A SOLUTION OF THE EQUATIONS OF MOTION OF VISCOUS FLUID FOR DIMINISHING VISCOSITY

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I. Introduction

The 2-dimensional problem of the Oseen's linearized equations of stationary motion of viscous fluid around a cylindrical solid body for the limiting case viscosity $\longrightarrow 0$, was reduced by Oseen into the following results. (Oseen, Hydrodynamik, Mathematik in Monographien und Lehrbüchern).

Let

$x_j (j = 1, 2)$: rectangular coordinates,

$U(>0)$: constant velocity of fluid for $x_1^2 + x_2^2 \longrightarrow \infty$ in the x_1 negative direction,

$u_1 - U, u_2$: components of velocity,

p : pressure,

ρ : density,

B_v : the region of fluid which is not passed by the solid body,

B_h : the region which is passed by the body,

S_v : the surface of the body facing to B_v,

S_h : the surface of the body facing to B_h.

Then the results are :

in B_v : $u_j = v_j, \quad (j = 1, 2)$,

in B_h : $u_1 = v_1 - (v_1)_h + U, \quad u_2 = v_2$,

and in both regions

$$p = \rho U v_1 - \frac{1}{2} \rho (u_1^2 + u_2^2),$$

where $(v_1)_h$ = value of v_1 on S_h with same x_2, and v_1 and v_2 are to be determined in the following way.

Let

$x_1 + i x_2 = z$,

$z = f(z')$: conformal representation of the outer region of $S = S_v + S_h$ into $|z'| > 1$ so that the boundary S_v is represented by $z' = z^* = e^{i\theta^*}, \ -\frac{\pi}{2} < \theta^* < \frac{\pi}{2}$ and S_h by $\frac{\pi}{2} < \theta^* < \frac{3}{2}\pi$ respectively.

Put

$$\infty = f(z_0'),$$

and make

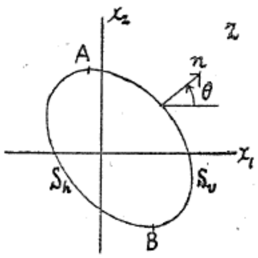

Fig. 1 a.

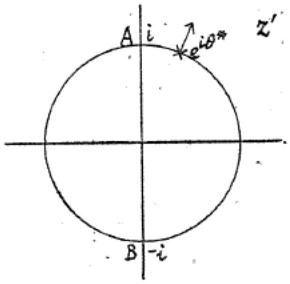

Fig. 1 b.

$$w(z') = \frac{1}{\pi}\int_{\theta^*=-\pi/2}^{\pi/2}\left(\theta(\theta^*) - \frac{\pi}{2}\right)\left(\frac{1}{z^* - z'} - \frac{1}{2\,z^*}\right)dz^*, \quad \cdots\cdots\cdots (\text{I}, 1)$$

where $\theta(\theta^*) = \operatorname{arc}(n, x_1), \qquad z^* = e^{i\theta^*},$

n: outward normal to S_v at $z = f(e^{i\theta^*})$,

and $-\frac{\pi}{2} < \theta(\theta^*) < \frac{\pi}{2}$ for $-\frac{\pi}{2} < \theta^* < \frac{\pi}{2}$.

Further put

$$W(z') = U + e^{w(z')}\cdot\left(k_1 + ik_2\frac{z' - i}{z' + i}\right), \quad \cdots\cdots\cdots (\text{I}, 2)$$

where the real constants k_1, k_2 are to be determined by

$$W(z_0') = 0.$$

Finally we obtain

$$W(z') = W(z) = v_1 - iv_2. \quad \cdots\cdots\cdots (\text{I}, 3)$$

The resistance and lift on the body can be calculated by the formulas

$$-\frac{1}{2}\rho\int_{S_h}(U - v_1)^2\,dx_2, \quad \cdots\cdots\cdots (\text{I}, 4)$$

$$2\pi B\rho U^2 - \frac{1}{2}\rho\int_{S_h}(U^2 - v_1^2)\,dx_1, \quad \cdots\cdots\cdots (\text{I}, 5)$$

respectively, where the integration must be taken in positive sense along S_h and B is determined by

$$W(z) = U\frac{A - iB}{z} + \cdots. \quad \cdots\cdots\cdots (\text{I}, 6)$$

The calculation of this problem has been carried out, as far as we know, in the following 2 cases: the curve $S = S_v + S_h$ is a circle or a line segment. We treat in this paper the lens-shaped case.

