ON THE LATERAL MASS-IMPACT APPLIED TO A LONG UNIFORM BAR WITH TWO FLEXUAL FREEDOMS—BENDING AND SHEARING

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Contents

1. Introduction

The problems of the longitudinal impacts of bars or of the struck strings can be solved easily by the well-known processes. But, on the contrary, the problems of lateral impacts of bars are not so simple.

In spite of many investigations reported up to this day, most of their results do not seem to be successful in the meaning of practical use. Fourier [1], St-Venant [2], Sezawa [3], Shibuya [4], etc. tried to solve this problem upon the base of the ordinary bending theory of thin beams. Some of them assumed the initial disturbances and some assumed the force variations applied to the bar, but, evidently, such assumptions are not suitable for the cases of impacts. A more preferable treatment for this problem is to assume the system as a freely vibrating mass-loaded bar. The ordinary theory of bending, however, gives no reasonable results especially for the contact pressures, the induced stresses and their propagations, even when this last treatment is applied.

Timoshenko [5], and his successors; Mason [6], Lennerz [7], Lees [8], etc. introduced an idea of the local deformation based on the Hertz's theory of contact [9], and succeeded in to get the impact pressure and the duration of contact, but this way of treatment is also not applicable to the investigation of the transmission of shock waves.

Another interesting work has been reported by Miyagi [10], but it seems to be far from the rigorous treatment of the dynamical elasticity.

The reason why the classical treatment of the lateral impact problems is not satisfactory is mainly caused by the fact that the ordinary beam theory is established upon only one flexual freedom neglecting all the higher strain components, and this shear rigid assumption induces some solid characteristics against the local shearing force; the expressions in series for stresses or strains do not converge to finite values. Obviously more rigorous equations of motion must be adopted. The pure mathematical treatment of the three dimensional elasticity — or even of the two dimensional one — is, however, so complicated for the practical use that it seems to be quite unapplicable for the cases of bars with arbitrary sectional forms. Thus, the author tried to apply a conventional equation for the lateral deflection of bars introducing the second freedom, which, theoretically, means the next higher order of component strains and, from another point of view, is considered to be the effective shearing deformation. Application of this idea to our problems leads us to very satisfactory solutions. The bar thus discussed means, however, somewhat a schematical construction and there might obviously remain some questions in pure physical meanings, but the defects in such meanings, if any, should be only of the order of errors.

From the view point of engineering, the most probable cases of shocks are of the momentum types, that is, the impacts of masses with some initial velocities. After the mass detaches from the surface, the motion will be in the state of quite free vibrations (non-stational ofcourse) of a uniform bar, while, until this moment, the mass is in contact with the face of the bar, and the problem is of a transcendental vibration of a mass-loaded bar. In this case, the sectional form of the bar is assumed to be unchanged throughout the duration of motion even at the struck point, and so the motion of the mass is considered to be coincident with that of the bar at this point.

2. The equations of motion and their general solutions for lateral vibrations of bars

Let

x = longitudinal coordinate, $x' = x/\kappa$,

y = lateral displacement,

t = time, $t' = tb/\kappa$,

β = effective angular rotation of a section,

$\overline{\Gamma}$ = effective shearing angle,

$\partial\beta/\partial x$ = bending strain,

ρ = density of the bar,

A = sectional area,

κ = radius of gyration of the section,

E = Young's modulus,
G = modulus of rigidity,
G_eA = effective shearing rigidity of the bar,
$\mu = E/G_e$,
$b = \sqrt{E/\rho}$ = longitudinal sound velocity,
S = shearing force,
M = bending moment.

Then the fundamental equations of motion in general are expressed as

$$\left.\begin{aligned} \rho A \frac{\partial^2 y}{\partial t^2} &= \frac{\partial S}{\partial x} - \text{(external resisting lateral force)}, \\ \rho A \kappa^2 \frac{\partial^2 \beta}{\partial t^2} &= S - \frac{\partial M}{\partial x} - \text{(external resisting moment)}. \end{aligned}\right\} \quad \cdots\cdots\cdots (1)$$

On the other hand, the shearing force S and the bending moment M are expressed by

$$\left.\begin{aligned} S &= G_e A \bar{\Gamma} + \text{(internal resisting force)}, \\ M &= -EA\kappa^2 \frac{\partial \beta}{\partial x} + \text{(internal resisting moment)}, \end{aligned}\right\} \quad \cdots\cdots\cdots (2)$$

if we use the effective shearing strain $\bar{\Gamma}$ defined as in Appendix I.

If there are no resistances at all, the equation of motion is reduced to

$$\frac{\partial^2 y}{\partial t^2} - \kappa^2(1+\mu)\frac{\partial^4 y}{\partial t^2 \partial x^2} + b^2\kappa^2 \frac{\partial^4 y}{\partial x^4} + \frac{\kappa^2}{b^2}\mu \frac{\partial^4 y}{\partial t^4} = 0.^{*} \quad \cdots\cdots\cdots (3)$$

When $\bar{\Gamma}$ is neglected or μ is assumed to be zero, Eq. (3) is reduced to

$$\frac{\partial^2 y}{\partial t^2} - \kappa^2 \frac{\partial^4 y}{\partial t^2 \partial x^2} + b^2\kappa^2 \frac{\partial^4 y}{\partial x^4} = 0.^{**} \quad \cdots\cdots\cdots (4)$$

Further, if the term of rotatory inertia is also neglected, the ordinary equation of bending vibration

$$\frac{\partial^2 y}{\partial t^2} + b^2\kappa^2 \frac{\partial^4 y}{\partial x^4} = 0.^{***} \quad \cdots\cdots\cdots (5)$$

is obtained.

On the contrary, when μ and E are assumed to be infinitely large, the equation for a pure shear bar

$$\frac{\partial^2 y}{\partial t^2} + b_s^2 \frac{\partial^2 y}{\partial x^2} = 0, \qquad b_s = \sqrt{G_e/\rho} \quad \cdots\cdots\cdots (6)$$

can be obtained.

In our problem, in which the components of higher frequencies are important, careful consideration is needed in neglecting any physical terms. Eq. (3) would be the minimum necessary construction. In this case, however, the effects of the external

* This form of equation has already been given by S. Timoshenko.[11]

, * Shown in Rayleigh's "Theory of Sound," Vol. 1 for example.

and internal resistances might also be important. It is ofcourse not easy to obtain simple and correct expressions for these resistances. The external resistance is consisted mainly of air resistance, in which the essential viscous resistance, the elastic damping resistance owing to the discipation of energy as sounds, virtual mass resistance, eddy loss resistance, etc. are contained. These might show all different types of characteristics. But, for convenience's sake, let us confine ourselves only to the case of resistance proportional to the velocity, with the same coefficient η for both linear and rotational motions. Thus we put

$$\frac{\text{external resistance}}{\text{mass}} = 2\,\eta \times \text{(displacement velocity).}$$

The physical constitution of the internal viscosity is not so clear to-day. But according to the experimental results reported up to this day [12], the damping is more likely to the type of constant logarithmic decrement than the type induced from the viscosity proportional to the strain velocity, though it is not asserted if this result can be extended or not to the case of shock including the elements of vibration of very high frequencies, in the form of a resultant of them moreover. The latter type of viscosity has been treated by Sezawa and many other authorities. Endô [13] has succeeded in to formalize the former type of viscosity, though there remains some physical question in his logic. In this paper, both cases shall be discussed for the sake of comparison.

Introducing the internal viscosity proportional to the strain velocity together with the external viscosity mentioned above, the fundamental equations of motion are reduced to

$$\left.\begin{aligned}&\left(\frac{\partial^2}{\partial t^2} + 2\,\eta\,\frac{\partial}{\partial t}\right)\left\{y - \kappa^2(1+\mu)\,\frac{\partial^2 y}{\partial x^2} + \kappa^2\,\frac{\partial \overline{\Gamma}}{\partial x}\right\} = -\,b^2\kappa^2\left(1 + 2\,\xi\,\frac{\partial}{\partial t}\right)\frac{\partial^4 y}{\partial x^4},\\&\left(\frac{\partial^2}{\partial t^2} + 2\,\eta\,\frac{\partial}{\partial t}\right)y = \frac{b^2}{\mu}\left(1 + 2\,\xi\,\frac{\partial}{\partial t}\right)\frac{\partial \overline{\Gamma}}{\partial x},\end{aligned}\right\} \quad \cdots\cdots (3a)$$

where the coefficient of viscosity ξ is defined as

$$\frac{\text{viscous stress}}{\text{elastic constant}} = 2\,\xi \times \text{(strain velocity),}$$

using the same value of ξ for both normal and shearing strains for the sake of convenience.

After the idea proposed by T. Endô, the force and the moment in a section including the resistance due to the solid viscosity of the type of constant logarithmic decrement may be assumed to be

$$\begin{aligned}S &= G_e A\{\sqrt{1-(2\,\xi)^2} + 2\,i\xi\}\overline{\Gamma},\\M &= -\,EA\kappa^2\{\sqrt{1-(2\,\xi)^2} + 2\,i\xi\}\left(\frac{\partial^2 y}{\partial x^2} - \frac{\partial \overline{\Gamma}}{\partial x}\right),\end{aligned}$$

adopting the complex expressions for y and $\overline{\Gamma}$ in periodic motions. With these expressions, the fundamental equations corresponding to (3a) become to

$$\left. \begin{aligned} &\left(\frac{\partial^2}{\partial t^2}+2\eta\frac{\partial}{\partial t}\right)\left\{y-\kappa^2(1+\mu)\frac{\partial y^2}{\partial x^2}+\kappa^2\frac{\partial \overline{\Gamma}}{\partial x}\right\}=-b^2\kappa^2\{\sqrt{1-(2\xi)^2}+2i\xi\}\frac{\partial^4 y}{\partial x^4}, \\ &\left(\frac{\partial^2}{\partial t^2}+2\eta\frac{\partial}{\partial t}\right)y=\frac{b^2}{\mu}\left\{\sqrt{1-(2\xi)^2}+2i\xi\right\}\frac{\partial \overline{\Gamma}}{\partial x}, \end{aligned} \right\} \quad (3b)$$

in which the coefficient of viscosity ξ is a dimensionless constant related to the statical hysteresis of the material used.

The general solutions of Eqs. (3)$\sim$(6) in series form

$$y=\sum e^{\pm i(\alpha x'\pm pt')}=\sum_\nu u_\nu(x)e^{\pm ip_\nu t'}$$

can always easily be obtained, where $u_\nu(x)$ is to be one of the normal functions satisfying the boundary conditions and p_ν is related with it by a proper relation.

In the following, auxiliary parameters ν and θ are used for convenience only.

