SYNTHESIS FOR FINITE SETTLED TIME RESPONSE IN SAMPLED-DATA CONTROLL SYSTEMS WITH NONLINEARITY

MICHIO SATO and KAHEI NAKAMURA

Automatic Control Laboratry

(Received October 30, 1959)

1. Introduction

This paper is concerned with the synthesis for finite settled time response in sampled data-control systems which consist of samplers, discrete compensator, hold circuit and controlled system having nonlinear characteristics—such as saturation, dead zone and hysteresis—and disturdances at the final control element.

The following sections show that the compensation for disturbances at the final control element is equal to the compensation for nonlinear characteristics at the same place by replacing nonlinear effects with equivalent disturbances which can be fixed by the order of hold circuit.

Especially, for the saturating system, this paper shows that it is very effective to use the saturation element of the same character in discrete compensator.

2. Disturbances at the final control element

Fig. 1 illustrates a representative linear sampled-data control system.

Let $R(s)$: reference input, $N(s)$: disturbances, $C(s)$: controlled variable, $D(z)$: sampling compensator, $H(z, s)$: hold element, $G(s)$: controlled system, $K(z) = \dfrac{D(z)\cdot HG(z)}{1+D(z)\cdot HG(z)}$: overall pulse transfer function.

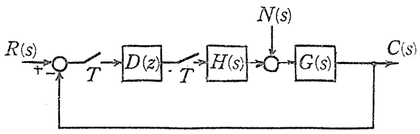


FIG. 1. A representative sampled-data control system.

Now our aim is that steady state error after a finite settled time should be zero for any reference input and disturbances.

2.1. For reference input $R(s)$

Finite settled time response syntheses for reference input have been made by some researchers. The results are as follows,

(1) $K(z)$ must be a finite polynomial in z^{-1},

(2) When $R(s)=1/s^n$, $HG(s)$ must have n-poles at the origin, and zero steady

state error condition must be satisfied, that is,

$$\frac{d^m}{d(z^{-1})^m}[1-k(z^{-1})]_{z^{-1}=1}=0 \quad (m=0,\ 1,\ \ldots,\ n-1). \tag{1}$$

2.2. For disturbances $N(s)$ which is added at sampling instant

When $R(s)=0$,

$$\begin{aligned} C(z) &= -C(z)\,D(z)\,HG(z)+NG(z) \\ \therefore\quad C(z) &= \{1-K(z)\}\,NG(z) \\ C(z,\ \tau) &= NG(z,\ \tau)-K(z)\cdot\frac{HG(z,\ \tau)}{HG(z)}\cdot NG(z), \end{aligned} \tag{2}$$

so if we choose $H(s)$ as

$$H(s)=N(s), \tag{3}$$

$$C(z,\ \tau)=\{1-k(z)\}\,NG(z,\ \tau). \tag{4}$$

So when it is desired that the system has zero steady state error after a finite settled time for disturbances $N(s)$, as well as $R(s)$, the following requirements must be satisfied.

(1) $H(s)=R(s)$,

(2) $1-K(z^{-1})$ must include the denominater of $NG(z)=HG(z)$.

2.3. For $N(s)$ which is added in the sampling interval

When $N(s)$ is actuated L seconds later than some sampling instant, ($L<T$, T: sampling period), and $H(s)=N(s)$,

$$\begin{aligned} C(z) &= -C(z)\cdot D(z)\cdot HG(z)+HG(z,\ -L) \\ \therefore\quad C(z) &= \{1-K(z)\}\,HG(z,\ -L). \end{aligned} \tag{5}$$

The equation (5) shows that the error is zero at sampling instant after some finite settled time, though the settled time is longer by one sampling period than in the previous case.

On the other hand, the output $Y_1(z)$ of compensator $D(z)$ is

$$Y_1(z)=-\frac{HG(z,\ -L)}{HG(z)}\,K(z). \tag{6}$$

So $y_1(nT)$ is zero after some time because $K(z)$ includes the numerater of $HG(z)$ and $Y_1(z)$ is a finite polynomial in z^{-1}.

After the instant $y_1(nT)=0$, the output of $H(s)$ is similar to $N(s)$ in opposite sign, and from the equation (5) the error must be zero in the sampling interval as well as at the sampling instant after some finite settled time.