II. Conformal Representation and the Determination of $w(z')$ and $W(z')$

We take the 2 side curves of the cross section as arcs of circles and put the cross section as is shown in Fig. 2.

$$0 < \gamma \leqq \frac{\pi}{2},$$

$$-\gamma \leqq \beta \leqq \alpha \leqq \gamma.$$

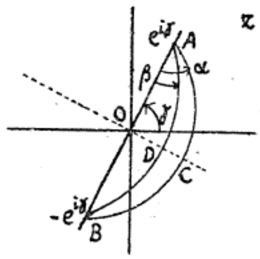

Fig. 2.

The conditions that the arcs ACB and ADB shall be S_v and S_h respectively are fulfilled.

We transform the z-plane successively by

$$\zeta = e^{i(\pi/2-\gamma)}\, z,$$

$$\xi = \frac{\zeta + i}{\zeta - i} \qquad \left(\zeta = i\frac{\xi+1}{\xi-1}\right),$$

$$u = \xi^{\pi/(2\pi-(\alpha-\beta))},$$

$$v = e^{i\delta}\, u,$$

$$\delta = \frac{\pi}{2} - (\pi - \alpha)\frac{\pi}{2\pi - (\alpha-\beta)} = \frac{1}{\varepsilon}\,\frac{\alpha+\beta}{2},$$

$$\varepsilon = 2 - \frac{1}{\pi}(\alpha - \beta),$$

$$z' = i\frac{v+1}{v-1} \qquad \left(v = \frac{z'+i}{z'-i}\right).$$

$$z'_0 = \operatorname{ctg}\frac{\delta}{2}.$$

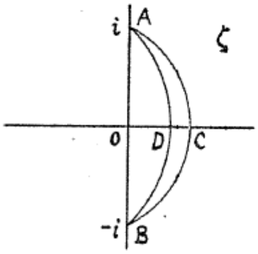

Fig. 3 a.

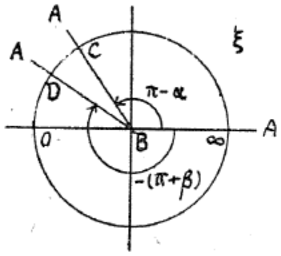

Fig. 3·b.

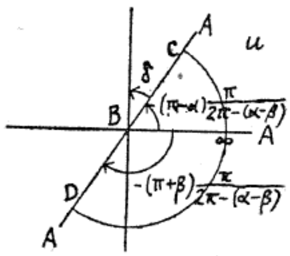

Fig. 3 c.

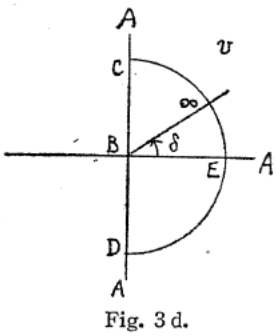

Fig. 3 d.

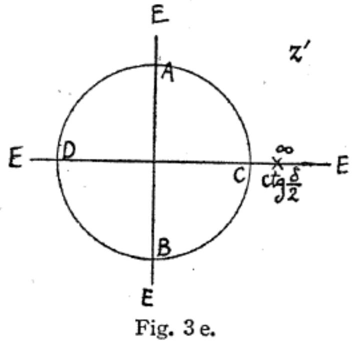

Fig. 3 e.

Calculation of $\theta(\theta^*)$.