Eq. (5):

$$p=\nu, \quad \alpha^4-\nu^2=0, \quad \alpha_1=\pm\sqrt{\nu}, \quad \alpha_2'=\pm\sqrt{\nu}, \qquad \cdots\cdots\cdots (7)$$

$$u_\nu(x)=A\,\mathrm{ch}\sqrt{\nu}\,x'+B\,\mathrm{ch}\sqrt{\nu}\,x'+C\cos\sqrt{\nu}\,x'+D\sin\sqrt{\nu}\,x', \quad \cdots\cdots (8)$$

$$y_\nu=u_\nu(x)\{P\cos\nu t'+Q\sin\nu t'\}.$$

Eq. (4):

$$p=\nu=1/\mathrm{sh}\,\theta\,\mathrm{ch}\,\theta,$$

$$\alpha^4-\nu^2\alpha^2-\nu^2=0, \quad \alpha_1=\pm 1/\mathrm{sh}\,\theta, \quad \alpha_2'=\pm 1/\mathrm{ch}\,\theta, \quad \cdots\cdots (9)$$

$$u_\nu(x)=A\,\mathrm{ch}\frac{x'}{\mathrm{ch}\,\theta}+B\,\mathrm{sh}\frac{x'}{\mathrm{ch}\,\theta}+C\cos\frac{x'}{\mathrm{sh}\,\theta}+D\sin\frac{x'}{\mathrm{sh}\,\theta}, \quad \cdots (10)$$

$$y_\nu=u_\nu(x)\left\{P\cos\frac{t'}{\mathrm{sh}\,\theta\,\mathrm{ch}\,\theta}+Q\sin\frac{t'}{\mathrm{sh}\,\theta\,\mathrm{ch}\,\theta}\right\}.$$

Eq. (3):

$$p=\nu, \quad \alpha^4-(1+\mu)\nu^2\alpha^2+(\mu\nu^2-1)\nu^2=0. \qquad \cdots\cdots\cdots (11)$$

If we choose the value $\mu=3$, (This corresponds to the case of rectangular cross sections and Poisson's ratio $\sigma=0.25$), putting $\nu=1/\mathrm{sh}\,\theta$,

$$\alpha_1=\sqrt{2+\mathrm{ch}\,\theta}/\mathrm{sh}\,\theta, \quad \alpha_2=\sqrt{2-\mathrm{ch}\,\theta}/\mathrm{sh}\,\theta, \quad \alpha_2'=\sqrt{\mathrm{ch}\,\theta-2}/\mathrm{sh}\,\theta, \quad (11')$$

$$u_\nu(x)=A\frac{\mathrm{ch}\,\alpha_2'x'}{\cos\alpha_2 x'}+B\frac{\mathrm{sh}\,\alpha_2'x'}{\sin\alpha_2 x'}+C\cos\alpha_1 x'+D\sin\alpha_1 x', \qquad \cdots\cdots\cdots (12)$$

$$y_\nu=u_\nu(x)\left(P\cos\frac{t'}{\mathrm{sh}\,\theta}+Q\sin\frac{t'}{\mathrm{sh}\,\theta}\right).$$

Eq. (3a):

$$\left. \begin{aligned} &\nu^2=\frac{1}{\mathrm{sh}^2\theta}=p^2\frac{1-2i\eta\kappa/bp}{1+2i\xi pb/\kappa}, \\ &p=i\zeta\pm q=\nu\{i(X+Y)\pm\sqrt{1-(X+Y)^2}\}, \\ &X=(b/\kappa\,\mathrm{sh}\,\theta)\xi, \quad Y=(\kappa\,\mathrm{sh}\,\theta/b)\eta, \end{aligned} \right\} \qquad \cdots\cdots\cdots (13)$$

(11), (11′), and (12),

$$y_\nu=u_\nu(x)e^{-\zeta t'}\{P\cos qt'+Q\sin qt'\} \qquad \cdots\cdots\cdots (14)$$

Eq. (3b):

$$\left.\begin{aligned} &\nu^2 = \frac{1}{\mathrm{sh}^2\theta} = p^2 \frac{1 - 2\,i\eta\kappa/bp}{\sqrt{1-(2\,\xi)^2 + 2\,i\xi}}, \\ &p = i\zeta \pm q = \nu\{iY \pm \sqrt{\sqrt{1-(2\,\xi)^2 + 2\,i\xi} - Y^2}\}, \\ &\zeta \doteqdot \frac{\kappa}{b}\eta + \frac{\xi}{\mathrm{sh}\,\theta}\Big/\sqrt{1-\xi^2-Y^2}, \\ &q \doteqdot \frac{1}{\mathrm{sh}\,\theta}\sqrt{1-\xi^2-Y^2}, \end{aligned}\right\} \qquad \cdots\cdots\cdots (15)$$

(11), (11′), (12) and (14).

In this last case, ξ^2 is always negligibly small as compared with unity, and also Y^2 for high frequencies. Thus we can further put

$$\zeta \doteqdot (\kappa/b)\eta + \xi/\mathrm{sh}\,\theta\,, \qquad q \doteqdot 1/\mathrm{sh}\,\theta = \nu\,.$$

The comparative differences between equations (3a) and (3b), when the external resistance is neglected, are summarised as follows.

	Damping factor	Frequency factor
Eq. (3a)	$\exp\{-(b/\kappa)\xi\nu^2\}t'$	$\nu\sqrt{1-(b/\kappa)^2\nu^2\xi^2}\,t'$
Eq. (3b)	$\exp\{-(b/\kappa)\xi\nu\}t'$	$\nu t'$

Eq. (6):

With notations $\quad x' = x/\kappa\,, \qquad t'' = tb_s/\kappa\,,$

$$\alpha = p = \nu\,,$$
$$y_\nu = (C\cos\nu x' + D\sin\nu x')(P\cos\nu t'' + Q\sin\nu t'')\,.$$

3. The velocity of propagation of the lateral waves in a bar

It has already been proved that a motion

$$y = \int_{\nu-\varepsilon}^{\nu+\varepsilon} y_\nu d\nu\,,$$

in which $y_\nu = P_\nu \sin\alpha_\nu x' \sin q_\nu t'$, is a wave motion with a group velocity $b_{g\nu}\cdot b = \pm\,(dq/d\alpha)_\nu b$, while the velocity of the wave component y_ν is $b_\nu'\cdot b = (q_\nu/\alpha_\nu)\cdot b$.

As α and q are now not in linear relation, the wave is of a dispersive type and changes its form from time to time, and it is propagated as an energy mass of a wave group.

Now, as the fundamental equation (3) (or including (3a) and (3b)) has two sorts of solutions in respect of the relation between α and q, it is easily seen that there are two group velocities for a same wave length or for a same wave frequency. The coefficients b' and b_g of the both types thus calculated are obtained as follows.

For the first group,

$$b_1' = \frac{1}{\sqrt{2+\mathrm{ch}\,\theta}} \qquad \text{or} \qquad b_1' = \frac{\sqrt{1-X_\theta^2}}{\sqrt{2+\mathrm{ch}\,\theta}}\,,$$

$$b_{g1} = \frac{2\,\mathrm{ch}\,\theta\sqrt{2+\mathrm{ch}\,\theta}}{\mathrm{ch}^2\,\theta + 4\,\mathrm{ch}\,\theta + 1}, \qquad \text{or} \qquad b_{g1} = \frac{2\,\mathrm{ch}\,\theta\sqrt{2+\mathrm{ch}\,\theta}\,(1-2X_0^2)}{(\mathrm{ch}^2\,\theta + 4\,\mathrm{ch}\,\theta + 1)\sqrt{1-X_0^2}},$$

and for the sound group,

$$b_2' = \frac{1}{\sqrt{2-\mathrm{ch}\,\theta}} \qquad \text{or} \qquad b_2' = \frac{\sqrt{1-X_0^2}}{\sqrt{2-\mathrm{ch}\,\theta}},$$

$$b_{g2} = \frac{2\,\mathrm{ch}\,\theta\sqrt{2-\mathrm{ch}\,\theta}}{\mathrm{ch}^2\,\theta - 4\,\mathrm{ch}\,\theta + 1} \qquad \text{or} \qquad b_{g2} = \frac{2\,\mathrm{ch}\,\theta\sqrt{2-\mathrm{ch}\,\theta}\,(1-2X_0^2)}{(\mathrm{ch}^2\,\theta - 4\,\mathrm{ch}\,\theta + 1)\sqrt{1-X_0^2}},$$

the left side relations of these being for Eqs. (3) and (3b) and the right side relations for Eq. (3a). The effects of the external viscosity are now all omitted for convenience. b_{g2} and b_2' do not exist for the range $\mathrm{ch}\,\theta > 2$.

Table 1 shows the comparison of these velocity coefficients with those obtained

Table 1. Comparison of α, b', b_g when $\xi = \eta = 0$

Equation	(5)	(4)	(3)	(6)
Assumption	$G_e = \infty$, $\ddot{\beta} = 0$	$G_e = \infty$	$\mu = E/G_e = 3$	$E = \infty$
Aux. Variable	ν	$\nu = 1/\mathrm{sh}\,\theta\,\mathrm{ch}\,\theta$	$\nu = 1/\mathrm{sh}\,\theta$	ν
q	ν	ν	ν	ν
α_1	$\sqrt{\nu}$	$1/\mathrm{sh}\,\theta$	$\sqrt{2+\mathrm{ch}\,\theta}/\mathrm{sh}\,\theta$	$\nu\sqrt{\mu}$ *
α_2	—	—	$\sqrt{2-\mathrm{ch}\,\theta}/\mathrm{sh}\,\theta$	—
α_2'	$\sqrt{\nu}$	$1/\mathrm{ch}\,\theta$	$\sqrt{\mathrm{ch}\,\theta-2}/\mathrm{sh}\,\theta$	—
b_1'	$\sqrt{\nu}$	$1/\mathrm{ch}\,\theta$	$1/\sqrt{2+\mathrm{ch}\,\theta}$	$1/\sqrt{\mu}$ *
b_2'	—	—	$1/\sqrt{2-\mathrm{ch}\,\theta}$	—
b_{g1}	$2\sqrt{\nu}$	$(\mathrm{ch}^2\,\theta+\mathrm{sh}^2\,\theta)/\mathrm{ch}^3\,\theta$	$2\,\mathrm{ch}\,\theta\sqrt{2+\mathrm{ch}\,\theta}/(\mathrm{ch}^2\,\theta+4\,\mathrm{ch}\,\theta+1)$	$1/\sqrt{\mu}$ *
b_{g2}	—	—	$2\,\mathrm{ch}\,\theta\sqrt{2-\mathrm{ch}\,\theta}/(\mathrm{ch}^2\,\theta-4\,\mathrm{ch}\,\theta+1)$	—
b_1' at $\nu = \infty$ ↓ at $\nu = 0$	∞ ↓ $\sqrt{\nu} = 0$	1.00 ↓ $\sqrt{\nu} = 0$	$1/\sqrt{3} = 0.577$ ↓ $\sqrt{\nu} = 0$	0.577 ↓ 0.577
b_2' at $\nu = \infty$ ↓ at ν_{cr}	—	—	1.00 ↓ ∞ $(1/\nu_{cr} = \sqrt{3})$	—
b_{g1} at $\nu = \infty$	∞	1.00	$1/\sqrt{3} = 0.577$	0.577
↓ $b_{g1\,\max}$ ↓	↓	↓ 1.086 $(1/\nu = 0.865)$ ↓	↓ 0.616 $(1/\nu = 1.955)$ ↓	↓
b_{g1} at $\nu = 0$	$2\sqrt{\nu} = 0$	$2\sqrt{\nu} = 0$	$2\sqrt{\nu} = 0$	0.577
b_{g2} at $\nu = \infty$ ↓ b_{g2} at ν_{cr}	—	—	1.00 ↓ 0 $(1/\nu_{cr} = \sqrt{3})$	—

Note: In the column (6), the velocity ratios b', b_g against $b = \sqrt{E/\rho}$ cannot be shown, but for the sake of comparison with other cases, $\mu^* = 3$ and $b = \sqrt{\mu^* G_e/\rho}$ are used.

from Eqs. (5) and (4). These coincide with each other for waves of lower frequencies in all cases. But for the higher frequencies, Eq. (5) gives too much unreasonable results, and Eq. (4) shows the existence of group velocities greater than the longitudinal sound velocity. On the other hand, with Eq. (3), the highest velocity of transmission is only about 62 % to it for the first type of waves, and just the same value with it for the second type of waves, respectively.

The results due to Eq. (6) is also added for comparison, in which the velocity $b'\cdot b$ and $b_g\cdot b$ both coincide with the pure shear velocity $b_s = \sqrt{G_e/\rho}$.

In Table 2, the effects of the internal viscosity proportional to the strain velocity are shown. In this case the motion becomes non-periodic and the velocity of propagation disappears for the range sh $\theta < \sqrt{2}\,\xi b/\kappa$. The maximum value of the velocity and the frequency corresponding to this maximum velocity becomes the smaller, the larger the value of ξ increases.

Table 2. The effect of internal viscosity proportional to strain velocity on b_{g1}

$\xi b/\kappa$	0	1	2	5	20
sh θ at $q = 0$	—	1.414	2.828	7.07	28.28
$b_{g1\,\max}$	0.616	0.535	0.47	0.35	0.205
sh θ at $b_{g1\,\max}$	1.955	4.65	7.9	16	60

The conclusive results on the velocity of propagation for the bar with two flexual freedoms, shearing and bending, are as follows: —

(i) Waves with long wave lengths or with low frequencies have the same characteristics as those obtained from the ordinary theory of bending.