3. Saturation at the final control element

The illustrative system in this case is shown in Fig. 2, and the characteristics equation is as follows,

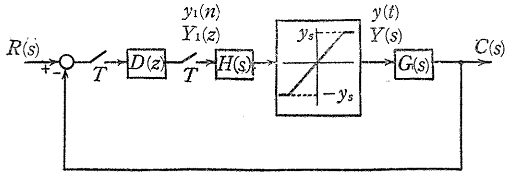


FIG. 2. Sampled-data control system with saturation.

$$|y| < y_s:\ y = y$$
$$|y| \geq y_s:\ y = y_s \text{ or } -y_s.$$

Now a property seen in sampled date control system is that the system can be solved as linear in such a case by expecting equivalent disturbances decided by the order of the hold element. Fig. 3 illustrates the appearance of saturation for zero order hold element and the equivalent disturbance. So for zero order hold element, if the system is synthesized for step disturbance at the final control element, it is also compensated for saturation.

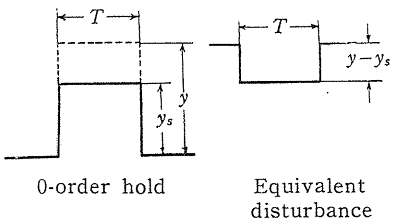


FIG. 3. Equivalent disturbance.

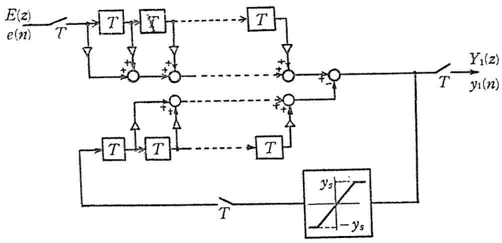


FIG. 4. Discrete compensator with Saturation element.

But the following examples show that it is better to use the compensator using saturation element in the feedback path as the construction of $D(z)$ as illustrated in Fig. 4, because there is desired to obtain the information of saturating as quickly as possible.

Example 1

$$G(s) = \frac{a}{s+a}, \quad H(z, s) = \frac{1-z^{-1}}{s}, \quad R(s) = 1/s$$

from §2,

$$D(z) = \frac{1+d-dz^{-1}}{(1-d)(1-z^{-1})} \qquad (d = e^{-aT}).$$

In the calculation of manipulated variable $y(t)$ if $D(z)$ such as Fig. 4 is used,

$$y_1^*(0) = \frac{1+d}{1-d} \geq y_s$$

$$\therefore \quad y(0) = y_s = \frac{1-e_1}{1-d}, \quad \text{where } e_1 = 1 - y_s(1-d)$$

$$y_1^*(1) = \frac{1+d}{1-d}e_1 - \frac{d}{1-d} + \frac{1-e_1}{1-d} = 1 + \frac{d}{1-d}e_1.$$

When $y_1^*(1) \leq y_s$, that is, $y_s \geqq \frac{1}{1-d^2}$,

$$y_1^*(2) = -\frac{d}{1-d}e_1 + y_1^*(1) = 1$$

$$y_1^*(n) = 1, \qquad (n \geq 2)$$

$$c(1) = 1 - e_1, \quad c(n) = 1, \quad (n \geq 2).$$

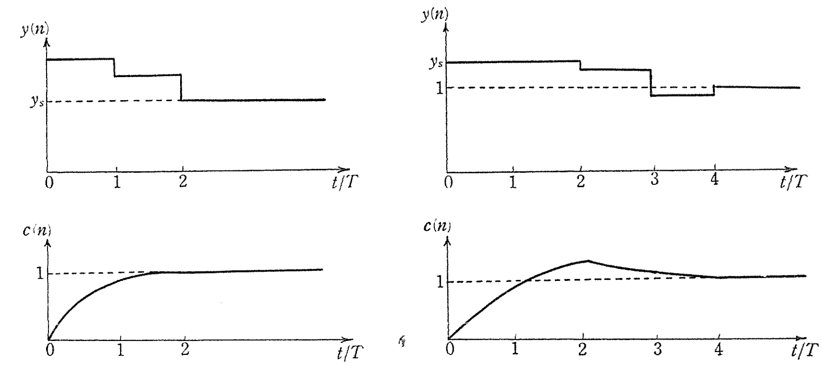
When the saturating continues till the time, $(n-1)T$,

$$y_1^*(n) = \frac{1+d}{1-d}e^*(n) - \frac{d}{1-d}e^*(n-1) + y_s$$

$$\therefore \quad y_1^*(n) \leq y_s, \quad \text{say,} \quad y_s \geq \frac{1}{1-d^{n+1}}. \tag{7}$$

So the steady state error is zero after the time $(n+1)T$, if the equation (7) is satisfied.

