# ON ELASTICITY PROBLEMS WHEN THE SECOND ORDER TERMS OF THE STRAIN ARE TAKEN INTO ACCOUNT, II

ZYURÔ SAKADI

*Department of Dynamics*

(Received April 30th, 1949)

## I. Preliminaries and Notations

The theory of elasticity considering the 2nd order terms of strain was developed by the author in the paper[1] wtih the same title. In this report the strain tensor with finite deformation will be adopted and the relation between stress and strain tensors (2, 1) in I will be obtained simply and the following 4 cases solved.

1. Bending of an elliptic cylinder,
2. Torsion of an elliptic cylinder,
3. Radial vibration of a sphere,
4. Torsional vibration of a circular cylinder.

Let, as in I,

$y_i$: coordinates of a material point in unstrained state,

$x_i$: coordinates of the same point in strained state,

$x_i - y_i = \eta_i(t, y) = \xi_i(t, x)$: components of displacement,

$A_{ij}(t, x)$: components of stress tensor at $(x_i)$,

$\rho_0$: density in unstrained state.

$$\sigma_{ij} = \frac{1}{2}\left(\frac{\partial \xi_j}{\partial x_i} + \frac{\partial \xi_i}{\partial x_j}\right), \qquad \sigma_{kk} = \frac{\partial \xi_k}{\partial x_k}, \qquad \omega_{ij} = \frac{1}{2}\left(\frac{\partial \xi_j}{\partial x_i} - \frac{\partial \xi_i}{\partial x_j}\right).$$

## II. The Equations of Motion and the Boundary Conditions

The equations of motion, when $\xi_i$'s are small and their 3rd order quantities are neglected, can be written easily as in (1, 1) in I as follows:

$$\rho_0\left(\frac{\partial^2 \xi_i}{\partial t^2} - \sigma_{kk}\frac{\partial^2 \xi_i}{\partial t^2} + 2\frac{\partial^2 \xi_i}{\partial t \partial x_k}\frac{\partial \xi_k}{\partial t} + \frac{\partial^2 \xi_k}{\partial t^2}\frac{\partial \xi_i}{\partial x_k}\right) = \frac{\partial A_{ij}}{\partial x_j}. \quad \cdots\cdots\cdots \text{(II, 1)}$$

We calculate for the finite deformation in the following way:

$$dx_i = dy_i + \frac{\partial \eta_i}{\partial y_k} dy_k, \quad dy_i = dx_i - \frac{\partial \xi_i}{\partial x_k} dx_k,$$

$$d\sigma^2 - ds^2 = dx_i^2 - dy_i^2 = \left(dy_i + \frac{\partial \eta_i}{\partial y_k} dy_k\right)\left(dy_i + \frac{\partial \eta_i}{\partial y_l} dy_l\right)$$

$$- dy_i^2 = \left(\frac{\partial \eta_j}{\partial y_i} + \frac{\partial \eta_i}{\partial y_j} + \frac{\partial \eta_k}{\partial y_i}\frac{\partial \eta_k}{\partial y_j}\right) dy_i dy_j$$

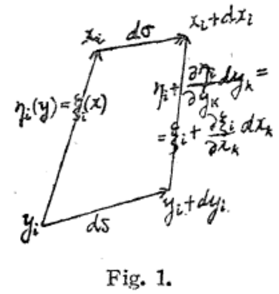


Fig. 1.

(1) Proc. Phys.-Math. Soc. Jap. 3rd Series, Vol. 22 (1940), p. 999. This paper shall be denoted as I.

$$= 2\,\varepsilon'_{ij}(t,y)dy_idy_j = dx_i^2 - \left(dx_i - \frac{\partial\xi_i}{\partial x_k}dx_k\right)\left(dx_i - \frac{\partial\xi_i}{\partial x_l}dx_l\right)$$

$$= \left(\frac{\partial\xi_j}{dx_i} + \frac{\partial\xi_i}{\partial x_j} - \frac{\partial\xi_k}{\partial x_i}\frac{\partial\xi_k}{\partial x_j}\right)dx_idx_j = 2\,\varepsilon_{ij}(t,x)dx_idx_j.$$

We can define the strain tensor by $\varepsilon_{ij}$ or by $\varepsilon'_{ij}$,

$$\varepsilon_{ij} = \sigma_{ij} - \frac{1}{2}(\sigma_{ik} + \omega_{ik})(\sigma_{jk} + \omega_{jk}), \qquad \varepsilon_{kk} = \sigma_{kk} - \frac{1}{2}\sigma_{kl}^2 - \frac{1}{2}\omega_{kl}^2,$$

and the general relations between 2 symmetric tensors $A_{ij}$ and $\varepsilon_{ij}$, which contain the 1st and 2nd order terms of $\varepsilon_{ij}$, can be written by:

$$\begin{aligned} A_{ij} &= \lambda\varepsilon_{kk}\delta_{ij} + 2\,\mu\varepsilon_{ij} + (c_1\varepsilon_{kk}\varepsilon_{ll} + c_2\varepsilon_{kl}^2)\delta_{ij} + c_3\varepsilon_{kk}\varepsilon_{ij} + c_4\varepsilon_{ik}\varepsilon_{jk} \\ &= \lambda\sigma_{kk}\delta_{ij} + 2\,\mu\sigma_{ij} + (a_1\sigma_{kk}\sigma_{ll} + a_2\sigma_{kl}^2 - \lambda/2\cdot\omega_{kl}^2)\delta_{ij} \\ &\quad + a_3\sigma_{kk}\sigma_{ij} + a_4\sigma_{ik}\sigma_{jk} - \mu(\omega_{ik}\omega_{jk} + \omega_{ik}\sigma_{jk} + \omega_{jk}\sigma_{ik}), \qquad \cdots\cdots\cdots(\text{II}, 2) \\ & a_1 = c_1, \quad a_2 = c_2 - \lambda/2, \quad a_3 = c_3, \quad a_4 = c_4 - \mu, \end{aligned}$$

coinciding with (2, 1) in I.

We put

$$\xi_i = \xi_i^0(t,x) + \xi_i'(t,x), \qquad \sigma_{ij} = \sigma_{ij}^0 + \sigma'_{ij}, \qquad \omega_{ij} = \omega_{ij}^0 + \omega'_{ij},$$

$$\begin{aligned} A_{ij} &= A_{ij}^0(t,x) + A'_{ij} + A_{ij}^{0,2} = (\lambda\sigma_{kk}^0\delta_{ij} + 2\,\mu\sigma_{ij}^0) + (\lambda\sigma'_{kk}\delta_{ij} + 2\,\mu\sigma'_{ij}) \\ &\quad + \{(a_1\sigma_{kk}^0\sigma_{ll}^0 + a^2\sigma_{kl}^{02})\delta_{ij} + a_3\sigma_{kk}^0\sigma_{ij}^0 + a_4\sigma_{ik}^0\sigma_{jk}^0 - \lambda/2\cdot\omega_{kl}^{02}\delta_{ij} \\ &\quad - \mu(\omega_{ik}^0\omega_{jk}^0 + \omega_{ik}^0\sigma_{jk}^0 + \omega_{jk}^0\sigma_{ik}^0)\}, \qquad \cdots\cdots\cdots(\text{II}, 3) \end{aligned}$$

where the usual equations are satisfied;

$$\rho_0\frac{\partial^2\xi_i^0}{\partial t^2} = \frac{\partial A_{ij}^0}{\partial x_j},$$

then we have from (II, 1):

$$\rho_0\left(\frac{\partial^2\xi_i'}{\partial t^2} - \sigma_{kk}^0\frac{\partial^2\xi_i^0}{\partial t^2} + 2\frac{\partial^2\xi_i^0}{\partial t\partial x_k}\frac{\partial\xi_k^0}{\partial t} + \frac{\partial^2\xi_k^0}{\partial t^2}\frac{\partial\xi_i^0}{\partial x_k}\right) = \frac{\partial A'_{ij}}{\partial x_j} + \frac{\partial A_{ij}^{0,2}}{\partial x_j}. \quad \cdots\cdot(\text{II}, 4)$$