(ii) Two types of wave groups with different velocities exist for higher frequencies. The first type is none other than the pure shear wave at the limit of $q = \infty$. The maximum velocity of this type is about 62 % to the longitudinal sound velocity, somewhat greater than the pure shear wave velocity, when μ is assumed to be 3. The wave group corresponding to this is repressented by $q = 1/1.955$ *. This type of wave corresponds to the Rayleigh's wave in the seismology.

(iii) When there is the internal viscosity proportional to the strain velocity, this maximum velocity decreases in its value.

(iv) The viscosity in the type of constant logarithmic decrement gives almost no effects on the velocity of propagation.

The variation of the values of α_1, α_2, α_2', b_{g1}, b_{g2} and $db_{g1}/d\alpha_1$ against sh θ is shown in Table 3 and Fig. 1.

* On this phenomena, see the discussion in Article 7.

Table 3. $\mathrm{sh}\,\theta$, $\mathrm{ch}\,\theta$, α_1, α_2, α_2', b_{g_1}, b_{g_2}, $db_{g_1}/d\alpha_1$

$\mathrm{sh}\,\theta$	$\mathrm{ch}\,\theta$	α_1	α_2	α_2'	b_{g_1}	b_{g_2}	$db_{g_1}/d\alpha_1$	
0	1.000	$\sqrt{3}/\theta$	$1/\theta$	.	0.577	1.000	0	
0.1	1.005	17.334	9.975	.	0.578	1.00	0	
0.2	1.020	8.689	4.950	.	0.579	0.99	−0.0004	
0.5	1.118	3.532	1.878	.	0.587	0.94	−0.0045	
1.0	1.414	1.848	0.765	.	0.605	0.81	−0.0166	
1.5	1.803	1.300	0.296	.	0.612	0.54	−0.0184	
1.732	2.000	1.154	0	0	0.616	0		
2.0	2.236	1.029	.	0.242	0.616	.	0.005	$\mathrm{sh}\,\theta = 1.955$
5.0	5.099	0.533	.	0.352	0.572	.	0.266	
10.0	10.05	0.347	.	0.284	0.491	.	0.682	$b_{g_1} = 0.616$
20	20.025	0.235	.	0.212	0.390	.	1.14	$db_{g_1}/d\alpha_1 = 0$
50	50.01	0.144	.	0.1385	0.267	.	1.58	
100	100.005	0.101	.	0.0990	0.194	.	1.76	$d^2b_{g_1}/d\alpha_1^2 = -0.0935$
200	200	0.071	.	0.0704	0.139	.	1.88	
500	500	0.0448	.	0.0447	0.089	.	1.96	
1000	1000	0.0317	.	0.03159	0.063	.	1.98	
2000	2000	0.0224	.	0.02235	0.0446	.	1.99	
5000	5000	0.01417	.	0.01415	0.0283	.	2.00	
10^4	10^4	0.0100	.	0.0100	0.0200	.	2.00	
10^5	10^5	0.00316	.	0.0316	0.0063	.	2.00	
∞	$\mathrm{sh}\,\theta$	$1/\sqrt{\mathrm{sh}\,\theta}$	.	$1/\sqrt{\mathrm{sh}\,\theta}$	$2/\sqrt{\mathrm{sh}\,\theta}$	.	2.00	

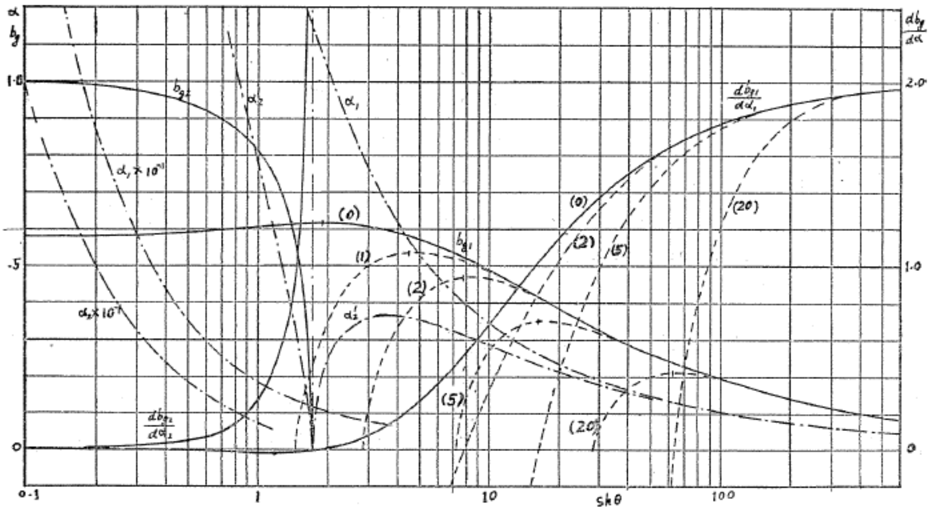

Fig. 1. α_1, α_2, α_2', b_{g_1}, b_{g_2}, $db_{g_1}/d\alpha_1$ and $db_{g_2}/d\alpha_2$ against $\mathrm{sh}\,\theta$. The broken lines are the curves for the cases with internal viscosity proportional to the strain velocity. The figures in brackets show the values of $b\xi/\kappa$.

4. The wave spectrum produced by a mass-impact applied laterally to an infinitely long bar

Take the origin of the coordinates at the midpoint of a bar with the length $2l$, and let a rigid particle with a mass M and with a velocity V strike laterally at this point. As already mentioned in Art. 1, during the mass is in contact with the face of bar, the whole system is considered as a freely vibrating uniform bar loaded with a mass at its center. By the law of Rayleigh for the linear vibrating system [14], this motion may be considered as a summation of many component vibrations expressed with proper normal coordinates.

According to the description shown in Art. 2, the general solution is given as follows,

$$\begin{aligned} y &= \sum Pu(x)e^{-\zeta t'}\sin(qt'-\psi) \\ u(x) &= u_1(x)+u_2(x), \\ u_1(x) &= C\cos\alpha_1 x' + D\sin\alpha_1 x', \\ u_2(x) &= A\,\mathrm{ch}\,\alpha_2' x' + B\,\mathrm{sh}\,\alpha_2' x' \qquad [\mathrm{ch}\,\theta > 2], \\ &= A\cos\alpha_2 x' + B\sin\alpha_2 x \qquad [\mathrm{ch}\,\theta < 2]. \end{aligned}$$

If we choose $\mu = 3$, as a typical case of interest, the relations between p, α_1, α_2 and α_2' are expressed by (11′) with the aid of an auxiliary variable θ. When l is finite, the ratio of the constant A, B, C and D and the value of θ can be determined from the conditions of the two points $x=0$ and $x=l$. When l becomes infinitely large, the conditions at $x=l$ vanish, and two of these constants can no more be determined. But, as far as the above type of solution is adopted, the normality of the function $u(x)$ must be kept always.

The conditions at $x=0$ are two; one of them is that the difference of the shearing forces at the both left- and right-hand adjacent sections is equal to the inertia force of the mass M, and the other is that the inclination of the sectional element at this point is always zero, or in other words the inclination of the neutral layer equals the shearing deformation there.

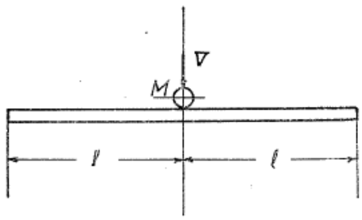

Fig. 2

Thus,

$$M\frac{\partial^2 y(0)}{\partial t^2} = 2G_e A\overline{\Gamma}(0) + (\text{resisting force}) \quad \text{and} \quad \left(\frac{\partial y}{\partial x}\right)_{x=0} - \overline{\Gamma}(0) = 0. \quad (16)$$

With the notations $\bar{\gamma}$, $\bar{\gamma}_1$, and $\bar{\gamma}_2$ for the component shear angles corresponding to u, u_1 and u_2, the following general relations are reduced from the fundamental equations (1).

The component shear angle:

$$\bar{\gamma}_1 = \frac{\mu\nu^2}{\alpha_1^2}\frac{du_1}{dx} = \frac{\mu}{\alpha_1^2\,\mathrm{sh}^2\theta}\frac{du_1}{dx},$$

$$\bar{\gamma}_2 = \frac{-\mu}{\alpha_1'^2\,\mathrm{sh}^2\theta}\frac{du_2}{dx} \quad \text{or} \quad \frac{\mu}{\alpha_2^2\,\mathrm{sh}^2\theta}\frac{du_2}{dx}.$$

The component bending rotation:

$$\frac{du_1}{dx} - \bar{\gamma}_1 = \frac{1}{\alpha_1^2(\mathrm{ch}\,\theta+1)}\frac{du_1}{dx},$$

$$\frac{du_2}{dx} - \bar{\gamma}_2 = \frac{1}{\alpha_2'^2(\mathrm{ch}\,\theta-1)}\frac{du_2}{dx} \quad \text{or} \quad \frac{-1}{\alpha_2^2(\mathrm{ch}\,\theta-1)}\frac{du_2}{dx}.$$

The conditions (16) are thus reduced to

$$\left.\begin{aligned} &k(A+C)+\left(\frac{-1}{\alpha_2'},\ \frac{1}{\alpha_2}\right)B+\frac{D}{\alpha_1}=0\,,\\ &\left(\frac{1}{\alpha_2'},\ \frac{-1}{\alpha_2}\right)\frac{B}{\operatorname{ch}\theta-1}+\frac{D}{\alpha_1(\operatorname{ch}\theta+1)}=0\,,\end{aligned}\right\}[\operatorname{ch}\theta \gtrless 2] \qquad \cdots\cdots\cdots (17)$$

where $k=M/2\,\rho A\kappa$ denotes the relative mass magnitude.

The normality of $u(x)$ can be realized by the condition that there is no escape of energy at the ends $x=\pm l$; that is,

$$y(l)S(l)\equiv 0 \qquad \text{and} \qquad \{(\partial y/\partial x)_l-\overline{\Gamma}(l)\}M(l)\equiv 0\,,$$

and all the ordinary end conditions satisfy these conditions. The condition of normality between u_m and u_n for our problem

$$ku_m(0)u_n(0)+\int_0^\infty\left[u_m(x)u_n(x)+\kappa^2\left(\frac{\partial u_m(x)}{\partial x}-\overline{\gamma}_m(x)\right)\left(\frac{\partial u_n(x)}{\partial x}-\overline{\gamma}_n(x)\right)\right]\frac{dx}{\kappa}$$
$$=0\,, \qquad [m\neq n] \qquad \cdots\cdots\cdots (18)$$

can be obtained from the fundamental equation (3) and above conditions. When $m=n$, the left-hand side of (18) does not vanish.

