ON THE NUMBER OF GENERA OF BOOLEAN FUNCTIONS OF n VARIABLES

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§ 1. Introduction

Expressions obtained from n variables, say, $x_1, x_2, \ldots, x_n$ by means of the Boolean operations, *i.e.*, additions, multiplications and complementations are called Boolean functions of n variables. As is well known, any Boolean function f can be expanded uniquely in the form:

$$f = \sum_{i=0}^{2^n-1} f_i p_i \tag{1.1}$$

where f_i take the values 0 and 1, and p_i are fundamental products, *i.e.*, products of the n variables complemented or not. The number of fundamental products p_i such that $f_i=1$ is called the dimension of the function f.

The operations of permuting and (or) complementing variables $x_1, x_2, \ldots, x_n$ are defined symmetric transformations. There are $2^n n!$ symmetric transformations in all and they form a group $\mathbf{O}_n$ called hyper-octahedral group. Under a symmetric transformation, a permutation of fundamental products and consequently a permutation of Boolean functions is induced. In connection with this fact, two classes of Boolean functions are defined as follows: "Two functions are of the same type if and only if they can be transformed from one to another by a symmetric transformation". "Two functions are of the same genus if and only if they can be transformed from one to another by a symmetric transformation and (or) a complementation".

There is a close relationship between the classes, type and genus. Roughly speaking, a genus consits of two complementary types, *i.e.*, two types which can be turned from one into another by complementing all their members. But this is not always true. Sometimes it happens that a genus is at the same time a type. Such a genus (type) and also functions belonging to it are defined as self-complementary. Clearly any self-complementary function is of the dimension 2^{n-1}.

Concerning the enumeration of types of Boolean functions, there are several contributions among which Slepian's[1] is the most important. He devised an ingenious method for this problem and computed the number of types of Boolean functions of n variables up to $n=6$.

The purpose of this paper is to compute the number of genera of Boolean functions of n variables by a method similar to that of Slepian. In § 2, the principle of the method is described and, in § 3, are shown the results obtained for moderate values of n.

§ 2. Principle of the Method

Let the numbers of types, genera and self-complementary genera (types) be denoted by T_n, G_n and S_n respectively. Then, the relation:

$$T_n = 2\,(G_n - S_n) + S_n,$$

i.e.,

$$G_n = \frac{1}{2}\,(T_n + S_n) \tag{2.1}$$

holds among these numbers.

Now, the combined operations of complementations and (or) symmetric transformations form a group $\mathbf{C}\times\mathbf{O}_n$, where $\mathbf{C}$ is the two element group consisting of the identity operation and a complementation γ. Elements of $\mathbf{C}\times\mathbf{O}_n$ induce permutations of Boolean functions, and matrices specifying these permutations constitute a representation of the group $\mathbf{C}\times\mathbf{O}_n$. It will be readily observed that this representation is reducible and contains as many identity representations as the genera of Boolean functions. Thus, from the elementary theory of group representation, we obtain

$$G_n = \frac{1}{2^{n+1}n!}\sum_{c} n_c(\chi_c + \chi_{\gamma c}), \tag{2.2}$$

where C denotes conjugate classes of $\mathbf{O}_n$, n_c is the number of elements of C and χ_c is the character of C in the representation. Here, as was shown by Slepian, T_n(N_n in the Slepian's paper) is given by

$$T_n = \frac{1}{2^n n!}\sum_{c} n_c\chi_c. \tag{2.3}$$

Hence we obtain

$$S_n = \frac{1}{2^n n!}\sum_{c} n_c\chi_{\gamma c} \tag{2.4}$$

from (2.1), (2.2) and (2.3). Since T_n was already given by Slepian, we may only compute S_n by means of (2.4) in order to obtain G_n.