On the other hand, if $D(z)$ without saturation element is used, the similar calculation in the real time domain shows that the finite settled time response can be also achieved, but that the settling time is longer. Fig. 5 explains the difference, when $\frac{1}{1-d^2} \leq y_s \leq \frac{1}{1-d}$.



(a) For the compensator with saturation element

(b) For the ordinary compensator

FIG. 5. Inditial response in example 1.

Fig. 6 illustrates the relations between y_s, a and settling time, appearing in the equation (7).

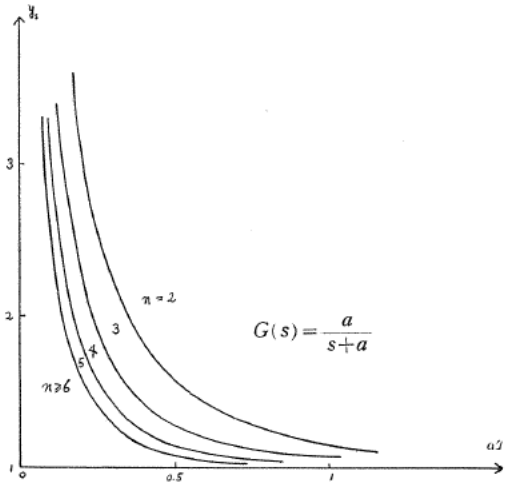


FIG. 6. Relationship in a, T, y_s, and n.

Example 2

$$G(s)=\frac{a}{s+a}e^{-Ts},\ H(z,s)=\frac{1-z^{-1}}{s},\quad R(s)=1/s.$$

From the equation (7) in § (2), $k=1$

$$D(z)=\frac{1+d+d^2-(d+d^2)z^{-1}}{(1-d)(1-z^{-1})\{1+(1+d)z^{-1}\}}.$$

The one after another calculation in real time domain as in the example 1 shows the following results when $D(z)$ as in Fig. 4 is used.

When $\frac{1+d+d^2}{1-d}\geq y_s\geq\frac{1}{1-d^2}$,

$$y_1^*(0)=\frac{1+d+d^2}{1-d}\geq y_s,\quad \therefore\ y(0)=y_s=\frac{1-e_2}{1-d}$$

$$e^*(0)=e^*(1)=1,\quad e^*(2)\equiv e_2=1-y_s(1-d)$$

$$y_1^*(1)=1+\frac{de_2}{1-d}\leq y_s,\quad \therefore\ y_1^*(2)=1$$

$$y_1^*(n)=1,\ (n\geqq 2)\qquad e^*(n)=0,\ (n\geq 3).$$

Furthermore when saturating continues till the time $(n-1)T$,

$$y_1^*(n)=\frac{1-d+d^2}{1-d}e^*(n)-\frac{d+d^2}{1-d}e^*(n-1)-dy_s+(1+d)y_s.$$

Next unequality must be satisfied for $|y_1^*(n)|\leq y_s$,

$$y_s\geq\frac{1}{1-d^{n+1}}. \tag{8}$$

If the equation (8) is satisfied, the system is settled after the time $(n+2)T$. Fig. 7 illustrates the response in this case.

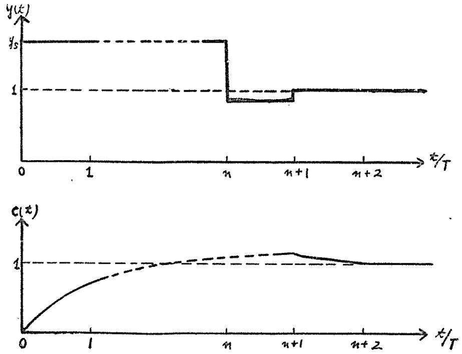


FIG. 7. Inditial response in example 2.

Example 3

$$G(s)=1/s, \quad H(z,\ s)=(1-z^{-1})/s, \quad R(s)=1/s$$

$$D(z)=\frac{\frac{1}{2}(5-3z^{-1})}{1+3/4\,z^{-1}}$$

$$y_1^*(n)=5/2\,e^*(n)-3/2\,e^*(n-1)-3/4\,y_1^*(n-1).$$

In this case $D(z)$ is identical with that by Jury and Schröder because $HG(z)$ has no pole other than one.

In this example maximum manipulated variable appears at the second sampling instant, as indicated in Table I, and the response is more complicated than in the previous examples. In Table II ~ VII, the condition of variables is stated when $y_s \geq \frac{1}{4}$, and the phase plane trajectory is shown in Fig. 9 when $y_s \geq \frac{1}{2}$, and $y_s = \frac{1}{4}$.

But in this calculation, a compensator is used such as in Fig. 4, and the similar one after another calculation was made in the real time domain.