Let us now consider only the simply supported end conditions, because of the fact that any combination should give the same results as the length tends to infinity. Thus,

$$u(l)=0\,, \qquad (\partial^2 u/\partial x^2)_l=0\,.$$

After some calculations, we get

$$\left.\begin{aligned} A&=-\operatorname{th}(\alpha_2'l')\cdot B=-B\,, &&[\operatorname{ch}\theta>2]\\ &=-\tan\alpha_2 l'\cdot B\,, &&[\operatorname{ch}\theta<2]\\ C&=-\tan\alpha_1 l'\cdot D\,. \end{aligned}\right\} \qquad \cdots\cdots\cdots (19)$$

From (17) and (19)

$\operatorname{ch}\theta>2$:

$$\tan\alpha_1 l'=\frac{\alpha_2'(\operatorname{ch}\theta-1)}{\alpha_1(\operatorname{ch}\theta+1)}+\frac{2\operatorname{ch}\theta}{k\alpha_1(\operatorname{ch}\theta+1)}\,, \qquad \cdots\cdots\cdots (20)$$

$$u=\frac{\alpha_2'(\operatorname{ch}\theta-1)}{\alpha_1(\operatorname{ch}\theta+1)}e^{-\alpha_2'x'}+\frac{1}{\cos\alpha_1 l'}\sin\alpha_1(x'-l')\,; \qquad \cdots\cdots\cdots (21)$$

$\operatorname{ch}\theta<2$:

$$\tan\alpha_1 l'+\frac{\alpha_2(\operatorname{ch}\theta-1)}{\alpha_1(\operatorname{ch}\theta+1)}\tan\alpha_2 l'=\frac{2\operatorname{ch}\theta}{k\alpha_1(\operatorname{ch}\theta+1)}\,, \qquad \cdots\cdots\cdots (20')$$

$$u=\frac{\alpha_2(\operatorname{ch}\theta-1)}{\alpha_1(\operatorname{ch}\theta+1)}\frac{\sin\alpha_2(x'-l')}{\cos\alpha_2 l'}+\frac{\sin\alpha_1(x'-l')}{\cos\alpha_1 l'}\,. \qquad \cdots\cdots\cdots (21')$$

For a finite value of l, a series of θ's can be determined from (20) or (20′) and the functions $u(x)$ are decided. When l tends to infinity, however, θ and α may have continuous values. Nevertheless, for the range such as $\operatorname{ch}\theta>2$, there shall be needed a promise $\delta\alpha_1=(\kappa/l)\pi$ from Eq. (20) and the value $\tan(\alpha_1 l')$ is determined. It is not so easy to determine the values of $\tan\alpha_1 l'$ or $\tan\alpha_2 l'$ from (20′). But considering the fact that the value of $\alpha_2(\operatorname{ch}\theta-1)/\alpha_1(\operatorname{ch}\theta+1)$ is always very small for the range such as $\mathrm{ch}\,\theta < 2$, and also with some statistical consideration, we can omit the second term in (20'), thus

$$\tan \alpha_1 l' \fallingdotseq 2\,\mathrm{ch}\,\theta / k\alpha_1(\mathrm{ch}\,\theta + 1)\,,$$

and again

$$\delta\alpha_1 = (\kappa/l)\pi\,.$$

There exists, obviously another set of roots of (20') which are mainly related to the value of $\tan \alpha_2 l'$, or $\alpha_2 l' \fallingdotseq (n+\frac{1}{2})\pi$, but we can exclude all such solutions because of the fact that they are not important in magnitudes.

When the normal functions $u(x)$ are determined, the amplitude P of each component vibration can be determined by Rayleigh's method.[14]

The initial conditions for our case are expressed as follows:

$$\text{At } t=0; \qquad y=0, \quad \beta = \partial y/\partial x - \overline{\Gamma} = 0, \quad \partial\beta/\partial t = 0,$$
$$(\partial y/\partial t)_{x\neq 0} = 0, \quad (\partial y/\partial t)_{x=0} = V.$$

Now the component amplitude P for each θ is calculated by the formula

$$\left.\begin{aligned} P &= \frac{\kappa}{b}\,\frac{kVu(0)}{qI}, \qquad \phi = 0, \\ I &= k\{u(0)\}^2 + \int_0^l \Big[\{u(x)\}^2 + \kappa^2\Big\{\frac{du(x)}{dx} - \overline{\gamma}(x)\Big\}^2\Big]\frac{dx}{\kappa}. \end{aligned}\right\} \quad \cdots\cdots\cdots (22)$$

The denominator I, when l tends to infinity, is reduced to

$$\mathrm{ch}\,\theta > 2: \qquad I = \frac{l}{2}\Big\{1 + \frac{1}{\alpha_1^2(\mathrm{ch}\,\theta+1)^2}\Big\}\frac{1}{\cos^2\alpha_1 l'},$$

$$\begin{aligned} \mathrm{ch}\,\theta < 2: \qquad I &= \frac{l}{2}\Big[\Big\{1 + \frac{1}{\alpha_1^2(\mathrm{ch}\,\theta+1)^2}\Big\}\frac{1}{\cos^2\alpha_1 l'} \\ &\qquad + \Big\{\frac{1+\alpha_2^2(\mathrm{ch}\,\theta-1)^2}{\alpha_1^2(\mathrm{ch}\,\theta+1)^2}\Big\}\frac{1}{\cos^2\alpha_2 l'}\Big]. \\ &\fallingdotseq \frac{l}{2}\Big\{1 + \frac{1}{\alpha_1^2(\mathrm{ch}\,\theta+1)^2}\Big\}. \end{aligned}$$

The second term of the latter equation has been omitted as its value is always less than one twelveth of the first term. Substituting q, $u(0)$ and I in expression (22), we get

$$P = \frac{4\kappa^2 V}{lb}\,\frac{-\,\mathrm{sh}\,\theta\,\mathrm{ch}\,\theta\,\alpha_1^3(\mathrm{ch}\,\theta+1)^3}{\{1+\alpha_1^2(\mathrm{ch}\,\theta+1)^2\}F(\theta,k)}\,\frac{1}{\sqrt{1-X^2}}, \qquad \cdots\cdots\cdots (23)$$

$$F(\theta,k) = \alpha_1^2(\mathrm{ch}\,\theta+1)^2 + \Big\{\frac{2\,\mathrm{ch}\,\theta}{k} + \binom{\alpha_2'(\mathrm{ch}\,\theta-1)}{0}\Big\}^2, \qquad \cdots\cdots\cdots (24)$$

where the factor $\sqrt{1-X^2}$ is necessary only for the case with the solid viscosity proportional to the strain velocity.

The term $\exp(-\alpha_2' x')$ in $u(x)$ shown by (21) is important only near the origin and can be neglected for the investigation of the propagating waves; the term $\sin \alpha_2(x'-l')$ in (21') is also negligibly small as cited above. Using the operation $\int_0^\infty (l/\pi\kappa)d\alpha_1$ instead of $\sum$, the complete solution, except for the neighbourhood of the struck point, is reduced to the expressions as follows.

Displacement:

$$y_1 = \frac{4\kappa V}{\pi b}\int_0^\infty \{-\Phi(\theta)\} \sin(\alpha_1 x' - \varphi_1) e^{-\zeta t'} \frac{\sin qt'}{\sqrt{1-X^2}} d\alpha_1, \quad \cdots\cdots (25)$$

Effective shearing strain:

$$\overline{\Gamma} = \frac{4V}{\pi b}\int_0^\infty \left\{-\Phi(\theta)\ \frac{\mu}{\alpha_1 \operatorname{sh}^2\theta}\right\} \cos(\alpha_1 x' - \varphi_1) e^{-\zeta t'} \frac{\sin qt'}{\sqrt{1-X^2}} d\alpha_1, \quad \cdots (26)$$

Bending strain:

$$\frac{\partial^2 y_1}{\partial x^2} - \frac{\partial \overline{\Gamma}_1}{\partial x} = \frac{4V}{\pi b\kappa}\int_0^\infty \frac{\Phi(\theta)}{\operatorname{ch}\theta + 1} \sin(\alpha_1 x' - \varphi_1) e^{-\zeta t'} \frac{\sin qt'}{\sqrt{1-X^2}} d\alpha_1. \quad \cdots (27)$$

In these expressions, the function $\Phi(\theta)$ and the phase angle φ_1 are defined as follows.

$$\left.\begin{aligned} \Phi(\theta) &= \frac{\operatorname{sh}\theta \operatorname{ch}\theta \alpha_1^2 (\operatorname{ch}\theta + 1)^2}{\{1+\alpha_1^2(\operatorname{ch}\theta+1)^2\}\{F(\theta_1 k)\}^{1/2}} = \frac{\operatorname{sh}\theta \operatorname{ch}\theta(\operatorname{ch}\theta+2)(\operatorname{ch}\theta+1)}{(\operatorname{ch}^2\theta + 4\operatorname{ch}\theta + 1)\{F(\theta, k)\}^{1/2}}, \\ \tan\varphi_1 &= \left\{\frac{2\operatorname{ch}\theta}{k} + \binom{\alpha_2'(\operatorname{ch}\theta - 1)}{0}\right\} \Big/ \alpha_1(\operatorname{ch}\theta + 1). \end{aligned}\right\} \quad (28)$$

In the case with solid viscosity proportional to the strain velocity, q and $\sqrt{1-X^2}$ may become imaginary when $\operatorname{sh}\theta$ becomes very small. In such case the following transformations are needed.

When $X = \xi b/\kappa \operatorname{sh}\theta = 1$;

$$\sin qt'/\sqrt{1-X^2} = t'/\operatorname{sh}\theta, \qquad e^{-\zeta t'} = e^{-t'/\operatorname{sh}\theta}.$$

When $X > 1$;

$$\sin qt'/\sqrt{1-X^2} = \operatorname{sh} q't'/\sqrt{X^2-1} \quad \text{where } q' = \sqrt{X^2-1}/\operatorname{sh}\theta.$$

When $\operatorname{sh}\theta \to 0$ or $X \gg 1$;

$$e^{-\zeta t'} \sin qt'/\sqrt{1-X^2} \longrightarrow \frac{\kappa \operatorname{sh}\theta}{2b\xi} e^{-(\kappa/(2b\xi))t'}.$$

For small α_1, i.e. for large θ, $\Phi(\theta)$ tends to $(k/2)\operatorname{sh}\theta$ or to $k/2\alpha_1^2$, and $\tan\varphi_1$ to $2/\alpha_1 k$. For large α_1, i.e. for small θ, $\Phi(\theta)$ tends to $\theta^2/2\sqrt{3}$ or to $\sqrt{3}/2\alpha_1^2$, and φ_1 to $1/\alpha_1 k$.

The velocity at $t = 0$, corresponding to above solutions, is expressed as

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \frac{4V}{\pi}\int_0^\infty \frac{-\operatorname{ch}\theta \cdot \alpha_1^3 (\operatorname{ch}\theta+1)^3}{\{1+\alpha_1^2(\operatorname{ch}\theta+1)^2\} F(\theta, k)} u(x) d\alpha_1,$$

$$\left(\frac{\partial y(0)}{\partial t}\right)_{t=0} = \frac{8V}{\pi k}\int_0^\infty \frac{\operatorname{ch}^2\theta \alpha_1^2 (\operatorname{ch}\theta+1)^2}{\{1+\alpha_1^2(\operatorname{ch}\theta+1)^2\} F(\theta, k)} d\alpha_1,$$

of which, the approximate coincidence with the initial condition is proved when we compare them with the results derived from the ordinary theory of bending, Eq. 5;

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \frac{4V}{\pi}\int_0^\infty \frac{-k^2\alpha\{e^{-\alpha x'} + \sin\alpha x' - (1+2/k\alpha)\cos\alpha x'\}}{2(k^2\alpha^2 + 2k\alpha + 2)} d\alpha = 0,$$

$$\left(\frac{\partial y(0)}{\partial t}\right)_{t=0} = \frac{4V}{\pi}\int_0^\infty \frac{d(k\alpha)}{(k\alpha)^2 + 2k\alpha + 2} = V\frac{4}{\pi}\Big|\tan^{-1}(1+k\alpha)\Big|_0^\infty = V.$$

5. The approximate integration of the solution by Sezawa-Kanai's method

Though the integration of the expression (25), (26) or (27) is never an easy work in general, an interesting method applicable to this problem is reported by Sezawa and Kanai [15] referring to their problem on the tidal wave transmission in a deep sea.

The shock wave is consisted of infinitely large numbers of components with whole range of wave length. But at the point x' and at the time t', only the component with the group velocity $x'/t' = b_g$ becomes conspicuous selectively. Thus, from this view point, it is required to trace accurately only the motions in the neighbourhood of this wave.