For the boundary conditions we also obtained in I:

$y_i = f_i(u,v)$: boundary surface of unstrained state,

$x_i = y_i + \eta_i(t,y) = x_i(t,u,v)$: boundary of strained state,

$$q_1^0(u,v) = \begin{vmatrix} \frac{\partial f_2}{\partial u} & \frac{\partial f_3}{\partial u} \\ \frac{\partial f_2}{\partial v} & \frac{\partial f_3}{\partial v} \end{vmatrix} = \frac{\partial(f_2,f_3)}{\partial(u,v)},\ q_2^0,\ q_3^0,$$

$$\beta_i(t,u,v) = \xi_i^0(t,f),$$

where we put $f_j(u,v)$ instead of $x_j$ in $\xi_i^0(t,x)$,

$$q_1'(t,u,v) = \frac{\partial(\beta_2,f_3)}{\partial(u,v)} + \frac{\partial(f_2,\beta_3)}{\partial(u,v)},\ q_2',\ q_3',$$

$F_i(t,u,v)$: components of external force at $x_i(t,u,v)$,

then, when the boundary conditions concerning the 1st order quantities are satisfied and hence

$$F_i - A_{ij}^0(t,f)\cdot\frac{q_j^0}{\sqrt{q_k^{02}}} = F_i'(t,u,v)$$

are small of 2nd order, we have:

$$F_i' = \frac{1}{\sqrt{q_k^{02}}}\left( -\frac{A_{ij}^0 q_i^0 q_l^0}{q_m^{02}} q_l' + A_{ij}' q_j^0 + A_{ij}^{0,2} q_j^0 + \frac{\partial A_{ij}^0}{\partial x_m}\beta_m q_j^0 + A_{ij}^0 q_j' \right), \cdots (\text{II}, 5)$$

where in $A_{ij}^0(t,x)$, $A_{ij}'$ and $A_{ij}^{0,2}$ we have to put $x_j = f_j(u,v)$.

## III. Bending of an Elliptic Cylinder.

$\sigma = \dfrac{\lambda}{2(\lambda+\mu)}$: Poisson's ratio,

$$\xi_1^0 = \frac{\sigma}{R} x_1 x_2, \quad \xi_2^0 = \frac{x_3^2}{2R} + \frac{\sigma(x_2^2 - x_1^2)}{2R}, \quad \xi_3^0 = -\frac{1}{R} x_2 x_3, \cdots (\text{III}, 1)$$

$R$: radius of curvature for the line $y_1 = y_2 = 0$,

$$\sigma_{11}^0 = \sigma_{22}^0 = \frac{\sigma}{R} x_2, \quad \sigma_{33}^0 = -\frac{1}{R} x_2, \quad \sigma_{kk}^0 = -\frac{1-2\sigma}{R} x_2, \quad \sigma_{ij}^0 = 0 \ (i \neq j),$$

$$\omega_{12}^0 = -\frac{\sigma}{R} x_1, \quad \omega_{23}^0 = -\frac{1}{R} x_3, \quad \omega_{13}^0 = 0, \qquad \cdots\cdots\cdots (\text{III}, 2)$$

$$A_{33}^0 = -\frac{2\mu(1+\sigma)}{R} x_2,$$

all other $A_{ij}^0$'s are zero. For the lateral surface we put:

$$\begin{aligned} &y_1 = f_1(u), \quad y_2 = f_2(u), \quad y_3 = v, \\ &q_1^0 = f_2', \quad q_2^0 = -f_1', \quad q_3^0 = 0, \end{aligned} \qquad \cdots\cdots\cdots (\text{III}, 3)$$

$$\beta_1 = \frac{\sigma}{R} f_1 f_2, \quad \beta_2 = \frac{1}{2R} v^2 + \frac{\sigma}{2R}\left(f_2^2 - f_1^2\right), \quad q_3' = \frac{1}{R} f_1' v. \quad \cdots\cdots (\text{III}, 4)$$

From (II, 5) we obtain for the free lateral surface the conditions:

$$A_{ij}' q_j^0 + A_{ij}^{0,2} q_j^0 + \frac{\partial A_{ij}^0}{\partial x_m}\beta_m q_j^0 + A_{ij}^0 q_j' = 0. \qquad \cdots\cdots\cdots (\text{III}, 5)$$

Using (III, 2) we calculate $A_{ij}^{0,2}$ into the following results:

$$\left.\begin{aligned} A_{11}^{0,2} &= -\frac{\lambda\sigma}{2R^2} x_1^2 + \frac{b_1}{R^2} x_2^2 - \frac{\lambda}{R^2} x_3^2, \\ A_{22}^{0,2} &= -\frac{\lambda\sigma}{2R^2} x_1^2 + \frac{b_1}{R^2} x_2^2 - \frac{\lambda+\mu}{R^2} x_3^2, \\ A_{33}^{0,2} &= -\frac{\lambda\sigma^2}{R^2} x_1^2 + \frac{b_2}{R^2} x_2^2 - \frac{\lambda+\mu}{R^2} x_3^2, \\ A_{12}^{0,2} &= 0, \quad A_{13}^{0,2} = \frac{\mu\sigma}{R^2} x_1 x_3, \quad A_{23}^{0,2} = -\frac{\mu(1+\sigma)}{R^2} x_2 x_3, \end{aligned}\right\} \cdots\cdots (\text{III}, 6)$$

where

$$b_1 = (1-2\sigma)^2 a_1 + (1+2\sigma^2) a_2 - \sigma(1-2\sigma) a_3 + \sigma^2 a_4,$$
$$b_2 = (1-2\sigma)^2 a_1 + (1+2\sigma^2) a_2 + (1-2\sigma) a_3 + a_4.$$

The equations of equilibrium are written by (II, 4) and (III, 6) in the form:

$$\frac{\partial A_{ij}'}{\partial x_j} = -\frac{\partial A_{ij}^{0,2}}{\partial x_j} = 6(\lambda+2\mu)(e_1 x_1, e_2 x_2, e_3 x_3), \qquad \cdots\cdots\cdots (\text{III}, 7)$$

with

$$e_1 = \frac{(\lambda-\mu)\sigma}{6(\lambda+2\mu)R^2}, \quad e_2 = \frac{\mu(1+\sigma) - 2b_1}{6(\lambda+2\mu)R^2}, \quad e_3 = \frac{2\lambda+3\mu}{6(\lambda+2\mu)R^2}. \qquad \cdots\cdots\cdots (\text{III}, 8)$$

For the lateral boundary we obtain from (III, 2~6) the conditions:

$$\left.\begin{aligned} A'_{11}f'_2 - A'_{12}f'_1 &= -A^{0,2}_{11}f'_2, \\ A'_{12}f'_2 - A'_{22}f_1' &= A^{0,2}_{22}f'_1, \\ A'_{13}f'_2 - A'_{23}f'_1 &= \frac{\mu}{R^2}v(-\sigma f_1 f'_2 + (1+\sigma)f'_1 f_2). \end{aligned}\right\} \cdots\cdots\cdots (\text{III}, 9)$$