We can conclude as follows from Table I ~ VII and Fig. 8.

TABLE I. Without Saturating

n	$y(n)$	$e(n)$	$\dot{e}(n)$
0	5/2	1	0
1	-4	$-1/4$	$-5/2$
2	3/2	$-3/4$	$-3/2$
≥ 3	0	0	0

TABLE II. $4 > y_s \geq 5/2$

n	$y(n)$	$e(n)$	$\dot{e}(n)$
0	5/2	1	0
1	$-y_s$	$-1/4$	$-5/2$
2	$2y_s-13/2$	$y_s/2-11/4$	$y_s-5/2$
3	$4-y_s$	$y_s/2-2$	$4-y_s$
≥ 4	0	0	0

TABLE III. $5/2 > y_s \geq 1$

n	$y(n)$	$e(n)$	$\dot{e}(n)$
0	y_s	1	0
1	$-y_s$	$1-y_s/2$	$-y_s$
2	$-(y_s-1)$	$-(y_s-1)$	0
3	y_s-1	$-(y_s-1)/2$	y_s-1
≥ 4	0	0	0

TABLE IV. $1 > y_s \geq 1/2$

n	$y(n)$	$e(n)$	$\dot{e}(n)$
0	y_s	1	0
1	$1-2y_s$	$1-y_s/2$	$-y_s$
2	$1(1-y_s)$	$(1-y_s)/2$	$-(1-y_s)$
≥ 3	0	0	0

TABLE V. $1/2 > y_s \geq 2/5$

n	$y(n)$	$e(n)$	$\dot{e}(n)$
0	y_s	1	0
1	$1-2y_s$	$1-y_s/2$	$-y_s$
2	$-y_s$	$(1-y_s)/2$	$-(1-y_s)$
3	$-2(1-2y_s)$	$-(1/2-y_s)$	$-(1-2y_s)$
4	$1-2y_s$	$-(1/2-y_s)$	$1-2y_s$
≥ 5	0	0	0

TABLE VI. $2/5 > y_s \geq 1/3$

n	$y(n)$	$e(n)$	$\dot{e}(n)$
0	y_s	1	0
1	$1-2y_s$	$1-y_s/2$	$-y_s$
2	$-y_s$	$(1-y_s)/2$	$-(1-y_s)$
3	$-y_s$	$-(1/2-y_s)$	$-(1-2y_s)$
4	$8y_s-3$	$-(3-7y_s)/2$	$3y_s-1$
5	$2-5y_s$	$-(2-5y_s)/2$	$2-5y_s$
≥ 6	0	0	0

TABLE VII. $1/3 > y_s \geq 1/4$

n	$y(n)$	$e(n)$	$\dot{e}(n)$
0	y_s	1	0
1	y_s	$1-y_s/2$	$-y_s$
2	$-y_s$	$1-2y_s$	$-2y_s$
3	$-y_s$	$1-11y_s/2$	$-y_s$
4	$-(4y_s-1)$	$-(4y_s-1)$	0
5	$4y_s-1$	$-1/2(4y_s-1)$	$4y_s-1$
≥ 6	0	0	0

(1) The system is settled when non-saturating state continues for two sampling periods.

(2) Macroscopically speaking, settling time becomes longer, but the overshoot becomes smaller according as the range of y_s becomes smaller.

(3) It should be noticed that the optimal response which is achieved in non-linear optimal control appears when $y_s=1$, and $\frac{1}{4}$.

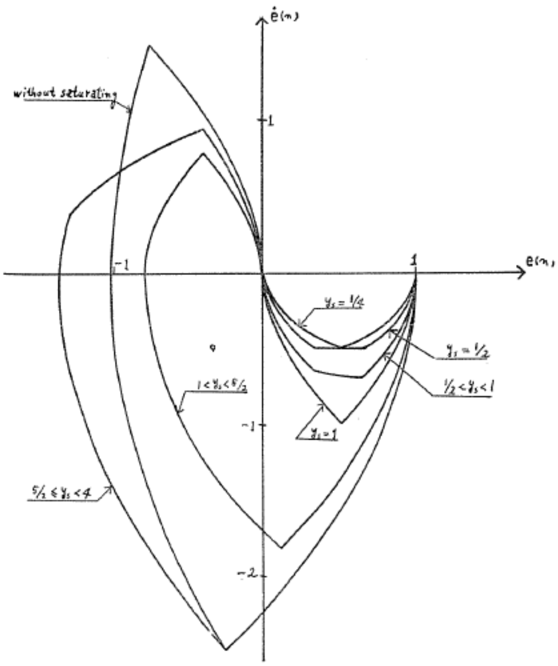


FIG. 8. Phase plane trajectory in example 3.