Now our solution is expressed in the type of

$$\begin{aligned} Z &= \int_0^\infty P \sin(\alpha x' - \varphi) e^{-\zeta t'} \sin qt' d\alpha \\ &= \frac{1}{2}\int_0^\infty P e^{-\zeta t'} \{\cos(\alpha x' - qt' - \varphi) + \cos(\alpha x' + qt' - \varphi)\} d\alpha . \quad \cdots\cdots (29) \end{aligned}$$

Let P, ζ and φ, the functions of α as they are, be assumed to vary their values very slowly compared with the cosine factors. When we apply the Taylor's expansion to $\alpha x' - qt'$, taking the origin of α at the point where $x'/t' = b_g = dq/d\alpha$, we get

$$\alpha x' - qt' = (\alpha x' - qt')_0 - \left(t' \frac{db_g}{d\alpha}\right)_0 \frac{(\alpha - \alpha_0)^2}{2} - \left(t' \frac{d^2 b_g}{d\alpha^2}\right)_0 \frac{(\alpha - \alpha_0)^3}{6} - \cdots (30)$$

The motion with the argument $\alpha x' + qt'$ can be omitted as it has no effect for the range $x' > 0$. Putting

$$\sigma = (\alpha - \alpha_0)\left\{\frac{t'}{2}\left|\frac{db_g}{d\alpha}\right|\right\}_0^{1/2} \qquad \text{or} \qquad d\alpha = d\sigma \Big/ \left\{\frac{t'}{2}\left|\frac{db_g}{d\alpha}\right|\right\}_0^{1/2},$$

and neglecting the third and higher terms, we obtain from Eq. (29)

$$\begin{aligned} Z = \frac{P_0}{2} e^{-(\zeta t')_0} \Bigg[& \frac{\cos(\alpha x' - qt' - \varphi)_0}{\left\{\frac{t'}{2}\left|\frac{db_g}{d\alpha}\right|\right\}_0^{1/2}} \int_{-\sigma_0}^{\infty} \cos\sigma^2 d\sigma \\ & \mp \frac{\sin(\alpha x' - qt' - \varphi)_0}{\left\{\frac{t'}{2}\left|\frac{db_g}{d\alpha}\right|\right\}_0^{1/2}} \int_{-\sigma_0}^{\infty} \sin\sigma^2 d\sigma \Bigg], \end{aligned}$$

so far as $db_g/d\alpha$ does not vanish. The notation [0] for these expressions shows the values at the point where the relation quoted above is kept between x' and t', and the top sign of $\mp$ means to be applied when $db_g/d\alpha > 0$, and the bottom sign when $db_g/d\alpha < 0$. The lower limit of integration can be expanded up to $-\infty$ without seious errors except when $\alpha_0 \to 0$, while it must be zero when $\alpha_0 = 0$. Further, introducing the mathematical relations

$$\int_{-\infty}^{\infty} \cos\sigma^2 d\sigma = \int_{-\infty}^{\infty} \sin\sigma^2 d\sigma = \sqrt{\frac{\pi}{2}},$$

the final result can be reduced to

$$Z = \sqrt{\frac{\pi}{2}} P_0 \left(e^{-\zeta t'} \Big/ \sqrt{t' \left|\frac{db_g}{d\alpha}\right|}\right)_0 \cos\left\{(\alpha x' - qt' - \varphi)_0 \pm \frac{\pi}{4}\right\}. \quad \cdots (31)$$

When $db_g/d\alpha \doteqdot 0$, the motion represents the wave of the first arrival. In this case the third term of Eq. (30) must not be omitted, and if we put

$$\sigma = (\alpha - \alpha_0)\left\{\frac{t'}{6}\left|\frac{d^2 b_g}{d\alpha^2}\right|\right\}_0^{1/3},$$

and with the relations

$$\int_{-\infty}^{\infty}\cos\sigma^3 d\sigma = \frac{2}{3}\Gamma\left(\frac{1}{3}\right)\cos\frac{\pi}{6} = 1.546\,, \qquad \int_{-\infty}^{\infty}\sin\sigma^3 d\sigma = 0\,,$$

the solution can be reduced to

$$Z \fallingdotseq 0.773\,P_0\left(e^{-\xi t'}\Big/\sqrt[3]{\frac{t'}{6}\frac{d^2 b_g}{d\alpha^2}}\right)_0\cos(\alpha x' - qt' - \varphi)_0 \quad \cdots\cdots\cdots (32)$$

It must be remarked that, on application of this method, x', t' and α are no more independent variables but are related with each other, and also that, when α becomes very small, the variation of P or φ might be comparable with that of the cosine term, so the approximate solutions shown above might contain considerable errors. In the latter case, however, our solution must coincide with that reduced from the fundamental equation (5), and correct calculations can be performed with the aid of the Fresnel's integral table. Further, the above solution must not be applied to the point $x = 0$, because the terms of α_2 or α_2' can not be neglected there.

The wave motion shown by Eq. (31) is damped out in inverse ratio to the square root of x' or t', even when there is no internal viscosity. On the contrary, the top wave shown by Eq. (32) is damped out in inverse ratio to the cubical root of them. It seems somewhat unreasonable. The physical meaning of Eq. (32) might only show the qualitative fact that the amplitude there is finite any way.

When we put

$$P_0 = \frac{4\,V}{\pi b\kappa}\left(\frac{\Phi(\theta)}{\mathrm{ch}\,\theta + 1}\right)_0 \quad \cdots\cdots\cdots (33)$$

in the above expressions, the solution of bending strains for our problem of lateral impact shown by Eq. (27), Art. 4, will be transformed into a form of practical use. Fig. 3 shows the time variations of the bending strain calculated by this method, with the magnitude of $k = 20$ for example, for the two ideal cases of internal frictions, one with the viscosity proportional to the strain velocity $b\xi/\kappa = 2$, and the other with that of constant logarithmic decrement type $2\pi\xi = 0.0031$. These two values were selected so as to give the same order of damping in case of the frequencies of ordinary vibration tests.

The motion at any position in the bar varies its frequency gradually and is damped out aperiodically as t' becomes large. The total number of cycles of it becomes smaller as x' approaches the origin. For the case with internal viscosity of constant logarithmic decrement type, the wave of the first arrival has the velocity $b \cdot b_{g\max}$ and still a large amplitude, but for the case of another type of viscosity, the notable top vibration arrives much later owing to the high damping effects for higher frequencies. The difference between them for larger t' can not be recognized.

The expressions for α, q, ζ, b_{g1}, are already given; the values of $db_{g1}/d\alpha_1$ and of $d^2b_{g1}/d\alpha_1^2$ at the point where $db_{g1}/d\alpha = 0$, are expressed as follows

$$\left.\begin{aligned} \frac{db_{g1}}{d\alpha_1} &= \frac{2\,\mathrm{sh}^3\theta(\mathrm{ch}^3\theta - 3\,\mathrm{ch}\,\theta - 4)}{(\mathrm{ch}^2\theta + 4\,\mathrm{ch}\,\theta + 1)^3}, \\ \left(\frac{d^2b_{g1}}{d\alpha_1^2}\right)_0 &= \frac{-12\sqrt{2+\mathrm{ch}\,\theta}\,\mathrm{sh}^8\theta}{(\mathrm{ch}^2\theta + 4\,\mathrm{ch}\,\theta + 1)^4}. \end{aligned}\right\} \quad \cdots\cdots\cdots (34)$$

The numerical values of these are shown in Table 3. The numerical values of $F(\theta, k)$, $\Phi(\theta)/(\mathrm{ch}\,\theta + 1)$ and $\varphi(\theta)$ are shown in Table 4.

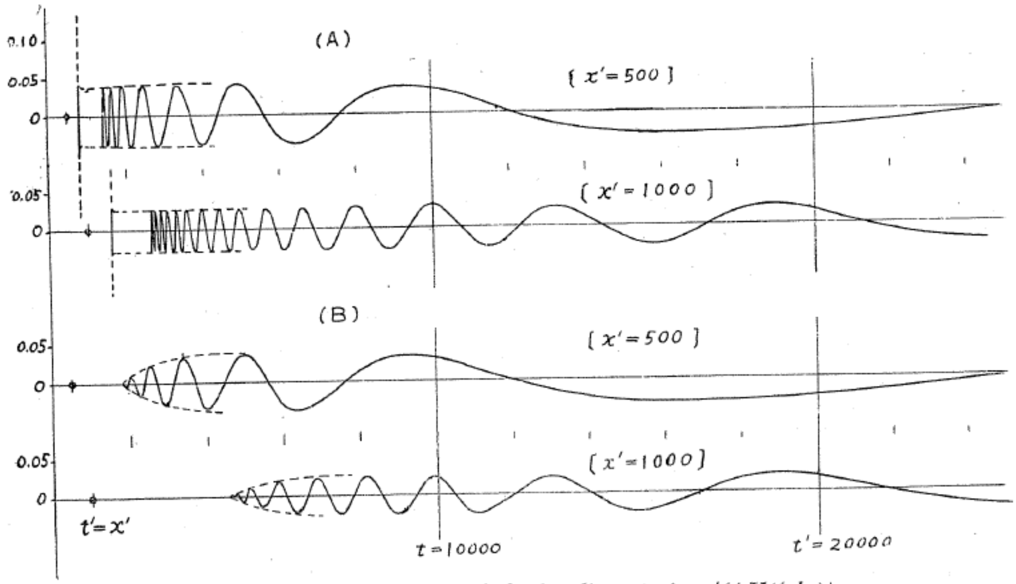

Fig. 3. The transmission of the bending strain $e/(4V/(\pi b\kappa))$;
(A) with internal friction $2\pi\xi = 0.0031$ of constant logarithmic decriment,
(B) with internal viscosity $b\xi/\kappa = 2$ proportional to the strain velocity.

6. The bending strain, shearing strain and the contact pressure at the impact point

The method of integration developed in Art. 5 can not be applied at the origin as the motion there is no more a propagating wave, and it is meaningless to consider the conspicuous wave only. Moreover, the components of the second group can not be neglected.

u, $\bar{\gamma}$, $\frac{du}{dx} - \bar{\gamma}$, $\frac{d^2u}{dx^2} - \frac{d\bar{\gamma}}{dx}$, at $x = 0$ derived from Eqs. (21) and (21′) are

$$\left.\begin{aligned} u(0) &= -\frac{1}{\alpha_1(\mathrm{ch}\,\theta+1)}\,\frac{2\,\mathrm{ch}\,\theta}{k}, \\ \bar{\gamma}(0) &= \frac{\mu}{\kappa\,\mathrm{sh}^2\theta}\,\frac{2\,\mathrm{ch}\,\theta}{\alpha_1(\mathrm{ch}\,\theta+1)}, \qquad \left(\frac{du}{dx}\right)_0 - \bar{\gamma}(0) \equiv 0, \\ \left(\frac{d^2u}{dx^2}\right)_0 - \left(\frac{d\bar{\gamma}}{dx}\right)_0 &= \frac{2\,\mathrm{ch}\,\theta}{\kappa^2\alpha_1(\mathrm{ch}\,\theta+1)^2}\left\{\frac{1}{k}\begin{array}{l} +\,\alpha_2' \\ -\,\alpha_2\tan\alpha_2 l' \end{array}\right\} \end{aligned}\right\} \quad \cdots\cdots (35)$$

Table 4. $F(\theta, k)$, $\Phi(\theta)/(\mathrm{ch}\,\theta + 1)$, φ/π.

