{1}{\gamma-1}\frac{dT}{T}\right)_0. \qquad \cdots\cdots\cdots(\text{VI. }21)$$

When the from of function $K_{c'}$ is given, it is easy for us to calculate c as a function of T. Namely the second term of right hand side of (VI. 6′) is able to be integrated.

Next, $K_{c'}$ being a function of T only, we get

$$dQ_v=RT^2\left\{\frac{d(\log K_{c'})}{dT}\right\}_0\cdot\frac{1}{\left(\sum\frac{\nu_\kappa^2}{c_\kappa}\right)_0}\cdot\left(\frac{dK_{c'}}{dT}\frac{dT}{K_{c'}}-\nu\frac{1}{\gamma-1}\frac{dT}{T}\right)_0,$$

inserting (VI. 21) into (VI. 17). Namely dQ_v is written as a derivative form of a function of T. Thus the first term of (VI. 6′) is calculated.

Further as $\gamma=c_p/c_v$, therefore

$$\begin{aligned}\frac{\gamma}{\gamma-1}&=\frac{c_p}{c_p-c_v}\\&=\frac{\sum c_{p\kappa}M_\kappa}{\sum Rc_\kappa}\\&=\frac{\sum c_{p\kappa}m_\kappa c_\kappa}{Rc},\end{aligned}$$

$$d\left(\frac{\gamma}{\gamma-1}\right)=\frac{\sum c_{p\kappa}m_\kappa dc_\kappa}{Rc}-\frac{dc\times(\sum c_{p\kappa}m_\kappa c_\kappa)}{Rc^2},$$

and referring to (VI. 21)

$$\begin{aligned}&=\frac{1}{Rc_0}\sum c_{p\kappa}m_\kappa\frac{\nu_\kappa}{\left(\sum\frac{\nu_\kappa^2}{c_\kappa}\right)_0}\left(\frac{dK_{c'}}{dT}\frac{1}{K_{c'}}dT-\nu\frac{1}{\gamma-1}\frac{dT}{T}\right)_0\\&\quad-\frac{1}{Rc_0^2}\sum c_{p\kappa}m_\kappa c_{\kappa0}\frac{\nu}{\left(\sum\frac{\nu_\kappa^2}{c_\kappa}\right)_0}\left(\frac{dK_{c'}}{dT}\frac{1}{K_{c'}}dT-\nu\frac{1}{\gamma-1}\frac{dT}{T}\right)_0.\end{aligned}$$

$$\cdots\cdots\cdots(\text{VI. }22)$$

Thus all the terms of right hand side of (VI. 6′) are calculated.

i). *Adiabatic condition.* In the present problem the equation corresponding to $p\rho^{-\gamma}=\text{const}$ is

$$-c_v dT+\frac{p}{\rho^2}d\rho=dQ_v. \qquad \cdots\cdots\cdots(\text{VI. }5)$$

Taking the equation of state in consideration, it follows that

$$-c_vT\left(\frac{dp}{p}-\frac{d\rho}{\rho}-\frac{dc}{c}\right)+\frac{p}{\rho^2}d\rho=dQ_v,$$

$$\frac{dp}{p}-\gamma\frac{d\rho}{\rho}=\frac{dc}{c}-\frac{1}{c_vT}dQ_v,$$

or

$$\frac{dp}{p}-\gamma\frac{d\rho}{\rho}-d\gamma\cdot\log\rho=\frac{dc}{c}-\frac{1}{c_vT}dQ_v-d\gamma\cdot\log\rho,$$

namely

$$d(\log p\rho^{-\gamma})=\frac{dc}{c}-\frac{dQ_v}{c_vT}-d\gamma\cdot\log\rho. \qquad \cdots\cdots\cdots(\text{VI. }23)$$

The right hand side terms of this equation is expressed in terms of T in the same way as in the previous paragraphs. In this case we must take c_0 for c, and c_{v0} for c_v.

ii). *Pressure.* From (VI. 23) the pressure is calculated by means of the following equation

$$\frac{dp}{p}=\frac{dc}{c}+\frac{\gamma}{\gamma-1}\frac{dT}{T}+\frac{dQ_v}{(c_p-c_v)T} \qquad \cdots\cdots\cdots(\text{VI. }24)$$

On the other hand in the usual stable case (adiabatic)

$$\frac{dp}{p}=\frac{\gamma}{\gamma-1}\frac{dT}{T}.$$

Now (VI. 24) is integrated as follows:

$$\int\frac{dp}{p}=\int\frac{dc}{c}+\frac{\gamma}{\gamma-1}\int\frac{dT}{T}-\int\frac{d\left(\frac{\gamma}{\gamma-1}\right)}{dT}\log T\cdot dT+\int\frac{dQ_v}{RcT}.$$

In this integration,

$$\int\frac{dc}{c}-\int\frac{d\left(\frac{\gamma}{\gamma-1}\right)}{dT}\log T\cdot dT+\int\frac{dQ_v}{RcT}$$

is the term of compensation and this is evaluated approximately as shown in last section.

