主論文

Nonstandard arithmetic of function fields over H-convex subfields of *Q

(代数関数体の超準整数論)

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1. Let $\mathbf{Q}$ be the set of rational numbers and $^*\mathbb{Q}$ an enlargement of $\mathbf{Q}$. A subfield Q_1 of $^*\mathbf{Q}$ is *H*-convex if $x \in Q_1$ and $H(x) \geq H(y)$ imply $y \in Q_1$ where H is the height function of $^*\mathbf{Q}$; i.e. $H(n/m) = \max(|n|, |m|)$. In this paper, we are concerned with a function fields F of one variable over a *H*-convex subfield Q_1 of $^*\mathbf{Q}$;

$$\mathbb{Q} \subseteqq Q_1 \subset F \subset {}^*\mathbf{Q}.$$

If $Q_1 = \mathbf{Q}$, then the situation is the same as in Robinson-Roquette[5]. Hence we assume that the reader is familiar with [5].

2. Main Theorems. Let $f(X, Y, T_1, \ldots, T_k)$ be a polynomial whose coefficients are contained in $\mathbf{Q}$. For each sequence $\boldsymbol{n} = (n_1, n_2, \ldots, n_k)$ of k integers, we let $C_{\boldsymbol{n}}$ be the plane algebraic curve defined by the equation

$$f(X, Y, n_1, n_2, \ldots, n_k) = 0.$$

Assume $C_{\boldsymbol{n}}$ is irreducible, then let $F_{\boldsymbol{n}} = \mathbb{Q}(x_{\boldsymbol{n}}, y_{\boldsymbol{n}})$ be the function field of $C_{\boldsymbol{n}}$ where $x_{\boldsymbol{n}}$ is transcendental over $\mathbb{Q}$ and $f(x_{\boldsymbol{n}}, y_{\boldsymbol{n}}, n_1, n_2, \ldots, n_k) = 0$. By a *functional prime* of $C_{\boldsymbol{n}}$, we mean an equivalence class of nontrivial valuations of $F_{\boldsymbol{n}}$ which are trivial on $\mathbb{Q}$. An *infinite prime* of $C_{\boldsymbol{n}}$ is a functional prime which is a pole of $x_{\boldsymbol{n}}$ or $y_{\boldsymbol{n}}$.

By a arithmetical prime divisor of $\mathbf{Q}$, we mean a prime number or the archimedean prime. Let $S = \{p_1, p_2, \ldots, p_s\}$ be a set of s prime divisors of $\mathbf{Q}$. We mean by a *S-integer* a rational number r such that $|r|_p > 1$ implies $p \in S$.

Theorem 1. *There exists $N \in \mathbb{N}$ such that if $C_{\boldsymbol{n}}$ is irreducible and has at least $s+1$ infinite primes, then any S-integer point (x, y) on $C_{\boldsymbol{n}}$ satisfies*

$$\max(H(x), H(y)) < \max(2, |n_1|, |n_2|, \ldots, |n_k|)^N.$$

It should be noted that N is determined by f only and independent from the choise of $\mathfrak{m}$ and S.

Let V be an irreducible surface defined over $\mathbb{Q}$. We denote by $V_{\mathbf{Q}}$ the set of all rational points on V. Let G be the function field of V over $\mathbb{Q}$.

Theorem 2. *Let $t\in G$ be transcendental over $\mathbb{Q}$ and $u,v\in G$ transcendental over $\mathbb{Q}(t)$. For any $\varepsilon>0$ there is $N\in\mathbb{N}$ such that for any $P\in V_{\mathbb{Q}}$ if*

$$\min(H(u(P)),H(v(P)))>H(t(P))^N>N^N,$$

then

$$\left|\frac{\log H(u(P))}{\log H(v(P))}-\frac{[G:\mathbb{Q}(t,u)]}{[G:\mathbb{Q}(t,v)]}\right|<\varepsilon.$$

3. First we prove the finiteness of S-integer points on an irreducible curve with at least $s+1$ infinite primes.

Proposition 1. *Let C be an irreducible curve defined over $\mathbb{Q}$. If C has at least $s+1$ infinite primes, then there are only finitely many S-integer points on C.*

Proof. If there are infinitely many S-integer points on C, then there is a nonstandard S-integer point (x,y) on C. By Lemma 4.1[5], any functional prime is induced by some arithmetical prime. Since x and y are S-integer, any infinite prime is induced by a prime from S. Therefore there are at most s infinite primes.