It is noted here that $\chi_{\gamma c}$ can be interpreted as the number of Boolean functions of n variables which are invariant under the combined operation of a complementation and an arbitrary element of the class C. Now, suppose that every function f is expressed in the form $f=\sum_{i=0}^{2^n-1} f_i p_i$, and fundamental products p_i are permuted with a cycle structure:

$$(i_1^{(1)}, i_2^{(1)}, \ldots, i_{\lambda_1}^{(1)})\,(i_1^{(2)}, i_2^{(2)}, \ldots, i_{\lambda_1}^{(2)})\ldots$$
$$(i_1^{(j)}, i_2^{(j)}, \ldots, i_{\lambda_j}^{(j)})\ldots(i_1^{(k)}, i_2^{(k)}, \ldots, i_{\lambda_k}^{(k)})$$

where λ_j, $j=1, 2, \ldots, k$ is the length of the j-th cycle and k is the number of cycles. Then the necessary and sufficient condition so that a function f may be invariant under the above mentioned operation is:

$$f_{i_1^{(j)}}=0,\quad f_{i_1^{(j)}}=1,\quad f_{i_1^{(j)}}=0,\ \ldots,\ \ldots,\ f_{i_{\lambda_j}^{(j)}}=1$$

or

$$f_{i_1^{(j)}}=1,\quad f_{i_1^{(j)}}=0,\quad f_{i_1^{(j)}}=1,\ \ldots,\ \ldots,\ f_{i_{\lambda_j}^{(j)}}=0$$

for every j. Since these two conditions imply that λ_j should be even, we see $\chi_{\gamma c}=0$ if λ_j are odd for some j, and $\chi_{\gamma c}=2^k$ if λ_j are even for all j. Further, it is seen that $\chi_{\gamma c}$ are identical for all classes c which induce permutations of fundamental products with the same cycle structure. Therefore, (2.4) can be rewritten as

$$S_n=\frac{1}{2^n n!}\sum_{\nu}2^{k_\nu}h_\nu, \tag{2.5}$$

where the summation should be taken over all cycle structures ν consisting exclusively of cycles of even lengths, k_ν is the number of cycles of the structure ν and h_ν is the number of elements of $\mathbf{O}_n$ inducing permutations of fundamental products with the structure ν.

A formula for n_c and a method for determining the cycle structures of permutations of fundamental products induced by classes of $\mathbf{O}_n$ were given by Slepian. By the help of the formula and the method, the results shown in Table 1 are obtained. In this table, the columns ν_i contain the numbers of cycles of the length i of the cycle structures identified by the numbers in the lefthand column. Now it is a simple matter to evaluate the expression (2.5) for $n=1$, 2, 3, 4, 5, 6 by the help of Table 1.

TABLE 1. Cycle Structures of Permutations of Fundamental Products induced by Elements of $\mathbf{O}_n$.

$n=1$, $2^n=2$, $2^n n!=2$.

ν	ν_1	ν_2	k_ν	h_ν
1	2	0	2	1
2	0	1	1	1

$n=2$, $2^n=4$, $2^n n!=8$.

ν	ν_1	ν_2	ν_4	k_ν	h_ν
1	4	0	0	4	1
2	2	1	0	3	2
3	0	2	0	2	3
4	0	0	1	1	2

$n=3$, $2^n=8$, $2^n n!=48$.

ν	ν_1	ν_2	ν_3	ν_4	ν_6	k_ν	h_ν
1	8	0	0	0	0	8	1
2	4	2	0	0	0	6	6
3	2	0	2	0	0	4	8
4	0	4	0	0	0	4	13
5	0	1	0	0	1	2	8
6	0	0	0	2	0	2	12

$n=4$, $2^n=16$, $2^n n!=384$.

ν	ν_1	ν_2	ν_3	ν_4	ν_6	ν_8	k_ν	h_ν
1	16	0	0	0	0	0	16	1
2	8	4	0	0	0	0	12	12
3	4	6	0	0	0	0	10	12
4	4	0	4	0	0	0	8	32
5	0	8	0	0	0	0	8	51
6	2	1	0	3	0	0	6	48
7	0	2	0	0	2	0	4	96
8	0	0	0	4	0	0	4	84
9	0	0	0	0	0	2	2	48

$n=5$, $2^n=32$, $2^n n!=3{,}840$.

