THE BOUNDARY LAYER OF AN INCOMPRESSIBLE FLUID WITH PERIODIC MOTION IN MAGNETOHYDRODYNAMICS

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Introduction

It has been shown in a periodic boundary layer flow that a potential flow which is periodic with respect to time induces a steady, secondary motion at a large distance from a body as a result of viscous effect.[1] If the fluid becomes to be electrically conducting, or to be magnetic, due to the interaction between the magnetic field and the flow field in the presence of the magnetic field, the motions of this fluid show various phenomena which are different from the non-magnetic fluid: the increase of the rigidity,[2] the tendency of stabilization,[3] and the reduction of the viscous effect,[4][5] etc.

This new field is called "magnetohydrodynamics" and has been considered to be important in connexion with the problems of nuclear fusion and of the aerospace science. Here as an example for the reduction of the viscous effect in magnetohydrodynamics we consider a periodic boundary layer flow of an electrically conducting fluid in the presence of the magnetic field.

The equations

Let us consider the unsteady plane boundary layer flow of an electrically conducting, viscous, incompressible fluid over the solid surface $y=0$, parallel to the x axis, in the uniform magnetic field B_0 applied normal to it. The oscillating stream with respect to time t is given by

$$u_\infty(x, t) = u_0(x)\, e^{i\omega t}, \tag{1}$$

where $u_0(x)$ is the amplitude of the oscillation, and ω is the frequency of the oscillation. With the assumption that the induced fields is negligible compared with the applied fields, the equations of motion are derived (in Appendix)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial u_\infty}{\partial t} + u_\infty \frac{\partial u_\infty}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} + \frac{\sigma B_0^2}{\rho}(u_\infty - u), \tag{3}$$

with the boundary conditions

$$u = v = 0 \text{ at } y=0, \text{ and } u = u_\infty \text{ at } y = \infty, \tag{4}$$

where u and v are velocity components, ρ the density, ν the kinematic viscosity, σ the electric conductivity.

Introducing the stream function defined in the form

$$\psi(x, y, t) = \sqrt{\frac{\nu}{\omega}} u_0 e^{i\omega t} F(x, \eta, t), \tag{5}$$

where

$$\eta = y\sqrt{\frac{\omega}{\nu}},$$

and hence

$$u(x, \eta, t) = u_0 e^{i\omega t} \frac{\partial F}{\partial \eta}, \tag{6}$$

$$v(x, \eta, t) = -\sqrt{\frac{\nu}{\omega}}\left(\frac{du_0}{dx} e^{i\omega t} F + u_0 e^{i\omega t} \frac{\partial F}{\partial x}\right), \tag{7}$$

we obtain from the equation (3),

$$\left(i + \frac{m}{\omega}\right)\frac{\partial F}{\partial \eta} + \frac{1}{\omega}\frac{\partial^2 F}{\partial t \partial \eta} - \frac{\partial^3 F}{\partial \eta^3} - \left(i + \frac{m}{\omega}\right) - \frac{1}{\omega}\frac{du_0}{dx} e^{i\omega t}\left\{1 - \left(\frac{\partial F}{\partial \eta}\right)^2 + F\frac{\partial^2 F}{\partial \eta^2}\right\}$$
$$+ \frac{1}{\omega} u_0 e^{i\omega t}\left(\frac{\partial F}{\partial \eta}\frac{\partial^2 F}{\partial x \partial \eta} - \frac{\partial^2 F}{\partial \eta^2}\frac{\partial F}{\partial x}\right) = 0, \tag{8}$$

with the boundary conditions

$$F = \frac{\partial F}{\partial \eta} = 0 \text{ at } \eta = 0, \text{ and } \frac{\partial F}{\partial \eta} = 1 \text{ at } \eta = \infty, \tag{9}$$

where

$$m = \frac{\sigma B_0^2}{\rho}.$$

We take for F the series

$$F(x, \eta, t) = F^0(\eta) + \frac{1}{\omega}\frac{du_0}{dx}\{F_1^1(\eta) e^{i\omega t} + F_2^1(\eta) e^{-i\omega t}\} + \cdots, \tag{10}$$

and then the functions F^0, F_1^1, F_2^1, . . . satisfy the ordinary differential equations

$$\left(i + \frac{m}{\omega}\right)\frac{dF^0}{d\eta} - \frac{d^3 F^0}{d\eta_3} - \left(i + \frac{m}{\omega}\right) = 0, \tag{11}$$

$$\left(2i + \frac{m}{\omega}\right)\frac{dF_1^1}{d\eta} - \frac{d^3 F_1^1}{d\eta^3} - \frac{1}{2}\left\{1 - \left(\frac{dF^0}{d\eta}\right)^2 + F^0\frac{d^2 F^0}{d\eta^2}\right\} = 0, \tag{12}$$