4. Let Q_1 be a H-convex subfield of $^*\mathbf{Q}$ and $Z_1=Q_1\cap{}^*\mathbb{Z}$.

Lemma 1. *Q_1 is algebraically closed in $^*\mathbf{Q}$.*

Proof. Let $\alpha/\beta\in{}^*\mathbf{Q}-Q_1$ where $\alpha,\beta\in{}^*\mathbf{Z}$ and $(\alpha,\beta)=1$. Then $\alpha\notin Z_1$ or $\beta\notin Z_1$. We may assume $\alpha\notin Z_1$. If α/β is algebraic over Q_1, then there are $\gamma_0,\gamma_1,\dots,\gamma_n\in Z_1$ such that $\gamma_n\neq 0$ and

$$\gamma_0\left(\frac{\alpha}{\beta}\right)^n+\gamma_1\left(\frac{\alpha}{\beta}\right)^{n-1}+\dots+\gamma_n=0.$$

Then

$$\gamma_0\alpha^n+\gamma_1\alpha^{n-1}\beta+\ldots+\gamma_n\beta^n=0.$$

Hence

$$\gamma_n\beta^n\equiv 0 \qquad \text{mod } \alpha.$$

Since $(\alpha,\beta)=1$

$$\gamma_n\equiv 0 \qquad \text{mod } \alpha.$$

Z_1 is convex and $\alpha\notin Z_1$, therefore $|\gamma_n|<|\alpha|$. Hence

$$\gamma_n=0.$$

This is a contradiction.

Let $R=\{\beta/\alpha\in{}^*\mathbf{Q} \mid \alpha\in Z_1,\ \beta\in{}^*\mathbf{Z}\}$ and I a maximal ideal of R. Then the local ring of I is a valuation ring which we denote by R_I. For, let $\beta/\alpha\notin R_I$ where $(\alpha,\beta)=1$. Then $\alpha\in I$. Since $(\alpha,\beta)=1$, $\beta\notin I$. Therefore $\alpha/\beta\in R_I$.

Let $R_\infty=\{\beta/\alpha\in{}^*\mathbf{Q} \mid |\beta/\alpha|<\gamma \text{ for some } \gamma\in Z_1\}$, then R_∞ is a valuation ring whose maximal ideal is $\{\beta/\alpha\in{}^*\mathbf{Q} \mid |\beta/\alpha|<1/\gamma \text{ for all } \gamma\in Z_1\}$.

Let F be a subfield of $^*\mathbf{Q}$ which is a function field of one variable with its constant field Q_1. If $F\cap R_\infty$ is not trivial, namely $F\not\subset R_\infty$, then $F\cap R_\infty$ is a valuation ring. Since $F\cap R_\infty\supset Q_1$, this valuation ring yields a functional prime P of F. We say that P is induced by the archimedean prime. Let I be a maximal ideal of R. If $F\cap R_I$ is not trivial, then $F\cap R_I$ also yields a functional prime P of F. We say that P is induced by I. For any standard or nonstandard prime number p, let

$$I_p=\{\beta/\alpha\in R \mid |\beta/\alpha|_p<1/|\gamma| \text{ for all } \gamma\in Z_1\}.$$

Then I_p is a maximal ideal of R. If a functiona prime P is induced by I_p, we simply say that P is induced by p.

Lemma 2. [c.f. [5], Lemma 4.1] *Every functional prime P is induced by the archimedean prime or a maximal ideal of R.*

Proof. By the theorem of Riemann-Roch, there exists $\beta/\alpha\in F$ which admits P as its only pole. If $|\beta/\alpha|>\gamma$ for all $\gamma\in Z_1$, then $\beta/\alpha\notin R_\infty$. Hence $\beta/\alpha\notin R_\infty\cap F$. Then the functional prime induced by the archimedean prime is a pole of β/α. Since P is the only functional prime which is a pole of β/α, P is induced by the archimedean prime. If $|\beta/\alpha|<\gamma$ for some $\gamma\in Z_1$, then $\alpha\in{}^*\mathbf{Z}-Z_1$ because β/α is a nonconstant. Let I_α be a maximal ideal of R which contains α. Then the local ring of I_α does not contain β/α. Then $\beta/\alpha\notin R_{I_\alpha}\cap F$. By the same arguments as above, P is induced by I.