$\mathrm{sh}\,\theta$	$F(\theta,k)$				$\Phi(\theta)/(\mathrm{ch}\,\theta+1)$				φ/π			
	$k=10$	$k=20$	$k=50$	$k=\infty$	$k=10$	$k=20$	$k=50$	$k=\infty$	$k=10$	$k=20$	$k \doteq 50$	$k=\infty$
0	$12/\theta^2$	$12/\theta^2$	$12/\theta^2$	$12/\theta^2$	$\theta^2/4\sqrt{3}$	$\theta^2/4\sqrt{3}$	$\theta_2/4\sqrt{3}$	$\theta^2/4\sqrt{3}$	0	0	0	0
0.1	1208	1208	1208	1208	0.0014	0.0014	0.0014	0.0014	0.002	0.001	0.000	0.000
0.2	308	308	308	308	0.0057	0.0057	0.0057	0.0057	0.004	0.002	0.001	0
0.5	56.05	56.01	56	56	0.0348	0.0348	0.0348	0.0348	0.010	0.005	0.002	0
1.0	19.98	19.92	19.9	19.9	0.125	0.125	0.125	0.125	0.020	0.010	0.004	0
1.5	13.43	13.33	13.30	13.3	0.245	0.246	0.246	0.246	0.031	0.016	0.006	0
1.732	12.16	12.04	12.01	12.0	0.306	0.307	0.307	0.307				0
2.0	11.65	11.36	11.25	11.18	0.372	0.376	0.378	0.379	0.07	0.05	0.04	0.03
5.0	16.67	14.40	13.30	12.67	0.935	1.006	1.047	1.072	0.21	0.17	0.15	0.13
10.0	35.7	27.5	23.54	21.32	1.423	1.622	1.752	1.844	0.28	0.24	0.21	0.19
20	89.1	60.9	47.8	40.65	1.937	2.345	2.65	2.87	0.32	0.28	0.25	0.22
50	337.7	193.5	131.7	100.5	2.64	3.47	4.20	4.81	0.368	0.322	0.277	0.236
100	994	496	294	200	3.175	4.40	5.72	6.93	0.395	0.348	0.300	0.243
200	3124	1364	690	400.6	3.54	5 36	7.54	9.90	0.416	0.373	0.317	0.246
500	15500	5744	2299	1002	4.00	6.57	10.39	15.72	0.440	0.405	0.345	0.250
1000	54800	18350	6119	2000	4.26	7.36	12.74	22.30	0.456	0.424	0.366	0.250
2000	2×10^5	62000	17550	4000	4.47	8.03	15.10	31.65	0.465	0.44	0.39	0.250
5000	1.155×10^6	3.31×10^5	7.84×10^4	10^4	4.65	8.70	17.85	50.00	0.475	0.46	0.42	0.250
10^4	4.42×10^6	1.22×10^6	2.6×10^5	2×10^4	4.76	9.05	19.62	70.7	0.48	0.47	0.44	0.250
10^5	4.15×10^8	1.067×10^8	1.88×10^7	2×10^5	4.905	9.68	23.07	224	0.49	0.49	0.48	0.250
∞	$\mathrm{sh}^2\,\theta/25$	$\mathrm{sh}^2\,\theta/100$	$\mathrm{sh}^2\,\theta/625$	$2\,\mathrm{sh}\,\theta$	5.00	10.00	25.00	$\sqrt{\mathrm{sh}\,\theta/2}$	0.50	0.50	0.50	0 25 $\sim$0.50

(*a*) *The bending strain*

As proved numerically by another example, say, by a shock problem when an initial velocity disturbance is given instead of a mass impact, the effect of the second wave group related to α_2 is negligibly small for the bending strain $\partial^2 y/\partial x^2 - \partial \overline{\Gamma}/\partial t$, so we can omit the term $\alpha^2 \tan \alpha_2 l'$ in the last expression of (35). On the contrary, the aperiodic group related to α_2' is considerably important. Thus, using the expression (33), we get

$$\left.\begin{aligned} e(0) &= (\text{bending strain at } x=0) = \frac{4V}{\pi b\kappa}\int_0^\infty \chi'(\theta)\sin qt'\,d\alpha_1 \\ \chi'(\theta) &= \Phi(\theta)\frac{2\,\mathrm{ch}\,\theta}{\mathrm{ch}\,\theta+1}\left(\frac{1}{k}+\overset{\alpha_2'}{0}\right)\sqrt{F(\theta,k)}\,, \\ F(\theta,k) &= \alpha_1^2(\mathrm{ch}\,\theta+1)^2\left\{2\frac{\mathrm{ch}\,\theta}{k}+\overset{\alpha_2'(\mathrm{ch}\,\theta-1)}{0}\right\}^2, \end{aligned}\right\} \quad \cdots\cdots (36)$$

in which only the case when $\xi = 0$ is considered for simplicity.

Now put

$$q = \frac{1}{\mathrm{sh}\,\theta} = \beta'^2, \qquad d\alpha = \frac{\mathrm{ch}^2\theta + 4\,\mathrm{ch}\,\theta + 1}{\mathrm{ch}\,\theta\sqrt{\mathrm{sh}\,\theta}\,(\mathrm{ch}\,\theta+2)}\,d\beta', \qquad \beta = \beta'\sqrt{t'}\,,$$

then (36) is further transformed to

$$\left.\begin{aligned} e(0) &= \frac{4V}{\pi b\kappa}\int_0^\infty \chi(\beta')\sin(\beta'^2 t')\,d\beta' = \frac{4V}{\pi b\kappa}\int_0^\infty \frac{1}{\sqrt{t'}}\chi\left(\frac{\beta}{\sqrt{t'}}\right)\sin\beta^2 d\beta\,, \\ \chi(\beta') &= 2\,\mathrm{ch}\,\theta\sqrt{\mathrm{sh}\,\theta\sqrt{\mathrm{ch}\,\theta+2}}\left(\frac{1}{k}+\overset{\alpha_2'}{0}\right)\Big/F(0,k)\,. \end{aligned}\right\}\cdots (37)$$

When k is finite, $\chi(\beta/\sqrt{t'})$ tends to $k/2$ at the limit of $t' \to \infty$, and (37) is then nothing but a Fresnel's integral;

$$e(0)_{t'\to\infty} = \frac{4V}{\pi b\kappa}\frac{k}{\sqrt{t'}}\frac{1}{4}\sqrt{\frac{\pi}{2}}\,, \qquad \cdots\cdots\cdots (38)$$

damping out proportionally to $1/\sqrt{t'}$. It also coincides with $e(x)$ at the limit of $t' \to \infty$ obtained in the previous article.

At the limit of $t' \to 0$,

$$e(0)_{t'\to 0} = \frac{4V}{\pi b\kappa}t'\int_0^\infty \chi(\beta')\beta'^2 d\beta'\,, \qquad \cdots\cdots\cdots (39)$$

the integral part in which converges. Thus, the bending strain at the beginning of the impact grows with the time.

In general, the integration (37) is mainly affected only by the larger value of θ, and the elements in the range of $\mathrm{ch}\,\theta < 2$ have almost no role in it.

The case when $k = \infty$ is identical with the case when a bar moving with uniform velocity collides with a fixed rigid point at $x = 0$. In this case, except for very large value of β', we can put $\chi(\beta') = 1/\beta'$ with no distinct error. Therefore, after the lapse of some short duration,

$$e(0)_{k\to\infty} = \frac{4V}{\pi b\kappa}\int_0^\infty \frac{\sin\beta^2}{\beta}d\beta = \frac{V}{b\kappa}\,, \qquad \cdots\cdots\cdots (40)$$

or the strain converges to a finite value without any damping. The calculation of

(37) in general can be performed more easily if we pay attention to the difference of it from (40).

The results of these calculations are shown in Tables 5 and 7 and Fig. 4. The maximum bending stress is proportional to $V/b\kappa$, and the effect of k upon it is not so remarkable; but the effect of k on the rate of damping is very evident, showing the larger damping for the smaller value of k.

As the bending strain at any finite value of x is always smaller than that at $x=0$, we see that the maximum bending stress is always produced at the point of impact as far as the bar is very long.

Table 5. The bending strain at $x=0$

t'	$e(0)/(4V/(\pi b\kappa))$			
	$k=10$	$k=20$	$k=50$	$k=\infty$
0	0	0	0	0
4		0.565		
16	0.58	0.61	0.64	0.67
100	0.24	0.416	0.62	
1000		0.148		
10000		0.063		0.785

(*b*) *The shearing strain and the contact pressure*

With the second expression of (35), the shearing strain at $x=0$ is expressed as follows,

$$\left.\begin{aligned}\overline{\Gamma}(0) &= \mu\frac{4V}{\pi b}\int_0^\infty \Psi'(\theta)\sin qt'd\alpha_1,\\ \Psi'(\theta) &= \Phi(\theta)\,\frac{2\,\mathrm{ch}\,\theta}{\mathrm{sh}^2\theta}\,\frac{1}{\sqrt{F(\theta,k)}}.\end{aligned}\right\}\qquad\cdots\cdots\cdots(41)$$

The value $\overline{\Gamma}(0)$ for a very small t' is mainly composed of the elements with very large α_1, and as

$$\Psi'(\theta)_{\alpha\to\infty}\to\frac{1}{2\sqrt{3}}\,\frac{\alpha}{\alpha^2+1/k^2},$$

the shearing strain at the limit $t'\to 0$, is reduced to

$$\overline{\Gamma}(0)_{t'\to 0}=\frac{V}{b\kappa}\,\frac{2\mu}{\sqrt{3}}\int^\infty\frac{\alpha}{\alpha^2+1/k^2}\sin\alpha\,\frac{t'}{\sqrt{3}}\,d\alpha=\frac{V}{b_S}\,e^{-(1/k)(t'/\sqrt{3})},\quad\cdots\cdot(42)$$

or quite the same solution as by the pure shear beam (Eq. (6)). The strain damps out as an exponential function of time.

The integration of (41) can be performed by a transformation of variables;

$$\overline{\Gamma}(0)=\frac{4V}{\pi b_S}\sqrt{\mu}\int_0^\infty\Psi(\beta')\sin(\beta'^2t')d\beta'=\frac{4V}{\pi b_S}\sqrt{\mu}\int_0^\infty\frac{1}{\sqrt{t'}}\Psi\left(\frac{\beta}{\sqrt{t'}}\right)\sin\beta^2d\beta,\quad(43)$$

$$\Psi(\beta')=\frac{2\,\mathrm{ch}\,\theta(\mathrm{ch}\,\theta+1)\sqrt{2+\mathrm{ch}\,\theta}}{\mathrm{sh}\,\theta\sqrt{\mathrm{sh}\,\theta}F(\theta,k)}.$$

The results of these calculations are shown in Tables 6 and 7 and in Fig. 5. The maximum strain occurs at the very moment of impact. The value of k does not affect this value, but the rate of damping incleases when k becomes smaller.

Table 6. The shearing strain $\overline{\Gamma}(0)/(V/b)$ at $x=0$

	$k=10$	$k=20$	$k=50$	$k=\infty$
$t'=0$	1.00	1.00	1.00	1.00
$t'=4$	0.65	0.75	0.83	0.84
$t'=16$	0.108	0.22	0.28	0.31

The contact pressure between the bar and the mass, expressed by $M(\partial^2 y/\partial t^2)_{x=0}$, is just twice the shearing force $G_e A\overline{\Gamma}(0)$ if ξ is assumed to be zero. When the viscosity exists, it can be calculated by

$$M\left(\frac{\partial^2 y}{\partial t^2}\right)_{x=0} = 2\,G_e A\left\{\overline{\Gamma}(0) + 2\,\xi\,\frac{\partial\overline{\Gamma}(0)}{\partial t}\right\},$$

if it is of the type proportional to the strain velocity. For the viscosity of the other type, the effect should be smaller than this, though there remains some physical question in the fundamental relations.

As the contact pressure by a lateral mass-impact for an infinitely long bar is aperiodic one, the collision occurs only one time with no rebound, and the duration of contact can not be clearly shown. This is a notable difference from the phenomena caused by a bar of finite length.