4. Dead zone and Hysteresis at the final control element

In this case it is also supposed that the same equivalent disturbance can be applied as shown in the previous section.

But a very laborious calculation shows that the system is not so easily settled as in saturation.

So the authors present here a new idea to accomplish finite settled time response in the system with dead zone and hysteresis. Suppose the reverse nonlinear element whose nonlinear characteristics is quite reverse with the nonlinear element in the system, that is, if $y=f(z)$, then $x=f(z)$ as shown in Fig. 9.

Let N be nonlinear element in the system, and N^{-1} be the inverse nonlinear element, and then Fig. 9-*c* illustrates this method.

When it is possible to realize this element, general nonlinearity can be compensated by connecting them in series, and can be reduced to saturation element as illustrated in Fig. 10. The saturation can be compensated by the method shown in § 3.

5. Conclusions

(1) In order to comenpensate both the disturbances $N(s)$ at the final control element, and reference input, $N(s)=H(s)$, and $1-K(z^{-1})$ must include the denominator of $HG(z)$ in addition to the Jury and Schröder conditions.

(2) Saturation at the final control element can be compensated by the same synthesis for disturbances there, and it is effective to use the saturation element in $D(z)$.

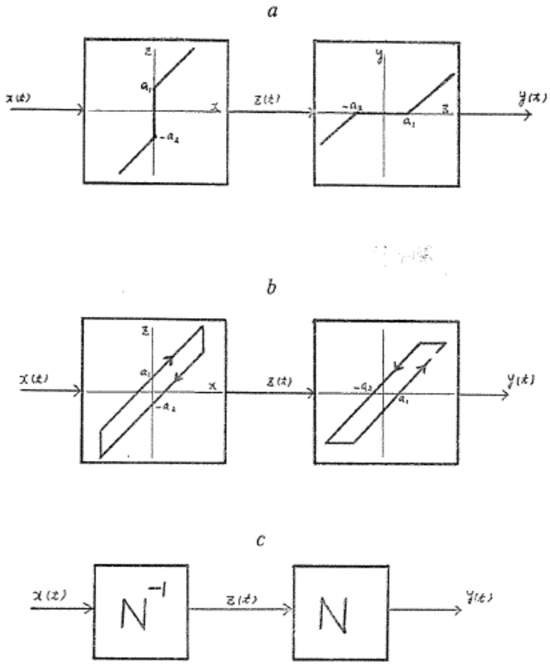


FIG. 9

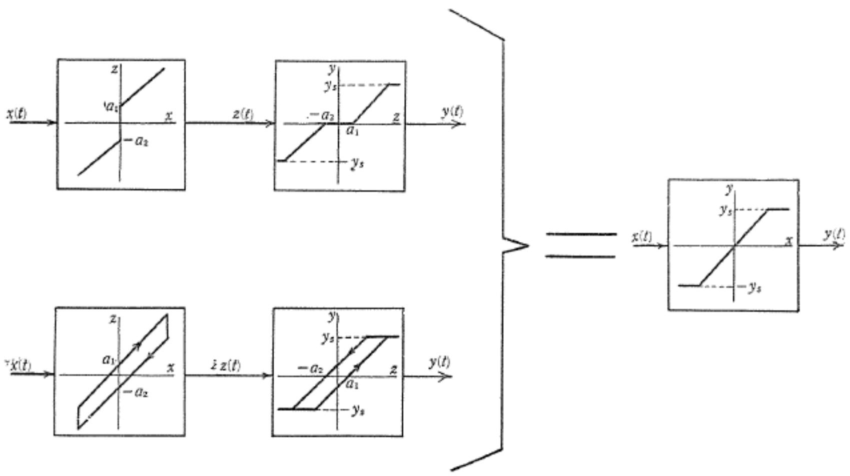


FIG. 10

Though there is a paper saying that there may be some hunting phenomena when the controlled system has delay or is of high order and synthesized by the procedure as proposed in §2, such oscillation can never be seen in these examples if we use the compensater having saturation element.

(3) Dead zone and Hysteresis can be reduced to saturation element by using the reverse nonlinear elements.

References

1) E. I. Jury and W. Schröder: AIEE Transactions vol. **76**, pt. II (1957).
2) M. Mori: Automatic Control vol. 4, No. 3 (1957).
3) M. Sato and K. Nakamura: Automatic Control vol **6**, No. 4 (1959).
4) Kondo and Soga: Automatic control Symposium **2**, No. 127 (1959).