5. *Proof of Theorem 1.* Suppose Theorem 1 is false. Then for any natural number N, there exist a finite set S of s prime divisors, a sequence $\boldsymbol{n}=(n_1,n_2,\ldots,n_k)$ of k integers and an S-integer point (x,y) on $C_{\boldsymbol{n}}$ such that $C_{\boldsymbol{n}}$ is irreducible and has at least $s+1$ infinite primes but

$$\max(H(x),H(y))>\max(2,|n_1|,|n_2|,\ldots,|n_k|)^N. \qquad (1)$$

If the number s of primes in S is larger than the degree d of the generic curve defined by $f(X,Y,t_1,t_2,\ldots,t_k)=0$ where $t_1,t_2,\ldots,t_k$ are indeterminates over $\mathbf{Q}$, then there is no $\boldsymbol{n}=(n_1,n_2,\ldots,n_k)$ such that $C_{\boldsymbol{n}}$ has at least $s+1$ infinite primes. Therefore s is not larger than d. By nonstandard principle, above assertion holds for any enlargement. We take $N\in{}^*\mathbb{N}$ to be nonstandard. Let Q_1 be the smallest H-convex subfields of ${}^*\mathbb{Q}$ which contains $n_1,n_2,\ldots,n_k$, namely

$$Q_1=\{z\in{}^*\mathbf{Q}\mid\ H(z)<\max(2,|n_1|,|n_2|,\ldots,|n_k|)^i \text{ for some } i\in{}^*\mathbb{N}\}.$$

By (1), x or y is not in Q_1. Hence $Q_1(x,y)$ is the function field of C_n contained in $^*\mathbf{Q}$ because Q_1 is algebraically closed in $^*\mathbf{Q}$. We will prove that any infinite prime is induced by some arithmetical prime in S. Let P be an infinite prime of C_n. We may assume that P is a pole of x. Since x is an S-integer,

$$x = \frac{\beta}{p_1^{\alpha_1}\cdots p_j^{\alpha_j}}$$

where $p_1,\ldots,p_j \in S$, $\beta \in {}^*\mathbf{Z}$ and $\alpha_1,\ldots,\alpha_j \in {}^*\mathbf{N}$. By lemma 2, P is induced by a maximal ideal of R or the archimedean prime. First, assume P is induced by a maximal ideal I. Since P is a pole of x, $1/x \in IR_I$. Hence $p_1^{\alpha_1}\cdots p_j^{\alpha_j} \in IR_I$. Since $j\,(\leq s \leq d)$ is standard and I is prime, $p_i^{\alpha_i} \in IR_I$ for some $i \leq j$. This means $I = I_{p_i}$. Second, assume P is induced by the archimedean prime. Since P is a pole at x,

$$\left|\frac{1}{x}\right| < \left|\frac{1}{\gamma}\right|$$

for all $\gamma \in Z_1$ because $\{\alpha \in {}^*\mathbf{Q} \mid |\alpha| < 1/\gamma \text{ for all } \gamma \in Z_1\}$ is the maximal ideal of R_∞. Since x is an S-integer, S contains the archimedean prime.

Since any infinite prime is induced by some prime in S, the number of infinite primes of C_n is not more than s. On the other hand, by nonstandard principle C_n has at least $s+1$ infinite primes in the nonstandard model. Since the number of infinite primes is absolute C_n has at least $s+1$ infinite primes. This is a contradiction.