In the ζ-plane let the center of the arc ACB be $O_1(-\operatorname{ctg}\alpha)$, $\left(\text{the radius} = \frac{1}{|\sin\alpha|}\right)$ then for ζ on the arc we have:

$$\zeta - (-\operatorname{ctg}\alpha) = \frac{1}{\sin\alpha}\, e^{i\theta_1}, \qquad \cdots\cdots\cdots(\text{II}, 1)$$

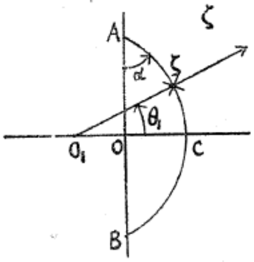

Fig. 4.

and

$$\theta(\theta^*) = \theta_1(\theta^*) - \left(\frac{\pi}{2} - \gamma\right),$$

$$z' = z^* = e^{i\theta^*},$$

$$v = \frac{z^* + i}{z^* - i} = e^{(\pi/2)\,i} \operatorname{ctg}\left(\frac{\pi}{4} - \frac{\theta^*}{2}\right),$$

$$\xi = e^{(\pi/2-\delta)\varepsilon i}\left\{\operatorname{ctg}\left(\frac{\pi}{4} - \frac{\theta^*}{2}\right)\right\}^{\varepsilon} = e^{(\pi-\alpha)i}\cdot\left\{\operatorname{ctg}\left(\frac{\pi}{4} - \frac{\theta^*}{2}\right)\right\}^{\varepsilon}.$$

Hence combining with (II, 1)

$$\zeta = i\,\frac{-e^{\alpha i}(\operatorname{ctg})^{\varepsilon} + 1}{-e^{-\alpha i}(\operatorname{ctg})^{\varepsilon} - 1} = \frac{e^{i\theta_1} - \cos\alpha}{\sin\alpha},$$

$$e^{i\theta_1} = \frac{(\operatorname{ctg})^{\varepsilon} + e^{-i\alpha}}{(\operatorname{ctg})^{\varepsilon} + e^{i\alpha}}\, e^{i\alpha} = e^{-2i\varphi + i\alpha}, \quad \text{(II, 2)} \quad \theta_1 = \alpha - 2\varphi,$$

with

$$0 \leqq \frac{\varphi}{\alpha} \leqq 1.$$

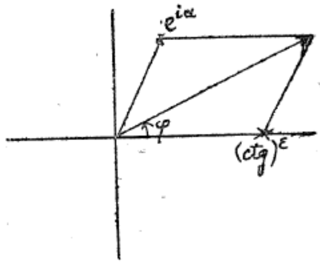

Fig. 5.

From (II, 2) or directly from the symmetry of the representation ($\zeta \longrightarrow z'$) it is clear that

$$\theta_1(-\theta^*) = -\theta_1(\theta^*). \qquad \cdots\cdots\cdots\text{(II, 3)}$$

From (I, 1) regarding (II, 3) we obtain:

$$w(z') = \frac{\gamma - \pi}{\pi}\int\left(\frac{1}{z^* - z'} - \frac{1}{2z^*}\right)dz^* + \frac{1}{\pi}\int\theta_1(\theta^*)\left(\frac{1}{z^* - z'} - \frac{1}{2z^*}\right)dz^*$$

$$= \frac{\gamma - \pi}{\pi}\lg\frac{z' - i}{z' + i} - \frac{i}{2}(\gamma - \pi) + \frac{1}{\pi}\int\theta_1(\theta^*)\frac{dz^*}{z^* - z'},$$

$$e^{w(z')} = ie^{-i\gamma/2}\cdot\left(\frac{z' + i}{z' - i}\right)^{1-(\gamma/\pi)}\cdot e^{a(z')},$$

where

$$a(z') = \frac{1}{\pi}\int\theta_1(\theta^*)\frac{dz^*}{z^* - z'} = \frac{2}{\pi}z'\int_0^{\pi/2}\theta_1(\theta^*)\frac{\sin\theta^* d\theta^*}{z'^2 - 2z'\cos\theta^* + 1}, \qquad \cdots\cdots\text{(II, 4)}$$

and

$$a(z_0') = a\left(\operatorname{ctg}\frac{\delta}{2}\right) = a_0 : \text{real}.$$