To solve (III, 7) with the conditions (III, 9) we put

$$\xi'_i = \xi''_i + (e_1x_1^3, e_2x_2^3, e_3x_3^3), \quad \sigma'_{ij} = \begin{cases}\sigma''_{ij} + 3(e_1x_1^2, e_2x_2^2, e_3x_3^2), & (i=j)\\ \sigma''_{ij}, & (i \neq j)\end{cases}$$

$$\sigma'_{kk} = \sigma''_{kk} + 3\sum e_k x_k^2,$$

$$A'_{ij} = \begin{cases} A''_{ij} + 3\lambda\sum e_k x_k^2 + 6\mu(e_1x_1^2, e_2x_2^2, e_3x_3^2), & (i=j)\\ A''_{ij}, & (i\neq j)\end{cases} \cdots (\text{III}, 10)$$

and the following equations and conditions are obtained:

$$\frac{\partial A''_{ij}}{\partial x_j} = 0, \qquad \cdots\cdots\cdots (\text{III}, 11)$$

$$\left.\begin{aligned} A''_{11}f'_2 - A''_{12}f'_1 &= \frac{\mu}{2(\lambda+2\mu)R^2}((\lambda+2\mu)\sigma f_1^2 - (\lambda(1+\sigma)+4b_1)f_2^2 + \lambda v^2)f'_2, \\ A''_{12}f'_2 - A''_{22}f'_1 &= \frac{\mu}{2(\lambda+2\mu)R^2}(-3\lambda\sigma f_1^2 + (\lambda+2\mu)(1+\sigma)f_2^2 - (3\lambda+4\mu)v^2)f'_1, \\ A''_{13}f'_2 - A''_{23}f'_1 &= \frac{\mu}{R^2}v(-\sigma f_1 f'_2 + (1+\sigma)f'_1 f_2). \end{aligned}\right\} \cdots\cdots\cdots (\text{III}, 12)$$

The equations (III, 11) have the following solution:

$$\left.\begin{aligned} \xi''_1 &= \varphi_{11} + \varphi_{12}\cdot x_3^2, \\ \xi''_2 &= \varphi_{21} + \varphi_{22}\cdot x_3^2, \\ \xi''_3 &= \varphi_{31}\cdot x_3 + \varphi_{32}\cdot x_3^3, \end{aligned}\right\} \cdots\cdots\cdots (\text{III}, 13)$$

where $\varphi$'s are functions of $x_1$ and $x_2$ only. Inserting the expressions (III, 13) into (III, 11) and (III, 12) we obtain:

$$\left.\begin{aligned} (\lambda+\mu)\frac{\partial P}{\partial x_1} + \mu(2\varphi_{12} + D\varphi_{11}) &= 0, \\ (\lambda+\mu)\frac{\partial P}{\partial x_2} + \mu(2\varphi_{22} + D\varphi_{21}) &= 0, \\ (\lambda+\mu)\frac{\partial Q}{\partial x_1} + \mu D\varphi_{12} &= 0, \\ (\lambda+\mu)\frac{\partial Q}{\partial x_2} + \mu D\varphi_{22} &= 0, \\ 2(\lambda+\mu)Q + \mu(6\varphi_{32} + D\varphi_{31}) &= 0, \end{aligned}\right\} \cdots\cdots\cdots (\text{III}, 14)$$

and

$$\left.\begin{aligned} E_1 &= (p_1f_1^2 - q_1f_2^2)f'_2, \\ E_2 &= -(p_2f_2^2 - q_2f_1^2)f'_1, \\ F_1 &= 2\mu r_1 f'_2, \\ F_2 &= -2\mu r_2 f'_1, \\ C_{13}f'_2 - C_{23}f'_1 &= 2s_1f_1f'_2 - 2s_2f_2f'_1, \end{aligned}\right\} \cdots\cdots\cdots (\text{III}, 15)$$

and further that

$$\varphi_{32} = \text{const.},$$

where

$$\left.\begin{aligned}
&D = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}, \\
&P = \frac{\partial \varphi_{11}}{\partial x_1} + \frac{\partial \varphi_{21}}{\partial x_2} + \varphi_{31}, \quad Q = \frac{\partial \varphi_{12}}{\partial x_1} + \frac{\partial \varphi_{22}}{\partial x_2} + 3\,\varphi_{32}, \\
&B = \frac{\partial \varphi_{22}}{\partial x_1} + \frac{\partial \varphi_{12}}{\partial x_2}, \quad C_{12} = \frac{\partial \varphi_{21}}{\partial x_1} + \frac{\partial \varphi_{11}}{\partial x_2}, \\
&C_{13} = \frac{\partial \varphi_{31}}{\partial x_1} + 2\varphi_{12}, \quad C_{23} = \frac{\partial \varphi_{31}}{\partial x_2} + 2\,\varphi_{22}, \\
&E_1 = \left(\lambda P + 2\,\mu \frac{\partial \varphi_{11}}{\partial x_1}\right) f_2' - \mu C_{12} f_1', \\
&E_2 = -\left(\lambda P + 2\,\mu \frac{\partial \varphi_{21}}{\partial x_2}\right) f_1' + \mu C_{12} f_2', \\
&F_1 = \left(\lambda Q + 2\,\mu \frac{\partial \varphi_{12}}{\partial x_1}\right) f_2' - \mu B f_1', \\
&F_2 = -\left(\lambda Q + 2\,\mu \frac{\partial \varphi_{22}}{\partial x_2}\right) f_1' + \mu B f_2',
\end{aligned}\right\} \cdots (\text{III}, 16)$$

and

$$\left.\begin{aligned}
&p_1 = \frac{\mu}{2\,R^2}\sigma, \quad p_2 = -\frac{\mu}{2\,R^2}\left(1+\sigma\right), \quad q_1 = \frac{\mu}{2\,R^2}\frac{\lambda(1+\sigma)+4\,b_1}{\lambda+2\,\mu}, \\
&q_2 = -\frac{\mu}{2\,R_2}\frac{3\,\lambda\sigma}{\lambda+2\,\mu}, \quad r_1 = \frac{1}{4\,R^2}\frac{\lambda}{\lambda+2\,\mu}, \quad r_2 = \frac{1}{4\,R^2}\frac{3\,\lambda+4\,\mu}{\lambda+2\,\mu}, \\
&s_1 = -\frac{1}{2\,R^2}\sigma, \quad s_2 = -\frac{1}{2\,R^2}(1+\sigma).
\end{aligned}\right\} (\text{III}, 17)$$

We now assume that the cross section is an ellipse:

$$y_1 = f_1 = a\cos u, \quad y_2 = f_2 = b\sin u,$$

then we can take:

$$\left.\begin{aligned}
&\varphi_{11} = \gamma_1 x_1 + \gamma_{111} x_1 x_2^2 + \gamma_{112} x_1^3, \quad \varphi_{12} = \gamma_{12} x_1, \\
&\varphi_{21} = \gamma_2 x_2 + \gamma_{211} x_2^3 + \gamma_{212} x_1^2 x_2, \quad \varphi_{22} = \gamma_{22} x_2, \\
&\varphi_{31} = \gamma_3 + \gamma_{311} x_2^2 + \gamma_{312} x_1^2, \quad \varphi_{32} = \gamma_{32},
\end{aligned}\right\} \cdots\cdots\cdots (\text{III}, 18)$$

and we calculate:

$$P = (\gamma_1+\gamma_2+\gamma_3) + (\gamma_{111}+3\,\gamma_{211}+\gamma_{311})\,x_2^2 + (3\,\gamma_{112}+\gamma_{212}+\gamma_{312})x_1^2,$$

$$Q = \gamma_{12}+\gamma_{22}+3\,\gamma_{32},$$

$$B = 0, \quad C_{12} = 2\,(\gamma_{212}+\gamma_{111}) f_1 f_2,$$

$$C_{13} = 2\,(\gamma_{12}+\gamma_{312}) f_1, \quad C_{23} = 2\,(\gamma_{22}+\gamma_{311}) f_2,$$

and from (III, 14 ~ 15) we obtain the equations for $\gamma$'s:

$$(\lambda+\mu)(3\,\gamma_{112}+\gamma_{212}+\gamma_{312}) + \mu(\gamma_{12}+\gamma_{111}+3\,\gamma_{112}) = 0,$$

$$(\lambda+\mu)(\gamma_{111}+3\,\gamma_{211}+\gamma_{311}) + \mu(\gamma_{22}+3\,\gamma_{211}+\gamma_{212}) = 0,$$

$$(\lambda+\mu)(\gamma_{12}+\gamma_{22}+3\,\gamma_{32}) + \mu(3\,\gamma_{32}+\gamma_{311}+\gamma_{312}) = 0,$$

and

$$\gamma_{12}+\gamma_{312} = s_1, \quad \gamma_{22}+\gamma_{311} = s_2,$$

$$(\lambda+2\,\mu)\gamma_{12}+\lambda\gamma_{22}+3\,\lambda\gamma_{32} = 2\,\mu r_1, \quad (\lambda+2\,\mu)\gamma_{22}+\lambda\gamma_{12}+3\,\lambda\gamma_{32} = 2\,\mu r_2,$$

$$\{(\lambda+2\,\mu)\gamma_1+\lambda\gamma_2+\lambda\gamma_3\} + 2\,\mu a^2(\gamma_{111}+\gamma_{212}) + b^2\{(\lambda+2\,\mu)\gamma_{111}+3\,\lambda\gamma_{211}+\lambda\gamma_{311}\} = -b^2 q_1,$$

$$\{(\lambda+2\,\mu)\gamma_1+\lambda\gamma_2+\lambda\gamma_3\} + a^2\{3\,(\lambda+2\,\mu)\gamma_{112}+\lambda\gamma_{212}+\lambda\gamma_{312}\} = a^2 p_1,$$

$$\{\lambda\gamma_1 + (\lambda + 2\mu)\gamma_2 + \lambda\gamma_3\} + 2\mu b^2(\gamma_{111} + \gamma_{212}) + a^2\{3\lambda\gamma_{112} + (\lambda + 2\mu)\gamma_{212} + \lambda\gamma_{312}\} = -a^2 q_2,$$

$$\{\lambda\gamma_1 + (\lambda + 2\mu)\gamma_2 + \lambda\gamma_3\} + b^2\{\lambda\gamma_{111} + 3(\lambda + 2\mu)\gamma_{211} + \lambda\gamma_{311}\} = b^2 p_2.$$

The solutions of these equations are the following:

$$\left.\begin{aligned}
\gamma_{32} &= -\frac{1}{3}\cdot\frac{1}{3\lambda + 2\mu}\{\lambda(r_1 + r_2) + (\lambda + \mu)(s_1 + s_2)\},\\
\gamma_{12} &= \frac{1}{3\lambda + 2\mu}\{2(\lambda + \mu)r_1 - \lambda r_2 + \frac{\lambda}{2}(s_1 + s_2)\},\\
\gamma_{22} &= \frac{1}{3\lambda + 2\mu}\{-\lambda r_1 + 2(\lambda + \mu)r_2 + \frac{\lambda}{2}(s_1 + s_2)\},\\
\gamma_{311} &= \frac{1}{3\lambda + 2\mu}\{\lambda r_1 - 2(\lambda + \mu)r_2 - \frac{\lambda}{2}s_1 + \frac{5\lambda + 4\mu}{2}s_2\},\\
\gamma_{312} &= \frac{1}{3\lambda + 2\mu}\{-2(\lambda + \mu)r_1 + \lambda r_2 + \frac{5\lambda + 4\mu}{2}s_1 - \frac{\lambda}{2}s_2\},
\end{aligned}\right\}\quad (\mathrm{III}, 19)$$

$$\left.\begin{aligned}
&2(\lambda + \mu)(3\lambda + 2\mu)(3a^4 + 2a^2b^2 + 3b^4)\gamma_{111} = -\lambda\{6(\lambda + \mu)a^4 + \lambda a^2b^2\\
&\quad + 3\lambda b^4\}r_1 + \lambda\{3\lambda a^4 + (5\lambda + 4\mu)a^2b^2 + 6(\lambda + \mu)b^4\}r_2\\
&\quad -\left\{\left(\frac{3}{2}\lambda^2 + 6\lambda\mu + 4\mu^2\right)a^4 + 2(2\lambda^2 + 6\lambda\mu + 3\mu^2)a^2b^2 - \frac{3}{2}\lambda^2b^4\right\}s_1\\
&\quad -\left\{\frac{3}{2}\lambda^2a^4 - 2(\lambda^2 + 5\lambda\mu + 3\mu^2)a^2b^2 + \lambda\left(\frac{9}{2}\lambda + 4\mu\right)b^4\right\}s_2\\
&\quad -\frac{3\lambda + 2\mu}{2\mu}\Big[\{(3\lambda + 4\mu)a^2 + 3(\lambda + 2\mu)b^2\}(a^2p_1 + b^2q_1)\\
&\quad -\{3(\lambda + 2\mu)a^2 - \lambda b^2\}(b^2p_2 + a^2q_2)\Big] = \omega(a, b)r_1 + \bar{\omega}(a, b)r_2\\
&\quad + \pi(a, b)s_1 + \bar{\pi}(a, b)s_2 + \frac{3\lambda + 2\mu}{2\mu}\Big\{\rho(a, b)(a^2p_1 + b^2q_1)\\
&\quad + \bar{\rho}(a, b)(b^2p_2 + a^2q_2)\Big\},\\
&2(\,〃\,)(\,〃\,)(\,〃\,)\gamma_{212} = \bar{\omega}(b, a)r_1 + \omega(b, a)r_2 + \bar{\pi}(b, a)s_1 + \pi(b, a)s_2\\
&\quad + \frac{3\lambda + 2\mu}{2\mu}\Big\{\bar{\rho}(b, a)(a^2p_1 + b^2q_1) + \rho(b, a)(b^2p_2 + a^2q_2)\Big\},\\
&6(\lambda + \mu)(3a^4 + 2a^2b^2 + 3b^4)\gamma_{112} = \lambda\left(6\frac{\lambda + \mu}{3\lambda + 2\mu}a^4 + a^2b^2 + 3b^4\right)r_1\\
&\quad - \lambda\left(\frac{3\lambda}{3\lambda + 2\mu}a^4 + a^2b^2\right)r_2 - \frac{1}{2}\Big\{\frac{21\lambda^2 + 28\lambda\mu + 8\mu^2}{3\lambda + 2\mu}a^4\\
&\quad + 4(2\lambda + \mu)a^2b^2 + 3(3\lambda + 2\mu)b^4\Big\}s_1 + \frac{1}{2}\Big\{\frac{3\lambda^2}{3\lambda + 2\mu}a^4 + 4\lambda a^2b^2\\
&\quad + (3\lambda + 2\mu)b^4\Big\}s_2 + \frac{1}{2\mu}\Big\{(\lambda + 2\mu)a^2 - 3\lambda b^2\Big\}(a^2p_1 + b^2q_1)\\
&\quad + \frac{1}{2\mu}\Big\{3\lambda a^2 + (3\lambda + 2\mu)b^2\Big\}(b^2p_2 + a^2q_2) = \varepsilon(a, b)r_1 + \bar{\varepsilon}(a, b)r_2\\
&\quad + \nu(a, b)s_1 + \bar{\nu}(a, b)s_2 + \frac{1}{2\mu}\Big\{\zeta(a, b)(a^2p_1 + b^2q_1)\\
&\quad + \bar{\zeta}(a, b)(b^2p_2 + a^2q_2)\Big\},\\
&6(\,〃\,)(\,〃\,)\gamma_{211} = \bar{\varepsilon}(b, a)r_1 + \varepsilon(b, a)r_2 + \bar{\nu}(b, a)s_1 + \nu(b, a)s_2
\end{aligned}\right\}\quad (\mathrm{III}, 20)$$