6. We denote by $^*\mathbb{R}$ the enlargement of $\mathbb{R}$. Let

$$R_1 = \{\alpha \in {}^*\mathbb{R} \mid |\alpha| < \beta \text{ for some } \beta \in Q_1\}$$

$$R_2 = \{\alpha \in {}^*\mathbb{R} \mid |\alpha| < \log|\beta| \text{ for some } \beta \in Q_1\}.$$

Then R_1 and R_2 are convex in $^*\mathbb{R}$. For any $\alpha, \beta \in {}^*\mathbb{R}$, we write

$$\alpha \overset{1}{\leqq} \beta \quad \textit{if } \alpha-\beta \leqq \gamma \textit{ for some } \gamma \in R_1$$

$$\alpha \overset{2}{\leqq} \beta \quad \textit{if } \alpha-\beta \leqq \gamma \textit{ for some } \gamma \in R_2.$$

In the case $Q_1=\mathbf{Q}$, $\overset{1}{\leqq}$ and $\overset{2}{\leqq}$ are the same as $\dot{\leqq}$ (c.f.[ ]). The factor groups of $^*\mathbb{R}$ by R_1 and R_2 are denoted by R^1 and R^2 respectively. Let D be the set of arithmetical divisors and $^*D$ its enlargement. Let $a=\sum_p \alpha_p \cdot p \in D$ where $\alpha_p \in {}^*\mathbf{Z}$ if p is a nonarchimedean prime and $\alpha_p \in {}^*\mathbb{R}$ if p is archimedean. For each prime divisor p, we define

$$\| a \|_p = Np^{-\alpha_p}$$

where $Np=e$ (base of the logarithm) if p is archimedean. The number

$$\sigma(a) = \sum_p -\log \| a \|_p = \sum_p w_p(a)$$

is called the additive size of a divisor a. Let D_2 be the set of divisors of $^*D$ whose additive size is in R_2, i.e.

$$D_2 = \{ a \in {}^*D \mid \sigma(a) \in R_2 \}. \tag{2}$$

D_2 forms an isolated subgroup and the factor group

$$D^2 = {}^*D/D_2$$

has a canonical order, i.e.

$$a \overset{2}{\leqq} b \quad \textit{if } a-b \leqq c \textit{ for some } c \in D_2.$$

By the definition of $^*D$, we have

$$^*D = {}^*D' + {}^*D''$$

where $^*D'$ is the archimedean part and $^*D''$ is the nonarchimedean part. Any element of $^*D''$ can be considered to be a fractional ideal of $^*\mathbb{Q}$. Hence we don't distinguish between divisors in $^*D''$ and fractional ideals of $^*\mathbf{Q}$. Let $a \in {}^*D''$ be such that $a \overset{2}{>} 0$. Then aR is an ideal of R. Remark (c.f.[5] Lemma 3.3) that it may happen that

$$w_p(a) \overset{2}{=} w_p(b) \quad \textit{for any prime } p$$

but

$$a \overset{2}{\neq} b.$$

7. According to section 1, we have

$$\mathbb{Q} \subseteqq Q_1 \subset F \subset {}^*\mathbb{Q}$$

where F is a function field of one variable over Q_1. Let P be a functional prime which is induced by an atithmetical prime p. Let $\pi \in F$ be a uniformizing variable at P; i.e., $v_P(\pi)=1$ and

$$\rho_p = v_p(\pi)\log(Np)/\deg P.$$

Then $\rho_p^2 > 0$ and

$$w_p(x) = \rho_p w_P(x) \qquad (3)$$

for all $x \in F$.

Let D be the set of divisors of F over Q_1. Our aim of this section is to construct a map

$$i : D \longrightarrow D^2$$

which is an injective homomorphism and satisfies strong order preservation, maximal preservation, minimam preservation and principal divisor preservation. We construct the map by the same way as in [5] lemma 4.2 but as pointed out at the last of section ., iA is not uniquely determined by its p-adic values.

For any functional divisor $A = \sum_P \alpha_P \cdot P$, we define

$$v_P(A) = \alpha_P$$

and

$$w_P(A) = v_P(A)\deg(P).$$

Let us assume $A > 0$. By the approximation theorem of valuation, there exist nonzero elements $x, y \in F$ which satisfies the following three conditions,

$$\left.\begin{array}{ll} w_P(x)=w_P(A) & \text{if } w_P(A)>0 \\ w_P(y)=w_P(A) & \text{if } w_P(A)>0 \\ w_P(y)=0 & \text{if } w_P(A)=0 \text{ and } w_P(x)>0. \end{array}\right\} \qquad (4)$$