$$\frac{m}{\omega}\frac{dF_2^1}{d\eta} - \frac{d^3 F_2^1}{d\eta^3} - \frac{1}{2}\left\{1 - \frac{dF^0}{d\eta}\frac{d\tilde{F}^0}{d\eta} + \frac{1}{2}\left(F^0\frac{d^2\tilde{F}^0}{d\eta^2} + \tilde{F}^0\frac{d^2 F^0}{d\eta^2}\right)\right\} = 0, \tag{13}$$

etc.,

the boundary conditions

$$F^0 = F_1^1 = F_2^1 = \cdots = \frac{dF^0}{d\eta} = \frac{dF_1^1}{d\eta} = \frac{dF_2^1}{d\eta} = \cdots = 0 \ at \ \eta = 0, \tag{14}$$

and

$$\frac{dF^0}{d\eta}=1,\ \frac{dF_1^1}{d\eta}=\cdots=0,\ \frac{dF_2^1}{d\eta}=\text{finite at } \eta=\infty,$$

where the symbol $\sim$ denotes a complex conjugate.

The solutions

These equations can be solved easily. The solution as the first approximation to the non-dimensional function $\frac{\partial F}{\partial \eta}$ is obtained from the equation (11) in the form

$$\frac{dF^0}{d\eta}=1-e^{-\frac{\eta}{\sqrt{2}}\left\{\sqrt{\sqrt{\left(\frac{m}{\omega}\right)^2+1}+\frac{m}{\omega}}+i\sqrt{\sqrt{\left(\frac{m}{\omega}\right)^2+1}-\frac{m}{\omega}}\right\}}. \tag{15}$$

Putting

$$\alpha=\frac{1}{\sqrt{2}}\sqrt{\sqrt{\left(\frac{m}{\omega}\right)^2+1}+\frac{m}{\omega}},\ \text{and}\ \beta=\frac{1}{\sqrt{2}}\sqrt{\sqrt{\left(\frac{m}{\omega}\right)^2+1}-\frac{m}{\omega}},$$

and from the solution (15) and the boundary condition, we obtain

$$F^0=\eta+\frac{1}{\alpha+\beta i}\{e^{-(\alpha+\beta i)\eta}-1\}, \tag{16}$$

$$\frac{d^2F^0}{d\eta^2}=(\alpha+\beta i)\,e^{-(\alpha+\beta i)\eta}. \tag{17}$$

Then the equations (12) and (13) become

$$\left(2i+\frac{m}{\omega}\right)\frac{dF_1^1}{d\eta}-\frac{d^3F_1^1}{d\eta^3}=\frac{1}{2}\{1+(\alpha+\beta i)\eta\}\,e^{-(\alpha+\beta i)\eta}, \tag{18}$$

$$\begin{aligned}\frac{d^3F_2^1}{d\eta^3}-\frac{m}{\omega}\frac{dF_2^1}{d\eta}&=-\frac{1}{2}e^{-(\alpha+\beta i)\eta}-\frac{1}{2}e^{-(\alpha-\beta i)\eta}+\frac{1}{2}e^{-2\alpha\eta}\\&-\frac{1}{4}(\alpha-\beta i)\,\eta e^{-(\alpha-\beta i)\eta}-\frac{1}{4}(\alpha+\beta i)\,\eta e^{-(\alpha+\beta i)\eta}\\&-\frac{1}{4}\frac{\alpha-\beta i}{\alpha+\beta i}\{e^{-2\alpha\eta}-e^{-(\alpha-\beta i)\eta}\}-\frac{1}{4}\frac{\alpha+\beta i}{\alpha-\beta i}\{e^{-2\alpha\eta}-e^{-(\alpha+\beta i)\eta}\}.\end{aligned} \tag{19}$$

Solving these equations, as the second approximation to $\frac{\partial F}{\partial \eta}$, we have

$$\frac{dF_1^1}{d\eta}=\left(\frac{m}{\omega}+\frac{i}{2}\right)\{e^{-(\alpha+\beta i)\eta}-e^{-\eta\sqrt{\frac{m}{\omega}+2i}}\}-\frac{i}{2}(\alpha+\beta i)\,\eta e^{-(\alpha+\beta i)\eta}, \tag{20}$$

$$\frac{dF_2^1}{d\eta}=2e^{-\alpha\eta}\sin\beta\eta-\frac{1}{2}\frac{\frac{m}{\omega}}{\sqrt{\left(\frac{m}{\omega}\right)^2+1}}e^{-\alpha\eta}\sin\beta\eta+2\frac{m}{\omega}e^{-\alpha\eta}\cos\beta\eta$$

$$+\frac{1}{2}\frac{1}{\sqrt{\left(\frac{m}{\omega}\right)^2+1}}e^{-\alpha\eta}\cos\beta\eta+\frac{1}{2}\frac{1}{2\sqrt{\left(\frac{m}{\omega}\right)^2+1+\frac{m}{\omega}}}\frac{\sqrt{\left(\frac{m}{\omega}\right)^2+1-\frac{m}{\omega}}}{\sqrt{\left(\frac{m}{\omega}\right)^2+1}}e^{-2\alpha\eta}$$