$$
\begin{aligned}
&+\frac{1}{2\mu}\Big\{\bar{\zeta}(b,a)\,(a^2p_1+b^2q_1)+\zeta(b,a)\,(b^2p_2+a^2q_2)\Big\},\\
&2\,(\lambda+\mu)\gamma_1=-\lambda\gamma_3+\frac{1}{2\mu}a^2\{(\lambda+2\mu)p_1+\lambda q_2\}-a^2\{6\,(\lambda+\mu)\gamma_{112}\\
&\quad+\lambda\gamma_{312}\}+\lambda b^2(\gamma_{111}+\gamma_{212}),\\
&2\,(\lambda+\mu)\gamma_2=-\lambda\gamma_3-\frac{1}{2\mu}a^2\{(\lambda+2\mu)q_2+\lambda p_1\}-a^2\{2\,(\lambda+\mu)\gamma_{212}\\
&\quad+\lambda\gamma_{312}\}-(\lambda+2\mu)b^2(\gamma_{111}+\gamma_{212}),
\end{aligned}
$$

here $\gamma_3$ is to be determined by the following condition (III, 23).

For the terminal surface $y_3=l$ we have:

$$y_1=u,\qquad y_2=v,$$

$$q_1{}^0=q_2{}^0=0,\qquad q_3{}^0=1,$$

$$\beta_1=\frac{\sigma}{R}uv,\qquad \beta_2=\frac{l^2}{2R}+\frac{\sigma(v^2-u^2)}{2R},\qquad q_3'=\frac{2\sigma}{R}v,$$

$$ds^2=dx_i^2=\left(1+2\frac{\partial\beta_1}{\partial u}\right)du^2+2\left(\frac{\partial\beta_1}{\partial v}+\frac{\partial\beta_2}{\beta u}\right)dudv+\left(1+2\frac{\partial\beta_2}{\partial v}\right)dv^2,$$

$$dS=\sqrt{1+2\left(\frac{\partial\beta_1}{\partial u}+\frac{\partial\beta_2}{\partial v}\right)}\,dudv=\left(1+2\frac{\sigma}{R}v\right)dudv:\qquad\cdots(\text{III},21)$$

surface element,

$$
\left.
\begin{aligned}
&A'_{i3}=A''_{i3}=\mu\left(2\varphi_{i2}+\frac{\partial\varphi_{31}}{\partial x_i}\right)x_3=2\mu x_3\begin{cases}(\gamma_{12}+\gamma_{312})x_1\\(\gamma_{22}+\gamma_{311})x_2\end{cases}=2\mu l\begin{cases}s_1u, & (i=1)\\ s_2v, & (i=2)\end{cases}\\
&A'_{33}=3\lambda\textstyle\sum e_kx_k{}^2+6\mu e_3x_3^2+A''_{33}=3\lambda(e_1u^2+e_2v^2)+3\,(\lambda+2\mu)e_3l^2\\
&\quad+\lambda(\gamma_1+\gamma_2+\gamma_3)+\lambda(\gamma_{111}+3\gamma_{211}+\gamma_{311})v^2+\lambda(3\gamma_{112}+\gamma_{212}+\gamma_{312})u^2\\
&\quad+\lambda(\gamma_{12}+\gamma_{22}+3\gamma_{32})l^2+2\mu\gamma_3+2\mu\gamma_{311}v^2+2\mu\gamma_{312}u^2+6\mu\gamma_{32}l^2\\
&=g_0+g_1u^2+g_2v^2,
\end{aligned}
\right\}(\text{III},22)
$$

$$F_i=F_i{}^0+F_i',\qquad F_1{}^0=F_2{}^0=0,\qquad F_3{}^0=A^0_{33}=-\frac{2\mu(1+\sigma)}{R}v,$$

$$F_1'=0,\qquad F_2'=-\frac{2\mu(1+\sigma)}{R^2}lv,$$

$$F_3'=A^{0,2}_{33}+A'_{33}-\frac{\mu(1+\sigma)}{R^2}\{l^2+\sigma(v^2-u^2)\}=h_0+h_1u^2+h_2v^2.$$

Hence we obtain for the total force on the surface:

$$X_i=\int F_idS=\int F_i{}^0\left(1+2\frac{\sigma}{R}v\right)du\,dv+\int F_i'dudv,$$

$$=\begin{cases}0, & (i=1,2)\\ \iint\left\{h_0+h_1u^2+\left(h_2-\dfrac{4\mu\sigma(1+\sigma)}{R^2}\right)v^2\right\}dudv, & (i=3)\end{cases}$$

and from the condition

$$X_3=0,\qquad\cdots\cdots\cdots(\text{III},23)$$

we have to determine $\gamma_3$. Similarly we calculate the moment of force on the surface:

$$G_1 = \int (x_2F_3 - x_3F_2)\, dS = \int (vF_3^0 - lF_2^0) dudv + \int (vF_3^0 - lF_2^0)\, 2\frac{\sigma}{R} vdudv$$
$$+ \int (\beta_2F_3^0 - \beta_3F_2^0)\, dudv + \int (vF_3' - lF_2')dudv = \int (vF_3^0 - lF_2^0)dudv = G_1^0,$$
$$G_2 = \int (x_3F_1 - x_1F_3)dS = 0, \qquad G_3 = 0.$$

By substituting (III, 17) in (III, 19) we get:

$$\gamma_{32} = \frac{\mu}{6(\lambda + 2\mu)R^2}, \qquad \gamma_{12} = -\frac{1}{4R^2}\frac{\lambda}{\lambda+\mu}, \qquad \gamma_{22} = \frac{1}{4R^2}\frac{\lambda+2\mu}{\lambda+\mu},$$
$$\gamma_{311} = -\frac{1}{R^2}, \qquad \gamma_{312} = 0.$$

The colculation of other $\gamma$'s by (III, 20) and (III, 23) is very troublesome and we omit the results.

## IV. Torsion of an Elliptic Cylinder

Let the cross section of the cylinder be

$$y_1 = f_1 = a\cos u, \qquad y_2 = f_2 = b\sin u$$

as in the foregoing case, further

$$\left.\begin{aligned}
&\xi_1^0 = -\tau x_2x_3, \qquad \xi_2^0 = \tau x_1x_3, \qquad \xi_3^0 = -\tau\frac{a^2-b^2}{a^2+b^2}x_1x_2,\\
&\sigma^0_{11} = \sigma^0_{22} = \sigma^0_{33} = \sigma^0_{kk} = \sigma^0_{12} = 0,\\
&\sigma^0_{13} = -\frac{\tau}{a^2+b^2}a^2x_2, \qquad \sigma^0_{23} = \frac{\tau}{a^2+b^2}b^2x_1,\\
&\omega^0_{13} = \frac{\tau}{a^2+b^2}b^2x_2, \qquad \omega^0_{23} = -\frac{\tau}{a^2+b^2}a^2x_1,
\end{aligned}\right\} \cdots\cdots(\text{IV}, 1)$$

all $A^0_{ij}$'s excepting $A^0_{13}$ and $A^0_{23}$ vanish. For the lateral surface we have:

$$\left.\begin{aligned}
&y_1 = f_1(u), \quad y_2 = f_2(u), \quad y_3 = v, \quad q_1^0 = f_2', \quad q_2^0 = -f_1', \quad q_3^0 = 0,\\
&\beta_1 = -\tau f_2v, \quad \beta_2 = \tau f_1v, \quad \beta_3 = -\tau\frac{a^2-b^2}{a^2+b^2}f_1f_2,\\
&q_1' = \tau f_1'v, \quad q_2' = \tau f_2'v, \quad q_3' = \tau(f_1f_1' + f_2f_2').
\end{aligned}\right\}(\text{IV}, 2)$$