Then for every functional prime P,

$$\max(0,\min(w_P(x),w_P(y)))=w_P(A).$$

Hence

$$\max(0,\min([x],[y]))=A$$

where $[x],[y]$ denote the functional principal divisors of x,y. We put

$$iA=\max(0,\min((x),(y)))$$

where $(x),(y)$ are the arithmetical principal divisors of x,y. We must show that the above construction of iA does not depend on the choice of x,y. Let s,t be elements of F satisfying the same conditions (4) of x,y. We will prove

$$\max(0,\min((x),(y)))\geq\max(0,\min((s),(t))).$$

Suppose not. If there exists a functional prime P induced by the archimedean prime p_∞, then

$$\begin{aligned}\max(0,\min(w_{p_\infty}(x),w_{p_\infty}(y)) &\geq \rho_{p_\infty}\max(0,\min(w_P(x),w_P(y))\\ &= \rho_{p_\infty}\max(0,\min(w_P(s),w_P(t))\\ &\geq \max(0,\min(w_{p_\infty}(s),w_{p_\infty}(t)).\end{aligned}$$

If there is no functional prime induced by the archimedean prime, then

$$\max(0,\min(w_{p_\infty}(x),w_{p_\infty}(y))\geq 0\geq\max(0,\min(w_{p_\infty}(s),w_{p_\infty}(t)).$$

Hence we have

$$\max(0,\min(\langle x\rangle,\langle y\rangle))\not\geq\max(0,\min(\langle s\rangle,\langle t\rangle))$$

where $\langle z\rangle$ denote the nonarchimedean part of the principal divisor (z) ; i.e., the principal ideal of z. Let

$$a_0-a_1=\max(0,\min(\langle x\rangle,\langle y\rangle))-\max(0,\min(\langle s\rangle,\langle t\rangle))$$

where a_0 and a_1 are positive divisors. We may assume $a_0\gneq 0$. Then a_0R is considered to be an ideal of R. We denote by $\langle z\rangle_0$ the positive part of $\langle z\rangle$; i.e., $\langle z\rangle_0=\max(0,\langle z\rangle)$. Then for some $a_0'\gneq 0$ with $a_0'\leq a_0$, one of $\langle x/s\rangle_0,\langle y/s\rangle_0,\langle x/t\rangle_0,\langle y/t\rangle_0\geq a_0'$. We may assume without loss of generality

$$<\frac{x}{s}>_0 \overset{2}{\geq} a'_0 \overset{2}{>} 0.$$

Let I be a maximal ideal of R including $a'_0R \supseteq a_0R$. Then $s/x \notin R_I$. Hence I induces a functional prime P appearing in $[x/s]_0$. By the choise of x and s, $[x/s]_0$ has no prime P in A. Hence P does not induce any prime in A. On the other hand,

$$\max(0,\min((x),(y))) \overset{2}{\geq} a_0 \overset{2}{>} 0$$

Then any maximal ideal I including a_0R induces a prime in $A=\max(0,\min([x],[y]))$. This is a contradiction. Therefore

$$\max(0,\min((x),(y))) \overset{2}{=} \max(0,\min((s),(t))).$$

Now we have proved that the map $i: D \longrightarrow D^2$ is well defined. Next we prove that i is a homomorphism; in other words

$$i(A+B)=i(A)+i(B).$$

Let P be a functional prime and x,y elements which satisfy the conditions (4) replacing A by P. Hence

$$\begin{aligned} i(nP) &\overset{2}{=} \max(0,\min((x^n),(y^n))) \\ &\overset{2}{=} n\max(0,\min((x),(y))) \qquad (5) \\ &\overset{2}{=} ni(P). \end{aligned}$$

Let A and B be positive divisors which have no common prime divisors. Let $x,y,z \in F$ be such that

$$\begin{aligned}
&w_P(x)=w_P(A) && \text{if } w_P(A)>0\\
&w_P(x)=0 && \text{if } w_P(B)>0\\
&w_P(y)=w_P(B) && \text{if } w_P(B)>0\\
&w_P(y)=0 && \text{if } w_P(A)>0\\
&w_P(y)=0 && \text{if } w_P(B)=0 \text{ and } w_P(x)>0\\
&w_P(z)=w_P(A) && \text{if } w_P(A)>0
\end{aligned}$$

$$w_P(z)=w_P(B) \qquad \text{if } w_P(B)>0$$

$$w_P(z)=0 \qquad \text{if } w_P(A)=w_P(B)=0 \text{ and } w_P(x)>0$$

$$w_P(z)=0 \qquad \text{if } w_P(A)=w_P(B)=0 \text{ and } w_P(y)>0$$

Then

$$A=\max(0,\min([x],[z]))$$

$$B=\max(0,\min([y],[z]))$$

$$A+B=\max(0,\min([xy],[z])).$$

Lemma 3. $\min((x)_0,(y)_0,(z)_0)\overset{2}{=}0$ *where* $(x)_0,(y)_0,(z)_0$ *are the positive parts of* $(x),(y),(z)$ *respectively.*