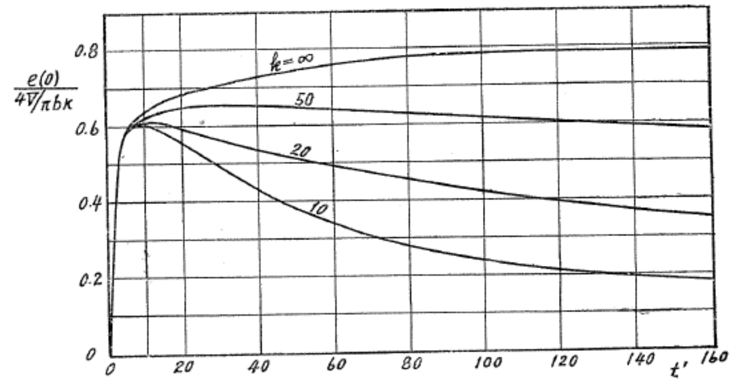

Fig. 4. Bending strain at $x=0$ $(\xi=0)$

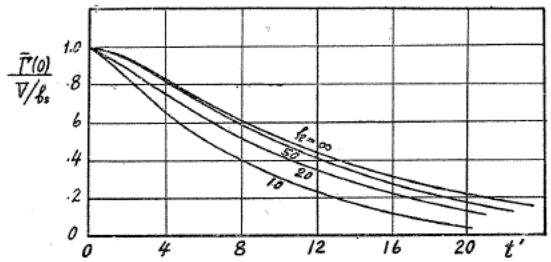

Fig. 5. Shearing strain at $x=0$ $(\xi=0)$

Table 7. β', $\chi(\beta')$ and $\Psi(\beta')$

sh θ	β'	$\chi(\beta')$				$\Psi(\beta')$			
		$k=10$	$k=20$	$k=50$	$k=\infty$	$k=10$	$k=20$	$k=50$	$k=\infty$
0	∞	0	0	0	0	0	0	0	0
0.1	3.162	0.0001	0	0	0	0.182	0.182	0.182	0.182
0.2	2.235	0.0005	0.0003	0.0001	0	0.260	0.260	0.260	0.260
0.5	1.414	0.0049	0.0025	0.0010	0	0.420	0.421	0.421	0.421
1.0	1.000	0.0262	0.0131	0.0052	0	0.628	0.630	0.630	0.630
2	0.707	0.382	0.335	0.303	0.282	0.902	0.925	0.934	0.940
5	0.447	1.649	1.698	1.701	1.690	0.888	1.028	1.112	1.169
10	0.316	2.360	2.663	2.833	2.92	0.683	0.886	1.037	1.145

20	0.223	2.943	3.61	4.08	4.37	0.495	0.724	0.922	1.083
50	0.1414	3.64	5.01	6.19	7.10	0.303	0.538	0.790	1.034
100	0.1000	4.00	6.01	8.09	9.90	0.205	0.411	0.694	1.018
200	0.0707	4.36	7.05	10.50	14.04	0.129	0.296	0.586	1.009
500	0.0447	4.66	8.25	14.09	22.3	0.065	0.175	0.436	1.002
1000	0.0316	4.80	8.90	16.87	31.6	0.036	0.109	0.327	1.000
2000	0.0223	4.89	9.33	19.26	44.7	0.020	0.065	0.228	1.000
5000	0.0141	4.93	9.68	21.89	70.7	0.009	0.030	0.127	1.000
10^4	0.0071	4.97	9.84	23.10	100	0.004	0.016	0.077	1.000
10^5	0.00316	4.98	9.97	24.66	316	0.0005	0.0016	0.0106	1.000
∞	$1/\text{sh}\,\theta$	5.00	10.00	25.00	$\sqrt{\text{sh}\,\theta}$ $\sim\sqrt{\text{sh}\,\theta}+k/2$	$50/\text{sh}\,\theta$	$200/\text{sh}\,\theta$	$1250/\text{sh}\theta$	1.000

7. Conclusions

Our theory of lateral impact developed above is based on the assumption that there are only two flexual freedoms neglecting the local deformation at the contact surface and also neglecting the higher order of strain components. As shown by the example 3 in Appendix I, our assumption is equivalent to the neglection of the components of the order of (beam depth/wave length)[4] and the higher. Therefore it might contain some errors. Our results that the maximum velocity of propagation of the waves of the first group occurs at $\text{sh}\,\theta = 1/1.955$ and that the critical value of the velocity at the limit of $\text{sh}\,\theta = 0$ is somewhat smaller than it, would be perhaps one of such errors. The result (6b), Appendix I, suggests us that, for the limiting wave of $\text{sh}\,\theta = 0$ (wave length $= 0$), it should be more preferable to use G instead of G_e, or $b_{g_1}b = \sqrt{G/\rho} = 0.632\,b$ instead of $\sqrt{G_e/\rho} = 0.577\,b$. Thus the top wave coincides with the critical wave.

Granted that such defects cited above were unavoidable, we have surely succeeded in solving the problem which can never be solved by the ordinary theory of bending. It seems unreasonable to discuss on the comparison with the investigation by Timoshenko reduced by introduction of Hertz's thoery, but if it is introduced to our theory, our solutions shall be more perfect.

Summaries:

(a) At the beginning of the impact, of course at the neighbourhood of the impact point, the motion starts with a character quite the same as by a pure shear beam.

(b) The disturbance propagates in both directions, but as the waves are of dispersive types, they propagate in a form of conspicuous wave groups with proper group velocities.

(c) The motion and strains at the impact point are aperiodic, while those at points apart from this point are periodic, beginning with the frequency of the top wave and damping out aperiodically. The total number of cycles grows larger as the distance from the impact point increases.

(d) The effects of the internal viscosity are not so clear, but, in any way, the motion of high frequency is damped out quickly, and only the waves of lower frequencies are transmitted causing the apparent top group velocity very much slower than the pure shear velocity.

This research on an infinitely long bar would be extended to the case of a finite length. But if we limit the question on the strength against shock waves produced by impacts, no considerations on the length will be needed because of the damping character, except when the impact be given near the terminals, in which case the reflected waves should be superposed. Experimental researches on bars with finite length show that the rebound occurs very soon, and that, in same cases, the collisions take place two or more times.

8. Experimental investigations

The apparatus and the way of Auther's experiments on the lateral impact for a long bar is illustrated schematically in Fig. 6, the detailed description of which shall be omitted for briefness.

Two samples of the experimental records obtained from the test (a) of Fig. 6, where the bar was supported near its one end and the impact was given at this end, are shown in Fig. 7 and 8. In Fig. 7, the displacement y magnified with mechanical and optical levers (at $x = 6$ m), and in Fig. 8, the surface strain extracted by a rochelle salt pick-up and electric amplifier (at $x = 17$ m) are shown. We see from these that, when we record the displacement y, the motion of high frequency does not come in evidence because of their small amplitudes, and the top velocity of propagation looks apparently very small compared with the theoretical value. On the contrary, the recording of strains shows very satisfactory results regarding to this point.

Figs. 9 to 13 show the strain oscillograms obtained from experiments (b), Fig. 6, also with rochelle salt pick-ups. The bar was suspended quite freely, and the impact was given at the middle point of it.

Generally speaking, good agreement with our theoretical conclusion is recognized, except that there are needed some discussions upon the features of the top wave. The top velocity of propagation recognized on the graphs seems to be about one fifth to $b = \sqrt{E/\rho}$, and the form of the graph shows better agreement with (B), Fig. 3 with the internal viscosity proportional to strain velocities, than with (A), Fig. 3 with that of the constant logarithmic decrement type. Author's opinion on this point is that, owing to the unsufficient sensibility of our measuring apparatus — finite length of the pick-up used (about 10 mm or twice the rod diameter) and the decomposing ability of the oscillograph vibrators (natural frequency about 6000 cycles per second *), — the portion of higher frequencies might have died out. In fact, when another set of vibrators with lower natural frequencies (about 1000 cycles per second *) was used, much smoother figures as shown in Fig. 14 were obtained.

* These values were estimated from the maker's catalogues.

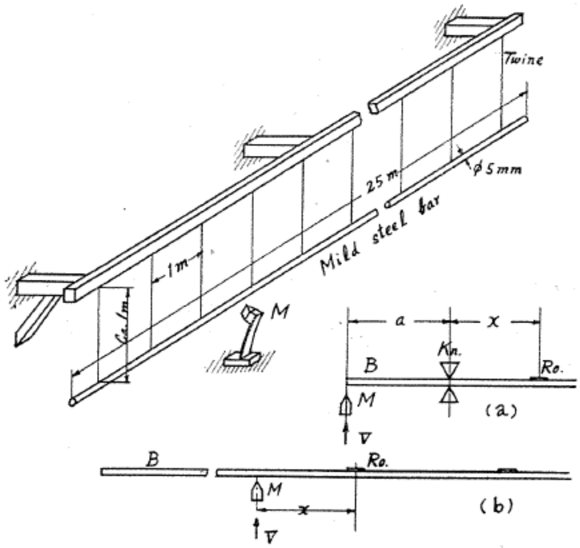

Fig. 6. Schematic illustration of the experiment
R_0: Rochelle salt strain pick-ups.