From (IV, 1) we calculate:

$$\left.\begin{aligned}
A^{0,2}_{11} &= \frac{\tau^2}{(a^2+b^2)^2}\Big[(2a_2b^4 - \lambda a^4)x_1^2 + \{(2a_2+a_4)a^4 - (\lambda+\mu)b^4 + 2\mu a^2b^2\}x_2^2\Big]\\
&\quad - (\lambda+\mu)\tau^2x_3^2,\\
A^{0,2}_{22} &= \quad {}'' \quad [\{(2a_2+a_4)b^4 - (\lambda+\mu)a^4 + 2\mu a^2b^2\}x_1^2 + (2a_2a^4 - \lambda b^4)x_2^2]\\
&\quad - \ {}'',\\
A^{0,2}_{33} &= \quad {}'' \quad [\{(2a_2+a_4)b^4 - (\lambda+\mu)a^4 - 2\mu a^2b^2\}x_1^2 + \{(2a_2+a_4)a^4\\
&\quad - (\lambda+\mu)b^4 - 2\mu a^2b^2\}x_2^2] - \lambda\tau^2x_3^2,\\
A^{0,2}_{12} &= \quad {}'' \quad \{-\mu(a^4+b^4) - (a_4-\mu)a^2b^2\}x_1x_2,\\
A^{0,2}_{13} &= -\mu\tau^2x_1x_3,\\
A^{0,2}_{23} &= -\mu\tau^2x_2x_3.
\end{aligned}\right\}(\text{IV}, 3)$$

The equations of equilibrium take the same form as (III, 7) with following values of $e$'s:

$$e_1 = \frac{\tau^2 A(a,b)}{3(\lambda+2\mu)(a^2+b^2)^2}, \qquad e_2 = \frac{\tau^2 A(b,a)}{〃}, \qquad e_3 = \frac{\tau_2(\lambda+\mu)}{3(\lambda+2\mu)}, \quad \cdots \text{(IV, 4)}$$

$$A(a,b) = (\lambda+\mu)a^4 + \frac{1}{2}(\mu+a_4)a^2b^2 + (\mu - 2a_2)b^4.$$

We put, as in (III, 10),

$$\xi_i' = \xi_2'' + (e_1x_1^3, e_2x_2^3, e_3x_3^3), \text{ etc.,}$$

then, with (IV, 1∼4) and (III, 5), we obtain the following conditions for the free lateral boundary:

$$A_{11}''f_2' - A_{12}''f_1' = \tau^2\Big[\frac{-1}{(a^2+b^2)^2}\Big\{\alpha f_1^2 + (\alpha\frac{a^2}{b^2} + \beta(a,b))f_2^2\Big\} + \frac{2\mu(\lambda+\mu)}{\lambda+2\mu}v^2\Big]f_2',$$

$$A_{12}''f_2' - A_{22}''f_1' = \tau^2\Big[\quad \frac{1}{〃}\quad \Big\{(\alpha\frac{b^2}{a^2} + \beta(b,a))f_1^2 + \alpha f_2^2\Big\} - \quad 〃 \quad\Big]f_1',$$

$$A_{13}''f_2' - A_{23}''f_1' = \frac{\mu\tau^2}{a^2+b^2}v\{(3a^2-b^2)f_1f_2' - (3b^2-a^2)f_1'f_2\},$$

$$\alpha = \mu(a^4+b^4) + \frac{1}{2}(\mu+a_4)a^2b^2, \qquad \beta(a,b) = \Big(\frac{2\mu(\lambda+\mu+2a_2)}{\lambda+2\mu} - \frac{1}{2}(\mu+a_4)\Big)a^4$$

$$-\Big(2\mu - \frac{\lambda(\mu+a_4)}{2(\lambda+2\mu)}\Big)a^2b^2 - 2\frac{\mu(\lambda+\mu)}{\lambda+2\mu}b^4,$$

these being of the same form as (III, 12). Hence we also assume (III, 13) and (III, 18) and the constants (III, 17) should be replaced by:

$$\left.\begin{aligned} & r_1 = r_2 = \tau^2\frac{\lambda+\mu}{\lambda+2\mu}, \qquad s_1 = \frac{\tau^2}{2(a^2+b^2)}(3a^2-b^2), \qquad s_2 = 〃 (3b^2-a^2), \\ & p_1 = p_2 = -\frac{\tau^2}{(a^2+b^2)^2}\alpha, \qquad q_1 = \frac{\tau^2}{(a^2+b^2)^2}\Big(\alpha\frac{a^2}{b^2} + \beta(a,b)\Big), \\ & q_2 = 〃 \{\alpha\frac{b^2}{a^2} + \beta(b,a)\}, \end{aligned}\right\} \text{(IV, 5)}$$

and by (III, 19∼20) we calculate the $r$'s.

Similarly for the terminal surface we take:

$$y_1 = u, \quad y_2 = v, \quad y_3 = l, \quad q_1^0 = q_2^0 = 0, \quad q_3^0 = 1,$$

$$\beta_1 = -\tau lv, \quad \beta_2 = \tau lu, \quad \beta_3 = -\tau\frac{a^2-b^2}{a^2+b^2}uv,$$

$$q_1' = \tau\frac{a^2-b^2}{a^2+b^2}v, \quad q_2' = 〃 u, \quad q_3' = 0.$$

$$dS = du dv : \text{surface element,}$$

the expressions (III, 22) hold without change and

$$F_i = F_i^0 + F_i',$$

$$F_1^0 = -\frac{2\mu\tau}{a^2+b^2}a^2v, \quad F_2^0 = \frac{2\mu\tau}{a^2+b^2}b^2u, \quad F_3^0 = 0,$$

$$F_1' = -\frac{2\mu l\tau^2}{a^2+b^2}b^2u, \quad F_2' = - \quad 〃 \quad a^2v,$$

$$F_3' = A_{33}^{0,2} + A_{33}' + \frac{2\mu(a^2-b^2)}{(a^2+b^2)^2}\tau^2(b^2u^2 - a^2v^2) = h_0 + h_1u^2 + h_2v^2,$$

$$X_i = \int F_i dS = \int F_i^0 dudv + \int F_i' dudv = \begin{cases} 0, \ (i = 1, 2) \\ \int (h_0 + h_1 u^2 + h_2 v^2) dudv, \ (i = 3), \end{cases}$$

here also $X_3 = 0$ determines $\gamma_3$,

$$G_1 = G_2 = 0, \qquad G_3 = G_3^0.$$

Finaly substitution of (IV, 5) in (III, 19) gives the results:

$$\gamma_{32} = -\frac{1}{3}\frac{\lambda+\mu}{\lambda+2\mu}\tau^2, \qquad \gamma_{12} = \gamma_{22} = \frac{\tau^2}{2}, \qquad \gamma_{311} = -\gamma_{312} = -\frac{a^2-b^2}{a^2+b^2}\tau^2. \quad \text{(IV, 6)}$$

In the special case $a = b$ also from (III, 20) we obtain the results:

$$\xi_i' = \left(\frac{\tau^2}{2}x_3^2 + c - \frac{\lambda}{2(\lambda+\mu)}\beta_3 + Cr^2\right)x_i, \ (i = 1, 2)$$

$$\xi_3' = \beta_3 x_3$$

$$C = \tau^2 \frac{2\lambda + \mu - 4a_2 + a_4}{32(\lambda + 2\mu)}, \qquad c = -\tau^2 a^2 \frac{3\mu^2 + 4\mu a_2 + (2\lambda + 3\mu)a_4}{32(\lambda+\mu)(\lambda+2\mu)},$$

and the condition (III, 23) gives:

$$\beta_3 = \tau^2 a^2 \left(\frac{3}{16} - \frac{1}{4}\frac{a_2}{3\lambda + 2\mu} - \frac{1}{16}\frac{a_4(\lambda+2\mu)}{\mu(3\lambda+2\mu)}\right).$$

The corresponding results in I do not contain the terms due to $\beta_3$ so that the condition (III, 23) is not satisfied.