We put

$$1 - \frac{\gamma}{\pi} = \eta, \quad \delta + \frac{\pi}{2} = \frac{\pi + \beta}{\varepsilon} = \omega,$$

then

$$e^{w(z_0')} = e^{i\eta\omega}\cdot e^{a_0},$$

and by the condition

$$W(z_0') = 0$$

we have

$$\left.\begin{aligned} k_1 &= -\frac{\sin(1-\eta)\,\omega}{\sin\omega}e^{-a_0}\cdot U, \\ k_2 &= \frac{\sin\eta\omega}{\sin\omega}\,e^{-a_0}\cdot U. \end{aligned}\right\} \qquad \cdots\cdots\cdots\text{(II, 5)}$$

Hence we have:

$$W(z') = U\Big\{1 - \frac{e^{-a_0}}{\sin\omega} e^{w(z')} \Big[\sin(1-\eta)\,\omega - i\sin\eta\omega \cdot \frac{z'-i}{z'+i}\Big]\Big\}$$
$$= U\Big\{1 - \frac{e^{a-a_0}}{\sin\omega} e^{i\eta\pi/2} v^{\eta} \Big[\sin(1-\eta)\,\omega - i\sin\eta\omega \cdot \frac{1}{v}\Big]\Big\}. \qquad \cdots\cdots\cdots\text{(II, 6)}$$

We expand (II, 6) as a function of z in power series of $\frac{1}{z}$.

$$\xi = 1 + \frac{2i}{\zeta} + \cdots,$$
$$u = 1 + \frac{2i}{\varepsilon}\frac{1}{\zeta} + \cdots, \qquad v = e^{i\delta}\Big(1 + \frac{2i}{\varepsilon}\frac{1}{\zeta} + \cdots\Big),$$
$$z' = \operatorname{ctg}\frac{\delta}{2} - \frac{1}{\varepsilon\sin^2\frac{\delta}{2}} \cdot \frac{1}{\zeta} + \cdots = \operatorname{ctg}\frac{\delta}{2} + \frac{ie^{i\gamma}}{\varepsilon\sin^2\frac{\delta}{2}}\frac{1}{z} + \cdots,$$
$$a = a_0 + \frac{2}{\varepsilon}\frac{ie^{i\gamma}}{\pi\sin^2\frac{\delta}{2}}\frac{1}{z}\int_0^{\pi/2}\theta_1(\theta^*)\sin\theta^* d\theta^* \times$$
$$\times\Big\{\frac{1}{z'^2 - 2z'\cos\theta^* + 1} + z'\frac{\partial}{\partial z'}\frac{1}{z'^2 - 2z'\cos\theta^* + 1}\Big\}_{z' = \operatorname{ctg}\frac{\delta}{2}}$$
$$= a_0 + \frac{2}{\varepsilon} ie^{i\gamma}\frac{a_1}{z} + \cdots,$$

with

$$a_0 = \frac{\sin\delta}{\pi}\int_0^{\pi/2}\frac{\theta_1(\theta^*)\sin\theta^* d\theta^*}{1 - \cos\theta^*\sin\delta},$$
$$a_1 = -\frac{\cos\delta}{\pi}\int_0^{\pi/2}\theta_1(\theta^*)\frac{\sin\theta^* d\theta^*}{(1-\cos\theta^*\sin\delta)^2}, \qquad (a_1 : \text{real}),$$
$$e^{w(z')} = e^{w(z_0')}\Big[1 + \frac{2}{\varepsilon}e^{i\gamma}\Big\{\Big(1 - \frac{\gamma}{\pi}\Big) + ia_1\Big\}\frac{1}{z} + \cdots\Big],$$
$$\frac{W(z)}{U} = \frac{2}{\varepsilon}e^{-i\eta\pi}\Big\{\eta - \frac{\sin\eta\omega}{\sin\omega}e^{-i(1-\eta)\omega} + ia_1\Big\}\frac{1}{z} + \cdots$$
$$= \frac{A - iB}{z} + \cdots,$$