Proof. If not, there exists $a\overset{2}{>}0$ such that

$$\min((x)_0,(y)_0,(z)_0)\overset{2}{\geqq}a\overset{2}{>}0.$$

Let $a=a"+rp_\infty$ where $a"$ is the nonarchimedean part of a and rp_∞ the archimedean part of a. If $a"\overset{2}{>}0$, then let I be a maximal ideal of R including $a"R$. Then $x^{-1},y^{-1},z^{-1}\notin R_I$. Hence I induces a prime P such that $\min(w_P(x),w_P(y),w_P(z))>0$. It is impossible by the choise of x,y,z. If $rp_\infty\overset{2}{>}0$, then let P be the prime induced by the archimedean prime p_∞. Then $\min(w_P(x),w_P(y),w_P(z))>0$. This is a contradiction. Now Lemma 3 is proved.

$$\begin{aligned} iA+iB &\overset{2}{=}\max(0,\min((x),(z)))+\max(0,\min((y),(z))) \\ &\overset{2}{=}\min((x)_0,(z)_0)+\min((y)_0,(z)_0) \\ &\overset{2}{=}\min((x)_0+(y)_0,(z)\ ) \qquad \text{(by Lemma 3)} \qquad (6) \\ &\overset{2}{=}\max(0,\min((xy),(z))) \\ &\overset{2}{=}i(A+B). \end{aligned}$$

Then by (5) and (6), i is a homomorphism.

Next we prove that i preserves strong order; i.e.,

$$iA \overset{2}{\leqq} iB \longrightarrow A\leqq B.$$

First we prove

Lemma 4. *For each functional prime* P,

$$iP \overset{2}{\geqq} 0.$$

Proof. Let $iP=a-b+rp_\infty$ where a and b are nonarchimedean positive divisors. If $rp_\infty \overset{2}{<} 0$, then p_∞ induces a functional prime P' with $w_{P'}(P)<0$, this is impossible. Hence $rp_\infty \overset{2}{\geqq} 0$.

Therefore it suffices to prove $b \overset{2}{=} 0$. Suppose $b \overset{2}{>} 0$. Then bR is an ideal of R. Let I be a maximal ideal including bR. Then there is a functional prime P' induced by I such that $w_{P'}(P)<0$. This is a contradiction.

Let $A=\sum_P \alpha_P P$. Since i is a homomorphism,

$$A \geqq 0 \longrightarrow i(A) \overset{2}{=} \sum_P \alpha_P i(P) \overset{2}{\geqq} 0.$$

Hence $B \geqq A$ implies $i(B) \overset{2}{\geqq} i(A)$. To prove the converse, we show

Lemma 5. *If $P \neq P'$, then*

$$\min(i(P), i(P')) \overset{2}{=} 0.$$

Proof. By Lemma 4, $\min(i(P), i(P')) \overset{2}{\geqq} 0$. If $\min(i(P), i(P')) \overset{2}{\neq} 0$, then there is a divisor $a \overset{2}{>} 0$ such that

$$\min(i(P), i(P')) \overset{2}{\geqq} a \overset{2}{>} 0.$$

Let I be a maximal ideal of R including aR. Then the prime P'' induced by I satisfies $w_{P''}(P)>0$ and $w_{P''}(P')>0$, this is a contradiction.