## V. Radial Vibration of a Sphere

Let

$$r^2 = x_k^2, \qquad \xi_j = x_j \cdot f(t, r),$$

then

$$\sigma_{ij} = f\delta_{ij} + \frac{1}{r}\frac{\partial f}{\partial r}x_i x_j, \qquad \sigma_{kk} = 3f + r\frac{\partial f}{\partial r}, \qquad \omega_{ij} = 0,$$

$$\begin{aligned} A_{ij} = \Big[ & (3\lambda + 2\mu)f + \lambda r \frac{\partial f}{\partial r} + \Big\{(9a_1 + 3a_2 + 3a_3 + a_4)f^2 \\ & + (6a_1 + 2a_2 + a_3)rf\frac{\partial f}{\partial r} + (a_1 + a_2)r^2\left(\frac{\partial f}{\partial r}\right)^2\Big\}\Big]\delta_{ij} \\ & + \Big[2\mu\frac{1}{r}\frac{\partial f}{\partial r} + \Big\{(3a_3 + 2a_4)\frac{1}{r}f\frac{\partial f}{\partial r} + (a_3 + a_4)\left(\frac{\partial f}{\partial r}\right)^2\Big\}\Big]x_i x_j \\ = {} & P\delta_{ij} + Qx_i x_j, \end{aligned}$$

and when we put:

$$\frac{\lambda + 2\mu}{\rho_0} = s^2, \qquad \frac{\lambda + 2\mu}{\mu} = \sigma^2,$$

$$2(3a_1 + a_2 + 2a_3 + a_4) = \rho_0 c_1, \qquad a_1 + a_2 + a_3 + a_4 = \rho_0 c_2,$$

$$2(4a_1 + 2a_2 + 4a_3 + 3a_4) = \rho_0 c_3, \quad 9a_1 + 3a_2 + 3a_3 + a_4 = \rho_0 c_4,$$

the equations of motion (II, 1) become:

$$\begin{aligned} & \frac{\partial^2 f}{\partial t^2} - 2f\frac{\partial^2 f}{\partial t^2} + 2\left(\frac{\partial f}{\partial t}\right)^2 + 2r\frac{\partial f}{\partial t}\frac{\partial^2 f}{\partial t \partial r} = \frac{1}{\rho_0}\left(\frac{1}{r}\frac{\partial P}{\partial r} + 4Q + r\frac{\partial Q}{\partial r}\right) \\ & = \frac{4s^2}{r}\frac{\partial f}{\partial r} + s^2\frac{\partial^2 f}{\partial r^2} + \left\{4c_1\frac{1}{r}f\frac{\partial f}{\partial r} + c_1 f\frac{\partial^2 f}{\partial r^2} + 2c_2 r\frac{\partial f}{\partial r}\frac{\partial^2 f}{\partial r^2} + c_3\left(\frac{\partial f}{\partial r}\right)^2\right\}. \quad \cdots\cdots \text{(V, 1)} \end{aligned}$$

For the free boundary surface we obtain:

$$y_k^2 = (x_k - \xi_k)^2 = r^2(1-f)^2 = a^2,$$
$$A_{ij}x_j = (P + r^2 Q)\, x_i,$$
$$\frac{1}{\rho_0}(P + r^2 Q) = s^2\left(3 - \frac{4}{\sigma^2}\right)f + s^2 r\frac{\partial f}{\partial r}$$
$$+\left\{c_1 r f\frac{\partial f}{\partial r} + c_2 r^2\left(\frac{\partial f}{\partial r}\right)^2 + c_4 f^2\right\} = 0, \qquad \cdots\cdots\cdots(\mathrm{V},2)$$

Let $f^0(t,r)$ be the solution of the 1st order quantities,

$$\left.\begin{aligned}&\frac{\partial^2 f^0}{\partial t^2} = \frac{4s^2}{r}\frac{\partial f^0}{\partial r} + s^2\frac{\partial^2 f^0}{\partial r^2},\\ &\left(3 - \frac{4}{\sigma^2}\right)f^0(t,\ a) + a\frac{\partial f^0}{\partial r}(t,a) = 0,\end{aligned}\right\} \qquad \cdots\cdots\cdots(\mathrm{V},3)$$

and putting

$$f = f^0 + f_1,$$

we obtain the following equation and condition for $f_1$:

$$\frac{\partial^2 f_1}{\partial t^2} - \left(\frac{4s^2}{r}\frac{\partial f_1}{\partial r} + s^2\frac{\partial^2 f_1}{\partial r^2}\right) = \left(2f^0\frac{\partial^2 f^0}{\partial t^2} - 2\left(\frac{\partial f^0}{\partial t}\right)^2 - 2r\frac{\partial f^0}{\partial t}\frac{\partial^2 f^0}{\partial t\partial r}\right)$$
$$+\left\{c_1\left(\frac{4}{r}f^0\frac{\partial f^0}{\partial r} + f^0\frac{\partial^2 f^0}{\partial r^2}\right) + 2c_2 r\frac{\partial f^0}{\partial r}\frac{\partial^2 f^0}{\partial r^2} + c_3\left(\frac{\partial f^0}{\partial r}\right)^2\right\}, \qquad \cdots\cdots(\mathrm{V},4)$$
$$\left(3 - \frac{4}{\sigma^2}\right)f_1 + a\frac{\partial f_1}{\partial r} = -a\left\{4\left(1 - \frac{1}{\sigma^2}\right)f_0\frac{\partial f_0}{\partial r} + af_0\frac{\partial^2 f_0}{\partial r^2}\right\}$$
$$-\frac{1}{s^2}\left\{c_1 a f^0\frac{\partial f^0}{\partial r} + c_2 a^2\left(\frac{\partial f^0}{\partial r}\right)^2 + c_4 f^{02}\right\}, \qquad \cdots\cdots\cdots(\mathrm{V},5)$$

where in (V, 5) we have to put $r = a$,

The general solution of (V, 3) is

$$f^0 = \sum C_n \cos(p_n t + \varepsilon_n)\,\psi_1\left(\frac{p_n}{s}r\right), \qquad \cdots\cdots\cdots(\mathrm{V},6)$$

where

$$\psi_1(z) = \frac{1}{z}\frac{d}{dz}\left(\frac{\sin z}{z}\right) = -\sqrt{\frac{\pi}{2}}\,\frac{1}{z^{3/2}}J_{3/2}(z),$$

and $\frac{p_n}{s}a$ is the $n$-th positive root of the equation:

$$\frac{tgx}{x} = \frac{1}{1 - \frac{\sigma^2}{4}x^2}.$$

When we put (V, 6) into (V, 4), the right side is of the form:

$$\sum F_{n,m}(r)\cos\{(p_n + p_m)t + (\varepsilon_n + \varepsilon_m)\} + {}^{*}\sum G_{n,m}(r)\cos\{(p_n - p_m)t + (\varepsilon_n - \varepsilon_m)\}$$
$$= F_0(r) + \sum_{l=1}^{\infty} F_l(r)\cos(q_l t + \varepsilon_l'),$$

$F_l(r)$ : regular functions of $r^2$, $(l \geqq 0)$.