ν	ν_1	ν_2	ν_3	ν_4	ν_5	ν_6	ν_8	ν_{10}	ν_{12}	k_ν	h_ν
1	32	0	0	0	0	0	0	0	0	32	1
2	16	8	0	0	0	0	0	0	0	24	20
3	8	12	0	0	0	0	0	0	0	20	60
4	8	0	8	0	0	0	0	0	0	16	80
5	0	16	0	0	0	0	0	0	0	16	231
6	4	2	4	0	0	2	0	0	0	12	160
7	4	2	0	6	0	0	0	0	0	12	240
8	0	4	0	6	0	0	0	0	0	10	240
9	2	0	0	0	6	0	0	0	0	8	384
10	0	4	0	0	0	4	0	0	0	8	720
11	0	0	0	8	0	0	0	0	0	8	520
12	0	1	0	0	0	0	0	3	0	4	384
13	0	0	0	2	0	0	0	0	2	4	320
14	0	0	0	0	0	0	4	0	0	4	480

TABLE 1. (*Continued*)

$n=6,\ 2^n=64,\ 2^n n!=46{,}080.$

ν	ν_1	ν_2	ν_3	ν_4	ν_5	ν_6	ν_8	ν_{10}	ν_{12}	k_ν	h_ν
1	64	0	0	0	0	0	0	0	0	64	1
2	32	16	0	0	0	0	0	0	0	48	30
3	16	24	0	0	0	0	0	0	0	40	180
4	8	28	0	0	0	0	0	0	0	36	120
5	16	0	16	0	0	0	0	0	0	32	160
6	0	32	0	0	0	0	0	0	0	32	1,053
7	8	4	8	0	0	4	0	0	0	24	960
8	8	4	0	12	0	0	0	0	0	24	720
9	4	0	20	0	0	0	0	0	0	24	640
10	4	6	0	12	0	0	0	0	0	22	1,440
11	0	8	0	12	0	0	0	0	0	20	2,160
12	4	0	0	0	12	0	0	0	0	16	2,304
13	0	8	0	0	0	8	0	0	0	16	5,280
14	0	0	0	16	0	0	0	0	0	16	4,920
15	2	1	2	0	0	9	0	0	0	14	3,840
16	0	2	0	0	0	10	0	0	0	12	1,920
17	0	2	0	0	0	0	0	6	0	8	6,912
18	0	0	0	4	0	0	0	0	4	8	3,840
19	0	0	0	0	0	0	8	0	0	8	5,760
20	0	0	0	1	0	0	0	0	5	6	3,840

§ 3. Conclusion

The number of genera of Boolean functions on n variables was obtained for the first six values of n. The following table contains the values of G_n and S_n obtained in this paper together with the values of T_n obtained by Slepian and those of $T_n^{(2^{n-1})}$ obtained by Pólya,[2] where $T_n^{(S)}$ is the number of types of n-variable Boolean functions of the dimension S. Comparing the values of S_n and $T_n^{(2^{n-1})}$, we can see that any function of the dimension 2^{n-1} is self-complementary for $n \leqq 3$, the same is not the case with $n \geqq 4$. For example, out of 74 types of four-variable Boolean functions of the dimension 8, 42 types are self-complentary but other 32 types are not and therefore the latter can be grouped, two by two, into 16 genera. It seems that this fact is overlooked in the table of Boolean functions of four variables given in the Appendix of the book "Synthesis of Electronic Computing and Control Circuits" by the Staff of the Computation Laboratory of Harvard University.

TABLE 2. Values of T_n, G_n, S_n and $T_n^{(2^{n-1})}$

n	1	2	3	4	5	6
T_n	3	6	22	402	1,228,158	400,507,806,843,728
G_n	2	4	14	222	616,126	200,253,952,527,184
S_n	1	2	6	42	4,094	98,210,640
$T_n^{(2^{n-1})}$	1	2	6	74	*	*

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