Let $A=\sum_P \alpha_P P$. If $A \not\geqq 0$, then there is a P' with $\alpha_{P'} < 0$. By Lemma 5,

$$\sum_{P \neq P'} \alpha_P P \overset{2}{\not\geqq} |\alpha_{P'}| P'.$$

Hence $i(A) = \sum_{P \neq P'} \alpha_P P + \alpha_{P'} P' \overset{2}{\not\geqq} 0$. Therefore $i(B) \geqq i(A)$ implies $B \geqq A$. We have proved that i preserves strong order. By the same way as above, we can easily prove that the map i satisfies maximum preservation, and minimum preservation. Now we have

Proposition 2. [c.f. [5] Lemma 4.2] *There is an injective homomorphism*

$$i: D \longrightarrow D^2$$

which satisfies strong order preservation

$$A \geqq B \longrightarrow i(A) \geqq i(B)$$

maximum preservation

$$i\max(A,B) \overset{2}{=} \max(i(A), i(B))$$

minimum preservation

$$i\min(A,B) \overset{2}{=} \min(i(A), i(B))$$

principal divisor preservation

$$i[x] \overset{2}{=} (x).$$

8. We identify D with its image $iD \subset D^2$. Then we obtain the size map $\sigma: D \longrightarrow {}^*\mathbb{R}^2$ which is induced by the size map $\sigma: {}^*D \longrightarrow {}^*\mathbb{R}$.

Lemma 6. [c.f. [5] Theorem 4.4] *There exists a* $\rho \in {}^*\mathbb{R}$ *with* $\rho \overset{2}{>} 0$ *such that*

$$\frac{\sigma(A)}{\rho} \simeq \deg(A)$$

for all functional divisor $A \in D$ *where the symbol* $\simeq$ *means "infinitely close".*

Proof. First we prove that $\sigma(A) \overset{2}{>} 0$ for every $A>0$. If $A>0$, then by Proposition 2, $A \overset{2}{>} 0$ *in* D^2. By the definition (2) of D_2, $\sigma(a) \overset{2}{>} 0$ for every divisor $a \overset{2}{>} 0$. Hence $\sigma(A) \overset{2}{>} 0$.

The rest of the proof is the same as that of [5] Theorem 4.4. Remark that we use $\overset{2}{>}, \overset{2}{=}$ instead of $\dot{>}, \doteq$ respectively

Corollary. *For any* $A, B \in D$

$$\frac{\sigma(A)}{\sigma(B)} \simeq \frac{\deg(A)}{\deg(B)} .$$

By the same way as in p155 [5], we have

Corollary. *For nonconstants* $x, y \in F$

$$\frac{\log H(y)}{\log H(x)} \simeq \frac{[F:Q_1(y)]}{[F:Q_1(x)]} .$$

9. *Proof of Theorem 2.* Suppose Theorem 2 false. Then there is is $\varepsilon>0$ which satisfies the following condition;

(*) for any $N\in\mathbf{N}$*, there is a rational point* P *on* V *such that*

$$(H(u(P)),H(v(P)))>H(t(P))^N>N^N$$

and

$$\left|\frac{\log H(u(P))}{\log H(v(P))}-\frac{[G:\mathbf{Q}(t,u)]}{[G:\mathbf{Q}(t,v)]}\right|>\varepsilon. \tag{7}$$

By nonstandard principle, *(*)* holds for any nonstandard model. Choose $N\in{}^*\mathbf{N}$ to be nonstandard. Since $H(t(P))^N>N^N$, P must be a nonstandard rational point of V; i.e. $P\ {}^*(V_{\mathbf{Q}})=V_{^*\mathbf{Q}}$. Let Q_1 be the smallest H-convex subfield of $^*\mathbf{Q}$ including $t(P)$;

$$Q_1=\{r\in{}^*\mathbf{Q}\mid H(r)<H(t(P))^n \text{ for some } n\in\mathbf{N}\}.$$

Let $x=u(P)$ and $y=v(P)$. Then $x,y\notin Q_1$ because $H(x),H(y)\ H(t(P))^N$. Let $F=Q_1(x,y)$.

$$\frac{\log H(x)}{\log H(y)}\simeq\frac{[F:Q_1(x)]}{[F:Q_1(y)]}=\frac{\deg([x]_\infty)}{\deg([y]_\infty)}=\frac{[G:K(u)]}{[G:K(v)]}=\frac{[G:\mathbf{Q}(t,u)]}{[G:\mathbf{Q}(t,v)]}$$

where K is the algebraic closure of $\mathbf{Q}(t)$ in G. This contradicts (7).

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