similarly the right side of (V,5) is

$$a_0 + \sum a_l \cos(q_l t + \varepsilon_l').$$

The particular solution of (V, 4), corresponding to $F_l\cos(q_l t + \varepsilon_l')$ and $a_l\cos(q_l t + \varepsilon_l')$, which is finite at $r = 0$ is given by:

$$f_1 = \cos(q_l t + \varepsilon_l')\cdot g(r),$$

$$g'' + \frac{4}{r} g' + \frac{q_l^2}{s^2} g = -\frac{1}{s^2} F_l,$$

for $l \geqq 1,\ q_l \neq 0,$

$$g = A\psi_1\left(\frac{q_l}{s} r\right) - \frac{1}{s^2}\psi_1\left(\frac{q_l}{s} r\right)\int_0^{(q_l/s)r} \rho^4 \Phi_1(\rho) F_l\left(\frac{s}{q_l}\rho\right) d\rho$$
$$+ \frac{1}{s^2}\Phi_1\left(\frac{q_l}{s} r\right)\int_0^{(q_l/s)r} \rho^4 \psi_1(\rho) F_l\left(\frac{s}{q_l}\rho\right) d\rho,$$
$$\Phi_1(\rho) = \frac{1}{\rho}\frac{d}{d\rho}\left(\frac{\cos\rho}{\rho}\right),$$

for $l = 0,\ q_l = 0,\ g = A - \frac{1}{3s^2}\int_0^r rF_0(r)\,dr + \frac{1}{3s^2}\frac{1}{r^3}\int_0^r r^4 F_0(r)\,dr,$

and $A$ is to be determined from (V, 5).

## VI. Torsional Vibration of a Circular Cylinder

$$x_1^2 + x_2^2 = r^2,$$
$$\xi_1^0 = -\zeta(t, r, x_3)x_2, \qquad \xi_2^0 = \zeta x_1, \qquad \xi_3^0 = 0,$$
$$\sigma_{11}^0 = -\sigma_{22}^0 = -\frac{1}{r}\frac{\partial\zeta}{\partial r}x_1 x_2, \qquad \sigma_{33}^0 = \sigma_{kk}^0 = 0,$$
$$\sigma_{12}^0 = \frac{1}{2}\frac{1}{r}\frac{\partial\zeta}{\partial r}(x_1^2 - x_2^2), \qquad \omega_{12}^0 = \zeta + \frac{1}{2} r\frac{\partial\zeta}{\partial r},$$
$$\sigma_{13}^0 = -\omega_{13}^0 = -\frac{1}{2}\frac{\partial\zeta}{\partial x_3}x_2, \qquad \sigma_{23}^0 = -\omega_{23}^0 = \frac{1}{2}\frac{\partial\zeta}{\partial x_3}x_1.$$

The left side of (II, 4) can be written:

$$\rho_0\left[\frac{\partial^2\xi_i'}{\partial t^2} - \left\{2\left(\frac{\partial\zeta}{\partial t}\right)^2 + \zeta\frac{\partial^2\zeta}{\partial t^2}\right\}x_i\right], \qquad (i = 1, 2)$$
$$\rho_0\frac{\partial^2\xi_3'}{\partial t^2}, \qquad (i = 3)$$

and we calculate for $A_{ij}^{0,2}$'s:

$$A_{11}^{0,2} = A + Bx_2^2, \qquad A_{22}^{0,2} = A + Bx_1^2, \qquad A_{33}^{0,2} = C,$$
$$A_{12}^{0,2} = -Bx_1x_2, \qquad A_{13}^{0,2} = Ex_1, \qquad A_{23}^{0,2} = Ex_2,$$
$$A = \frac{1}{4}(2a_2 + a_4)r^2\left(\frac{\partial\zeta}{\partial r}\right)^2 + \frac{1}{4}(2a_2 - \lambda)r^2\left(\frac{\partial\zeta}{\partial x_3}\right)^2 - (\lambda + \mu)\omega_{12}^{02} - \mu r\frac{\partial\zeta}{\partial r}\omega_{12}^0,$$
$$C = \frac{1}{2}a_2 r^2\left(\frac{\partial\zeta}{\partial r}\right)^2 + \frac{1}{4}(2a_2 + a_4 - \lambda - 3\mu)r^2\left(\frac{\partial\zeta}{\partial x_3}\right)^2 - \lambda\omega_{12}^{02},$$
$$B = \frac{1}{4}(a_4 + \mu)\left(\frac{\partial\zeta}{\partial x_3}\right)^2 + 2\mu\frac{1}{r}\frac{\partial\zeta}{\partial r}\omega_{12}^0,$$
$$E = \left\{\frac{1}{4}(a_4 - \mu)r\frac{\partial\zeta}{\partial r} - \mu\omega_{12}^0\right\}\frac{\partial\zeta}{\partial x_3}.$$

With these values the equations of motion become:

$$\left.\begin{aligned} \rho_0\frac{\partial^2\xi_j'}{\partial t^2} &= \frac{\partial A_{ij}'}{\partial x_j} + x_i\left[\rho_0\left\{2\left(\frac{\partial\zeta}{\partial t}\right)^2 + \zeta\frac{\partial^2\zeta}{\partial t^2}\right\} + \left(\frac{1}{r}\frac{\partial A}{\partial r} - B + \frac{\partial E}{\partial x_3}\right)\right], \quad (i = 1, 2) \\ \rho_0\frac{\partial^2\xi_3'}{\partial t^2} &= \frac{\partial A_{3j}'}{\partial x_j} + \left(\frac{\partial C}{\partial x_3} + 2E + r\frac{\partial E}{\partial r}\right). \end{aligned}\right\} \qquad \cdots\cdots\cdots(\mathrm{VI}, 1)$$

The equation of the surface of lateral boundary is:

$$a^2 = y_1^2 + y_2^2 = (x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 = (x_1 - \xi_1^0)^2 + (x_2 - \xi_2^0)^2$$
$$= x_1^2 + x_2^2 - 2(x_1\xi_1^0 + x_2\xi_2^0) = x_1^2 + x_2^2,$$

similarly those for the terminal boundaries are:

$$\left.\begin{matrix} l, \\ 0 \end{matrix}\right\} = y_3 = x_3 - \xi_3^0 = x_3,$$

so that the boundaries do not change their position and the boundary conditions are:

$$\left.\begin{array}{l} A'_{j1}x_1 + A'_{j2}x_2 = -(A^{0,2}_{j1}x_1 + A^{0,2}_{j2}x_2) \quad (j = 1, 2, 3) \quad \text{at} \quad r = a \\ \text{and} \\ A'_{j3} = A^{0,2}_{j3} \ (j = 1, 2, 3) \quad \text{at} \quad x_3 = \begin{cases} l, \\ 0. \end{cases} \end{array}\right\} \qquad \cdots\cdots\cdots (\text{VI}, 2)$$

We take for $\zeta$ the sinple case:

$$\zeta = C_n \cos\left(\frac{n\pi}{l}\sqrt{\frac{\mu}{\rho_0}}\, t\right)\cos\frac{n\pi}{l}x_3, \qquad \frac{\pi}{l}\sqrt{\frac{\mu}{\rho_0}} = \kappa,$$

then the inhomogeneous equations (VI, 1) have the following particular solution:

$$\xi'_i = x_i\left\{p_1 r^2 + p_2\cos\frac{2n\pi}{l}x_3 + p_3\cos 2n\kappa t + p_4 r^2\cos 2n\kappa t\cos\frac{2n\pi}{l}x_3\right\}, \quad (i = 1, 2)$$

$$\xi'_3 = \{(p_5 + p_6 r^2) + (p_7 + p_8 r^2)\cos 2n\kappa t\}\sin\frac{2n\pi}{l}x_3,$$

where $p_1, \cdots, p_8$ are constants, and we have to add the solution of the homogeneous equations to satisfy the boundary conditions (VI, 2).