$$\left.\begin{aligned} A &= \frac{2}{\varepsilon}\Big\{\eta\cos\eta\pi - \frac{\sin\eta\omega}{\sin\omega}\cos(\eta\omega - \omega - \eta\pi) + a_1\sin\eta\pi\Big\},\\ B &= \frac{2}{\varepsilon}\Big\{\eta\sin\eta\pi + \frac{\sin\eta\omega}{\sin\omega}\sin(\eta\omega - \omega - \eta\pi) - a_1\cos\eta\pi\Big\}.\end{aligned}\right\} \cdots\cdots\text{(II, 7)}$$

A and B determine, in one sense, the radial outflow and circulation.

III. Resistance and Lift on the Solid

When z moves in the z-plane along the arc ADB we get

$$dz = e^{-i(\pi/2-\gamma)}\,d\zeta,$$
$$d\zeta = \frac{-2i}{(\xi-1)^2}d\xi = \frac{-2i\varepsilon e^{i\varepsilon\delta}}{(v^{\varepsilon} - e^{i\varepsilon\delta})^2}v^{\varepsilon}\frac{dv}{v},$$
$$dz = 2e^{-i\eta\pi}\frac{e^{i\varepsilon\omega}}{(y - e^{i\varepsilon\omega})^2}dy \qquad \cdots\cdots\cdots\text{(III, 1)}$$

by putting

$$v = e^{-i\pi/2} y^{1/\varepsilon}, \qquad (y: \ +\infty \longrightarrow +0),$$

and from (II, 6)

$$W(z) = U\Big\{1 - \frac{e^{a-a_0}}{\sin\omega}[\sin(1-\eta)\,\omega \cdot y^{\eta/\varepsilon} + \sin\eta\omega \cdot y^{(\eta-1)/\varepsilon}]\Big\}. \quad \cdots\cdots(\text{III}, 2)$$

Here we see from (I, 3) by putting $v_2 = 0$ or from (III, 2) and (II, 4) that W is real along S_h.

By (I, 4) and (I, 5) we have to calculate:

$$-\int_{S_h} (U - v_1)^2\, dx_2 = -\int (U-W)^2\, dx_2 = -\Im\int (U-W)^2\, dz,$$

and

$$-\int (U^2 - v_1^2)\, dx_1 = -\Re\int (U^2 - W^2)\, dz$$

$$= \Re\int (U-W)^2\, dz - 2U\Re\int (U-W)\, dz. \quad \cdots\cdots(\text{III}, 3)$$

Using (III, 1) and (III, 2) we get:

$$\int (U-W)^2\, dz = \frac{2U^2 e^{-i(\eta\pi-\varepsilon\omega)}}{\sin^2\omega}\int_\infty^0 \frac{e^{2(a-a_0)}}{(y-e^{i\varepsilon\omega})^2}\times$$

$$\times\{\sin^2(1-\eta)\,\omega \cdot y^{2\eta/\varepsilon} + 2\sin(1-\eta)\,\omega \cdot \sin\eta\omega \cdot y^{(2\eta-1)/\varepsilon} + \sin^2\eta\omega \cdot y^{(2\eta-2)/\varepsilon}\}\, dy,$$

$$U\int (U-W)\, dz = \frac{2U^2}{\sin\omega} e^{-i(\eta\pi-\varepsilon\omega)} \cdot \int_\infty^0 \frac{e^{a-a_0}}{(y-e^{i\varepsilon\omega})^2}\times$$

$$\times\{\sin(1-\eta)\,\omega \cdot y^{\eta/\varepsilon} + \sin\eta\omega \cdot y^{(\eta-1)/\varepsilon}\}\, dy. \quad \cdots\cdots(\text{III}, 4)$$

Behaviour of e^a for $y = 0$ and $y = \infty$.