$$-\frac{1}{2}\eta e^{-\alpha\eta}(\beta\cos\beta\eta-\alpha\sin\beta\eta)+ce^{-\eta\sqrt{\frac{m}{\omega}}}, \tag{21}$$

where

$$c=-\left\{2\frac{m}{\omega}+\frac{1}{2\sqrt{\left(\frac{m}{\omega}\right)^2+1}}+\frac{1}{2}\frac{1}{2\sqrt{\left(\frac{m}{\omega}\right)^2+1+\frac{m}{\omega}}}\frac{\sqrt{\left(\frac{m}{\omega}\right)^2+1-\frac{m}{\omega}}}{\sqrt{\left(\frac{m}{\omega}\right)^2+1}}\right\}.$$

The result given by Schlichting[1)] is obtained from these solutions by carrying out the limiting process $m\to 0$. But as $\eta\to\infty$, we find the solution (21) vanish. It seems to show that due to the effect of the magnetic field, the viscous effect is reduced and that the periodic free stream does not induce a secondary flow at a large distance from the body.

Appendix

The fundamental equations in magnetohydrodynamics neglecting the displacement current and excess charges are as follows:

the equation of continuity

$$\frac{\partial\rho}{\partial t}+\mathrm{div}\,(\rho\mathbf{V})=0, \tag{A1}$$

the modified Navier-Stokes' equation

$$\frac{\partial\mathbf{V}}{\partial t}+\mathbf{V}\cdot\mathrm{grad}\,\mathbf{V}=-\frac{1}{\rho}\mathrm{grad}\,p+\nu\nabla^2\mathbf{V}+\frac{1}{\rho}\mathbf{J}\times\mathbf{B}, \tag{A2}$$

the equations of the electromagnetic field

$$\mathrm{curl}\,\mathbf{H}=\mathbf{J}, \tag{A3}$$

$$\mathrm{div}\,\mathbf{B}=0, \tag{A4}$$

$$\mathrm{curl}\,\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}, \tag{A5}$$

$$\mathrm{div}\,\mathbf{D}=0. \tag{A6}$$

As a complementary relation to these equations, the general Ohm's law is assumed:

$$\mathbf{J}=\sigma(\mathbf{E}+\mathbf{V}\times\mathbf{B}). \tag{A7}$$

Here we adopt the rationalized M.K.S. system. There are the following two relations which connect the magnetic induction $\mathbf{B}$ with the magnetic field $\mathbf{H}$ and the electrical displacement $\mathbf{D}$ with the electric field $\mathbf{E}$, respectively:

$$\mathbf{B}=\mu\mathbf{H}, \tag{A8}$$

$$\mathbf{D} = \varepsilon \mathbf{E}, \tag{A9}$$

where the constants μ and ε are the magnetic permeability and the dielectric constant, respectively. The notations in the above equations are as follows: ρ the density; $\mathbf{V}$ the velocity, p the pressure; ν the kinematic viscosity; σ the electric conductivity, $\mathbf{J}$ the current density.

In the case of the two dimensional incompressible fluid, the velocity and the magnetic induction, *i.e.* the magnetic field, are assumed to have the following components:

$$\mathbf{V} = (u,\ v,\ 0), \tag{A10}$$

$$\mathbf{B} = (B_x,\ B_y,\ 0). \tag{A11}$$

Putting $\frac{\partial}{\partial z} = 0$, the current density $\mathbf{J}$ and the electric field $\mathbf{E}$ have only the z components, j_z and E_z, respectively.

If we choose the uniform magnetic field B_0 applied in the y direction, the induced magnetic fields occur in the x and y directions. But these induced magnetic fields are negligible compared with the uniform applied magnetic field.[4] The ratio of the induced magnetic field to the applied magnetic field R_m is called "magnetic Reynolds' number", and in many practical situations magnetic Reynolds' number is very small.[6] As for the induced electric field, we may presume that it is of the order $\sqrt{R_m}$, except some factors.[7] If R_m is very small, the fundamental equations are derived as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{A12}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{\sigma B_0^2}{\rho} u, \tag{A13}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). \tag{A14}$$

In the case of the boundary layer flows, we may carry out the boundary layer approximation in the equations from (A12) to (A14). And then the required boundary layer equations in magnetohydrodynamics are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{A15}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2}{\rho} u, \tag{A16}$$

$$\frac{\partial p}{\partial y} = 0. \tag{A17}$$

Using the free stream $u_\infty(x,\ t)$ outside the boundary layer, we can eleminate the pressure gradient. The equation (A16) becomes

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial u_\infty}{\partial t} + u_\infty\frac{\partial u_\infty}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} + \frac{\sigma B_0^2}{\rho}(u_\infty - u). \tag{A18}$$

The boundary conditions are imposed only on the flow field.

References

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