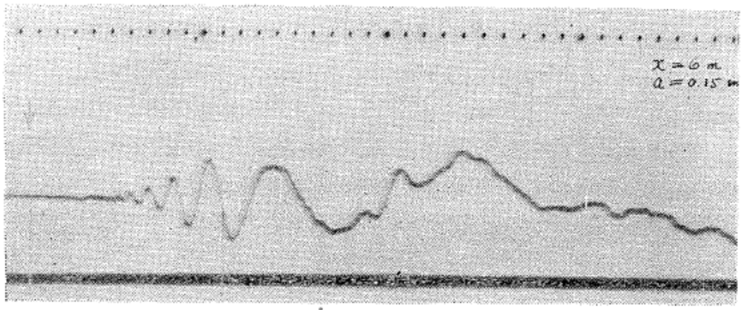

Fig. 7

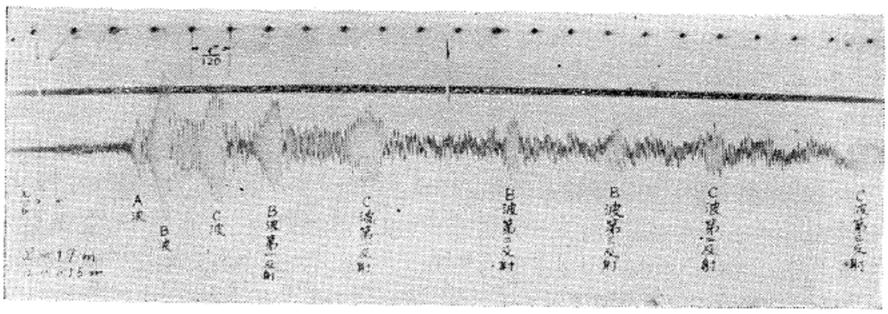

Fig. 8

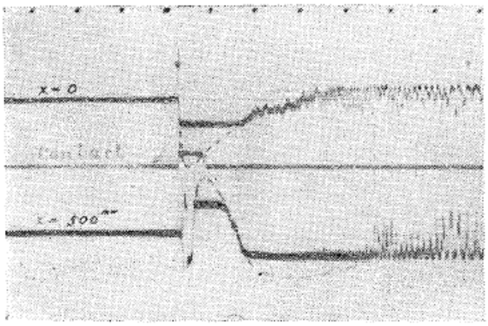

Fig. 9

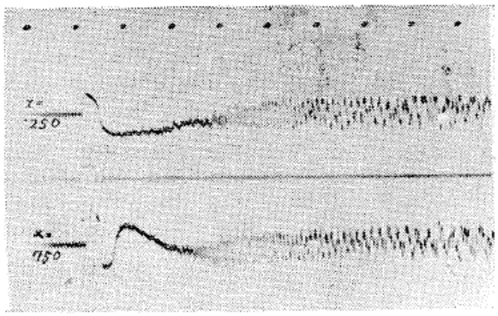

Fig. 10

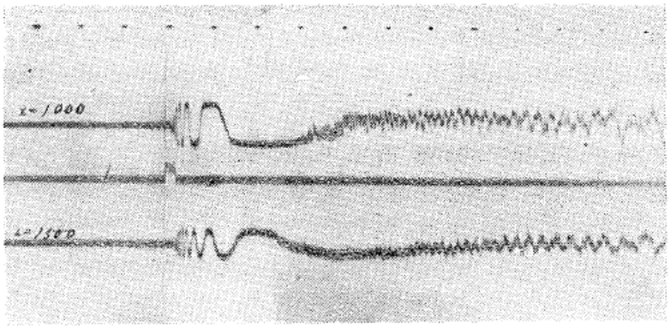

Fig. 11

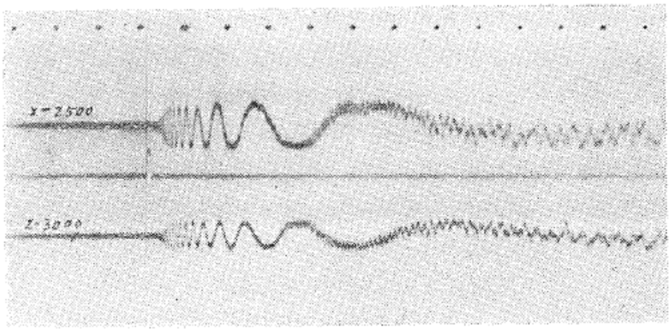

Fig. 12

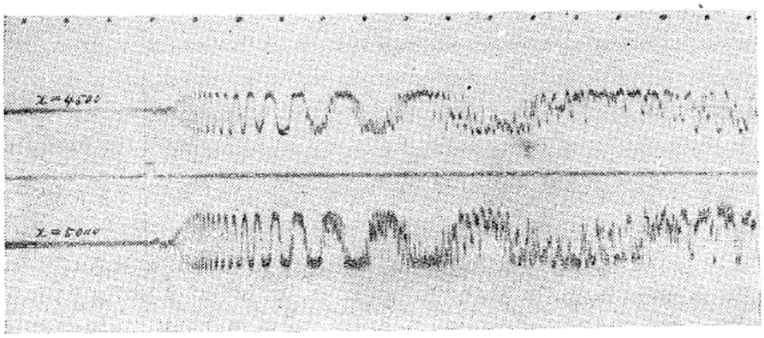

Fig. 13

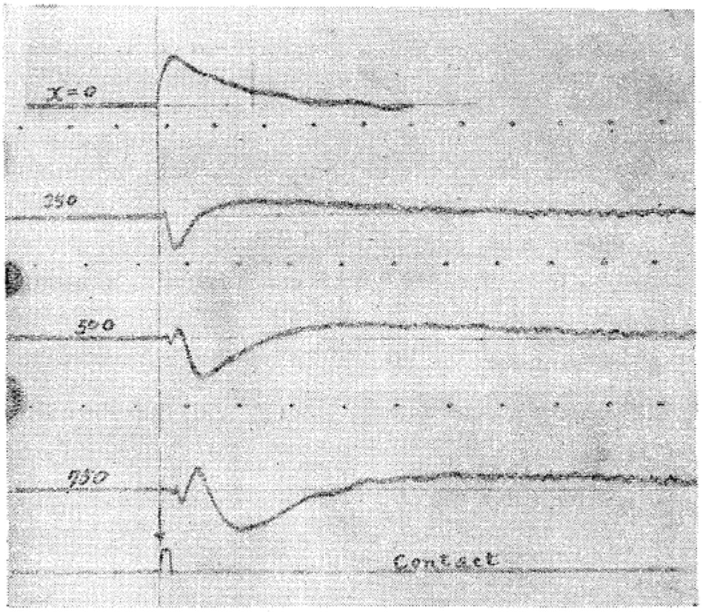

Fig. 14

Appendix I. On the conventional equation of bending for bars and plates

It is very troublesome to obtain the exact solutions for the problems regarding to the deflection of bars, and rather impossible for bars with arbitrary cross sections. But there are some occasions where it is desirable to get more correct solutions than those obtained from the ordinary theory of bending. And this is the reason why the conventional equation of bending should be proposed. The expression

$$\frac{\partial^2 v}{\partial x^2} = -\frac{M}{EI} + \frac{1}{G_e A}\frac{dS}{dx}, \qquad \text{or} \qquad \frac{d^2 v}{dx^2} - \frac{d\overline{\Gamma}}{dx} = -\frac{M}{EI} \qquad \cdots\cdots\cdots (1)$$

is not only very practical form from engineer's point of view, but also theoretically reasonable when the effective shear rigidity G_eA is estimated reasonably. This fact shall be described below.

The equation of this form has already been used by S. Timoshenko [16], etc. His estimation on G_e, however, seems too large, say $(2/3)G$ for rectangular cross sections, and he himself recognizes an error of 20% in the 4th example below.

To determine G_e, it should be reasonable to consider the matter according to the energy theory, assuming the stress distribution to remain the same as by the ordinary beam theorem. As the bending moment M and the shearing force S are two sectional forces independent with each other, M must not do any work by the displacement due to S. Thus,

$$\int_{y_{\min}}^{y_{\max}} (\text{axial displacement} - \beta y) \times \sigma_x \cdot t dy = 0\,, \qquad \cdots\cdots\cdots (2)$$

where y denotes the perpendicular coordinate, t the thickness, and $\beta = \partial v/\partial x - \overline{\Gamma}$ the effective rotational angle or the bending deformation of the section.

After the operation (2), we obtain

$G_e = (5/6)G$ for a rectangular bar or a plate,

$G_e \doteqdot (11/13)G$ for a circular bar,

and so on, and such operation can be applied to any sectional form.

For plates, the conventional equation of bending similar to (1) is obtained as

$$\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \overline{\Gamma}_x}{\partial x}\right) + \nu\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial \overline{\Gamma}_y}{\partial y}\right) = -\frac{M_x}{D}, \text{ etc.,} \quad \cdots (3)$$

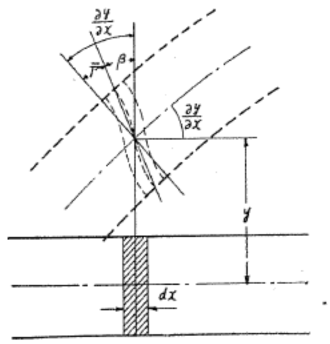

Fig. 15

or, expressed with the equivalent lateral load q_e and the shearing deflection w_s, $(\overline{\Gamma}_x = \partial w_s/\partial x\,,\ \overline{\Gamma} = \partial w_s/\partial y)$,

$$\nabla^4(w - w_s) = q_e/D\,; \qquad \nabla^2 w_s = -\,q_e/G_e\,2\,h\,. \qquad \cdots\cdots\cdots (3a)$$

Example 1:— A bar with constant shearing force.

A cantilever with an end load, or any beam with concentrated loads and reactions, is included in this category. In practice, the condition of constant shearing force does not hold at the loaded points or at the supported points, but if the small disturbance near these sections are neglected, the theoretical solution considered as a two-dimensional problem is expressed by

$$v = \bar{v} - \frac{6}{5}\frac{S}{GA}\left(1 - \frac{\nu}{4}\frac{G}{E}\right)(l - x) \qquad \cdots\cdots\cdots (4)$$

for a cantilever (loaded point $x = 0$, fixed end $x = l$), while the solution due to the equation (1) gives

$$v = \bar{v} - \frac{6}{5}\frac{S}{GA}(l - x)\,. \qquad \cdots\cdots\cdots (4a)$$

In these, $\bar{v}$ denotes the deflection according to the ordinary beam theory. The difference between these solutions is only 3% to the correcting term itself.

Example 2:— Both ends supported beam with uniformly distributed load $q = kA$, k being a constant body force.

The theoretical solution, also neglecting the small ends disturbances, is (length $2\,l$, origin at the center)

$$v = \bar{v} - \frac{q}{EA}\left(\frac{6}{5} + \frac{5}{4}\nu\right)\left(\frac{l^2}{2} - x^2\right), \qquad \cdots\cdots\cdots (5)$$

while the solution due to (1) is

$$v = \bar{v} - \frac{q}{EA}\left(\frac{6}{5} + \frac{6}{5}\nu\right)\left(\frac{l^2}{2} - x^2\right), \qquad \cdots\cdots\cdots (5a)$$

in which the error is only 1.5%. The coefficient related to ν in expression (5) is slightly changed if the load q is caused by surface pressures.

Example 3:— Bars with a lateral load $q = -\,k\cos(\pi/2\,l)x$.

A common equation holds for an infinitely long bar, a bar both ends supported with a length $2l$ and a bar both ends fixed with a length $4l$. Put $\alpha = \pi/2l$, $\alpha' = \sqrt{2+\nu}\cdot\alpha$ and $2h$ = depth of the bar. The vertical displacements obtained from the theory of elasticity are as follows.

(i) If the load is consisted of the surface pressures evenly distributed on both top and bottom surfaces;

$$v = -\frac{k\cos\alpha x}{2E\alpha}\left\{\frac{2\,\mathrm{ch}\,\alpha h\cdot\mathrm{ch}\,\alpha y + (1+\nu)(\alpha h\cdot\mathrm{sh}\,\alpha h\cdot\mathrm{ch}\,\alpha y - \mathrm{ch}\,\alpha h\cdot\alpha y\cdot\mathrm{sh}\,\alpha y)}{\mathrm{ch}\,\alpha h\cdot\mathrm{sh}\,\alpha h - \alpha h}\right\}. \quad \cdots\cdots\cdots (6)$$

(ii) If the load is consisted of a body force uniformly distributed in y-direction;

$$v = -\frac{k\cos\alpha x}{2E\alpha}\left\{\frac{(1+\nu)(\alpha h\cdot\mathrm{ch}\,\alpha h\cdot\mathrm{ch}\,\alpha y - \mathrm{sh}\,\alpha h\cdot\alpha y\cdot\mathrm{sh}\,\alpha y) + (1-\nu)\mathrm{sh}\,\alpha h\cdot\mathrm{ch}\,\alpha y}{\alpha h(\mathrm{ch}\,\alpha h\cdot\mathrm{sh}\,\alpha h - \alpha h)} - 2(1+\nu)/\alpha h\right\}. \quad \cdots\cdots\cdots (6')$$

(iii) If the load is consisted of a body force distributed proportionally to $\partial\tau/\partial x$ there;

$$v = -\frac{k\cos\alpha x}{2E\alpha}\sqrt{2+\nu}\left\{\frac{(2+\nu)\mathrm{ch}\,\alpha' h + \nu(\mathrm{ch}\,\alpha' h - \mathrm{ch}\,\alpha' y)}{\mathrm{sh}\,\alpha' h - \alpha' h\cdot\mathrm{ch}\,\alpha' h}\right\}. \quad \cdots\cdots (6'')$$

When αh is small, all these equations can uniformly be reduced to the approximate form

$$v = -\frac{k}{EI}\frac{\cos\alpha x}{\alpha^4}\left\{1 + \frac{4}{5}(\alpha h)^2 + \varepsilon\nu(\alpha h)^2\right\}, \quad \cdots\cdots\cdots (6a)$$

in which the coefficient ε changes its value according to the vertical distribution of the load and to the ordinate y. If we put $\varepsilon = 4/5$ in (6a), it coincides with the solution obtained from (1). The difference $4/5 - \varepsilon$ are always very small. The approximation (6a) fails when αh becomes large, but, even in such cases, (6′) and (6″) tend to the expression

$$v(y=0)_{\alpha h\to\infty} \longrightarrow -\frac{k}{GA}\frac{\cos\alpha x}{\alpha^2}, \quad \cdots\cdots\cdots (6b)$$

and this coincides with the result obtained from (1), if we substitute G instead of G_e.

(iv) *Example 4*:— Circular disk supported simply at its circumference with uniform surface pressure.

The mathematical solution due to H. Love [17] is

$$w = -\frac{q}{D}(R^2 - r^2)\left\{\frac{1}{64}\left(\frac{5+\nu}{1+\nu}R^2 - r^2\right) + \frac{1}{40}\frac{8+\nu+\nu^2}{1-\nu^2}h^2\right\}, \quad \cdots\cdots (7)$$

while the solution due to our conventional equation (3a) is

$$w = -\frac{q}{D}(R^2 - r^2)\left\{\frac{1}{64}\left(\frac{5+\nu}{1+\nu}R^2 - r^2\right) + \frac{1}{5}\frac{1+\nu}{1-\nu^2}h^2\right\}. \quad \cdots\cdots (7a)$$

The error is also very small and negligible if we, after Timoshenko, add the negative correction $-\nu h^2/6(1-\nu^2)$ in the brackets owing to the surface loading.

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