When $z^* = e^{i\theta^*}$ lies on the arc ACB of the z'-plane and is near to A, we have:

$$z' = z^* = i + t,$$

$$v = \frac{2i+t}{t} = \frac{2i}{t}\Big(1 + \frac{t}{2i}\Big),$$

$$u = e^{i(\pi/2-\delta)} \cdot \frac{2}{t}\Big(1 + \frac{t}{2i}\Big),$$

$$\xi = e^{i(\pi-\alpha)} \cdot \frac{2^\varepsilon}{t^\varepsilon}\Big(1 + \frac{\varepsilon}{2i}t + \cdots\Big),$$

$$\zeta = i(1 - e^{i\alpha} \cdot 2^{1-\varepsilon} t^\varepsilon + \cdots),$$

and by (II, 1)

$$e^{i\theta_1} = \sin\alpha \cdot (\zeta + \operatorname{ctg}\alpha) = e^{i\alpha}(1 - i\sin\alpha \cdot 2^{1-\varepsilon} t^\varepsilon + \cdots),$$

$$\theta_1(\theta^*) = \alpha + \frac{1}{i}\lg(1 - i\sin\alpha \cdot 2^{1-\varepsilon} t^\varepsilon + \cdots)$$

$$= \alpha - \sin\alpha \cdot 2^{1-\varepsilon}(z^* - i)^\varepsilon + \cdots,$$

$$(1 \leqq \varepsilon \leqq 2).$$

This last result can also be obtained from (II, 2), and we see from (II, 4) easily that

$$a(z') = \frac{\alpha}{\pi} \cdot \lg(z' - i) + \text{bounded function},$$

when z' is on the arc ADB and near to A. Similarly when z' is on the arc ADB and near to B we have

$$a(z') = \frac{\alpha}{\pi}\lg(z' + i) + \text{bounded function}.$$

Hence we have:

$$e^{a} = \begin{cases} y^{-\alpha/(\varepsilon\pi)} \cdot F(y), & \text{when } y: \text{real and } y \longrightarrow +\infty, \\ y^{\alpha/(\varepsilon\pi)} \cdot F(y), & \text{when } y: \text{real and } y \longrightarrow +0, \end{cases}$$

where $F(y)$ represents functions with F and $\frac{1}{F}$ bounded, and e^{a} is bounded in the other part of the line $y=$ real positive in the y-plane.

Conditions for convergence of integrals in (III, 4).

The integrals in (III, 4) are of the form:

$$I_{s,k} = \int_{\infty}^{0} \frac{e^{s(a-a_0)}\, y^{(s\eta-k)/\varepsilon}}{(y-e^{i\varepsilon\omega})^2}\, dy, \qquad \cdots\cdots\cdots \text{(III, 5)}$$

with

$$s=1, \quad k=0, 1,$$

$$\text{or} \quad s=2, \quad k=0, 1, 2.$$

The condition of convergence of (III, 5) in the neighbourhood of $y=0$, that is, in the neighbourhood of B in the z-plane is

$$\frac{2(\eta-1)}{\varepsilon}+\frac{2\,\alpha}{\varepsilon\pi} > -1, \qquad \cdots\cdots\cdots \text{(III, 6)}$$

and that in the neighbourhood of $y=\infty$ (A in z-plane) is

$$\frac{2\,\eta}{\varepsilon}-\frac{2\,\alpha}{\varepsilon\pi} < 1. \qquad \cdots\cdots\cdots \text{(III, 7)}$$

(III, 6) and (III, 7) can be written in the form

$$(\gamma-\alpha)+(\gamma-\beta) < 2\,\pi,$$

$$(\gamma+\alpha)+(\gamma+\beta) > 0,$$

respectively, that is, the first is satisfied except for

$$-\alpha = -\beta = \gamma = \frac{\pi}{2},$$

and the second except for $\qquad \cdots\cdots\cdots \text{(III, 8)}$

$$-\alpha = -\beta = \gamma > 0.$$

These 2 exceptional cases are shown in Fig. 6.