40. *Example: The flow of CO_2 gas.* The gas of CO_2 is dissociated partially at high temperature into CO and O_2. The rate of dissociation is shown in the following table according to the formula:[1]

$$K_p=p_{CO_2}p_{CO}^{-1}p_{O_2}^{-1/2}.$$

This is the case at one atmospheric pressure. But we know that K_p is independent of pressure approximately. Taking the unit of pressure as dyne/cm², the contents of this table are expresed in a formula;

$$\log_e p_{CO_2}p_{CO}^{-1}p_{O_2}^{-1/2}=-23.33+34800\frac{1}{T}$$
$$+0.7387\log_e T.$$

At the temperature of 2000°K and one atmospheric pressure

$$c_1\ (CO_2)=0.02238\ \text{mol/g},$$
$$c_2\ (CO)=0.00035\ \text{mol/g},$$
$$c_3\ (O_2)=0.00018\ \text{mol/g}.$$

In general the experimentally measured value of specific heat of a gas depends more or less

Table 8

T(°K)	$\log_{10} K_p$ (under 1 atmospheric pressure)
300	44.735
400	32.408
600	20.065
800	13.894
1000	10.199
1200	7.743
1400	5.994
1750	3.903
2000	2.864
2500	1.424
3000	0.475
3500	−0.193

[1] Landolt-Börnstein, *Physikalisch-Chemische Tabellen* (1936), Eg, IIIc, S. 2619.

on temperature and pressure.[1] This may be caused by the fact that the freedom of a molecule of gas is not constant. In the present case we assume that each component has its own constant value of specific heat as shown in Table 9.[2]

Table 9

	CO_2	CO	O_2
c_p(erg/g °C)	0.818×10^7	1.040×10^7	0.909×10^7
c_v	0.630×10^7	0.743×10^7	0.649×10^7
c_p-c_v	0.188×10^7	0.297×10^7	0.260×10^7
γ	1.30	1.40	1.40

Under these circumstances we calculate the case where the gas CO_2 (so-called) including the dissociated parts at the temperature of 2,000°K and one atmospheric pressure at rest is accelerated in a flow. The result is shown in Fig. 18. The abscissa represents temperature, and in the case of the chain line the ordinate represents the ratio of pressure in flow to that in tank or in combustion chamber. The dotted line is of p'/p_0 obtained assuming no chemical reaction. The full line represents δB:

$$\delta B \equiv \frac{\frac{\gamma}{\gamma-1}\frac{p}{\rho}+\frac{1}{2}q^2-\left(\frac{\gamma}{\gamma-1}\frac{p}{\rho}\right)_0}{\left(\frac{\gamma}{\gamma-1}\frac{p}{\rho}\right)_0}. \qquad \cdots\cdots\cdots (\text{VI. }25)$$

This reqresents the deviation from the usual Bernouill's Eq. (Cf. (VI. 6')). Here we assume that the change of state in sound wave is so quick and slight that chemical reaction does not exert any influence on the propagation of wave. Let a denote the sound velocity, and then

$$a^2=\frac{dp}{d\rho}=\gamma\frac{p}{\rho}.$$

The state where the flow velocity is equal to a is determined as follows: Substituting q in the left hand side of (VI. 25) with a and plotting this as a fuction of T, we get the point in question as the intersecting point of this curve and δB. Letting T_c denote thus obtained temperature,

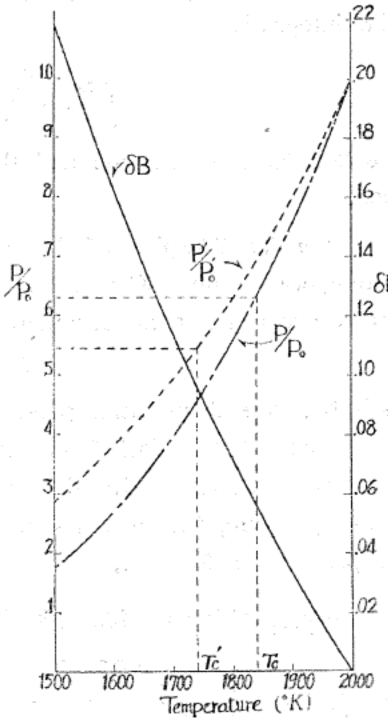

Fig. 18

$$T_c=1838\,\text{K}^\circ$$

in the present example. In the case where $\delta B=0$, i.e., there is no chemical reaction, the corresponding temperature T_c' is

$$T_c'=1738\,\text{K}^\circ.$$

[1] *International Critical Table* (1929), V. p. 80.

[2] Loc. cit. 1), pp. 82-83.

Hereafter let us represent the variables of state with no chemical reaction by the corresponding notations annexed with dash. Comparing the two cases, we get

$$\frac{q_c^2}{q_c'^2}=1.056,$$

$$\frac{p_c}{p_0}=0.632,$$

$$\frac{p_c'}{p_0}=0.541,$$

$$\frac{\rho_c}{\rho_c'}=1.10,$$

therefore

$$\frac{\rho_c q_c}{\rho_c' q_c'}=1.13.$$

41. *Appendix.* Let us consider the case where more than two kinds of reactions take place. We assume f kinds of reactions occur. The mol number of each element in unit mass is denoted by $c_1, c_2, \cdots, c_n$. The increase of them by each kind of reaction is represented as $d_j c_\kappa$, where $j=1, 2, \cdots f$. Thus

$$dc_\kappa=\sum_j d_j c_\kappa,$$

$$\frac{d_j c_1}{(\nu_j)_1}=\frac{d_j c_2}{(\nu_j)_2}= \cdots =\frac{d_j c_n}{(\nu_j)_n}=\frac{d_j c}{\nu_j}.$$

Further,

$$\prod_\kappa c_\kappa^{(\nu_j)_\kappa}=K_{pj}(RT)^{-\nu_j}\rho^{-\nu_j}.$$

Thus, corresponding to (VI. 14)

$$\frac{dK_{c'j}}{K_{c'j}}-(\nu_j)\frac{d\rho}{\rho}=\sum_\kappa(\nu_j)_\kappa\frac{d_j c_\kappa}{c_\kappa}.$$

The absorbed heat is

$$dQ_v=\sum_{j=1}^{j=f}RT^2\frac{d\log K_{c'j}}{dT}\cdot\frac{d_j c}{\nu_j}.$$