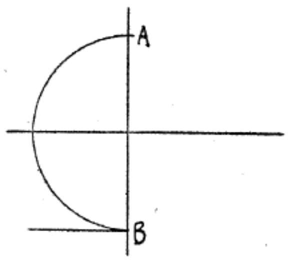

Fig. 6 a.

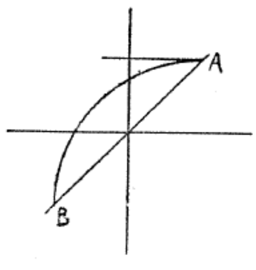

Fig. 6 b.

Calculation of (III, 5) for the case $\alpha=0$.

Because of the factor $e^{s(a-a_0)}$, which is identically $=1$ for $\alpha=0$, it seems difficult to evaluate (III, 5) in the general case.

When $\alpha=0$ the exceptional cases (III, 8) do not occur and (III, 5) becomes simply

$$I_k = \int_{\infty}^{0} \frac{y^k}{(y - e^{i\varepsilon\omega})^2} dy, \qquad k = \frac{2\eta}{\varepsilon}, \ \frac{2\eta - 1}{\varepsilon}, \ \frac{2(\eta - 1)}{\varepsilon}, \ \frac{\eta}{\varepsilon}, \ \frac{\eta - 1}{\varepsilon},$$

$$\frac{\pi}{2} \leqq \varepsilon\omega = \pi + \beta \leqq \pi.$$

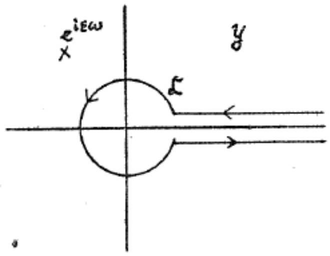

Fig. 7.

To evaluate I_k we put

$$I = \int_{\mathfrak{C}} \frac{y^k}{(y - e^{i\varepsilon\omega})^2} dy = (1 - e^{i2k\pi}) \cdot I_k$$

and by method of residue we obtain:

$$I = -i2k\pi e^{i(k-1)\varepsilon\omega},$$

$$e^{-i\eta\pi + i\varepsilon\omega} \cdot I_k = \frac{k\pi e^{i(k\varepsilon\omega - k\pi - \eta\pi)}}{\sin k\pi}.$$

Inserting these values into (III, 4) we have the results:

$$\int (U - W)^2 dz = \frac{2U^2}{\sin^2\omega} \Big\{ \sin^2(1-\eta)\omega \frac{2\eta(\pi - \omega)}{\sin 2\eta\,(\,''\,)} e^{i\eta(4\omega - 3\pi)} -$$

$$- 2\sin(1-\eta)\omega \sin\eta\omega \frac{(2\eta - 1)(\pi - \omega)}{\sin(2\eta - 1)(\,''\,)} e^{i\eta(4\omega - 3\pi) - i2\omega} +$$

$$+ \sin^2\eta\omega \frac{2(\eta - 1)(\pi - \omega)}{\sin 2(\eta - 1)(\,''\,)} e^{i\eta(\,''\,) - i4\omega} \Big\},$$

$$U \int (U - W)\, dz = \frac{2U^2}{\sin\omega} \Big\{ \sin(1-\eta)\omega \frac{\eta(\pi - \omega)}{\sin\eta(\pi - \omega)} e^{i2\eta(\pi - \omega)} -$$

$$- \sin\eta\omega \frac{(\eta - 1)(\pi - \omega)}{\sin(\eta - 1)(\pi - \omega)} e^{i2\eta(\pi - \omega) - i2\omega} \Big\}$$

and putting into (III, 3) we can finally obtain the components of force on the cylinder.