主論文 ベクトル束の理論における いくつかの結果 梅村浩

Some results in the theory of vector bundles

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We have several definitions of the positivity of a vector bundle, differentiable definitions, an algebro-geometric definition, a topological definition etc. In §1 we review the definitions and the relations between them. For a line bundle all the definitions are equivalent and every one agrees that they are reasonable. For a vector bundle, however, the definitions are not necessarily equivalent. One of the main results of this paper is the equivalence of the definitions over a complete non-singular curve. The proof is given in §2. We proved this over an elliptic curve in Umemura [18]. In this case the proof was based on Atiyah's classification. To prove the equivalence over a curve of genus ≥ 2, the fundamental lemma is ; A stable bundle of positive degree is positive in the sense of Nakano. The tool used to prove this lemma is the theory of stable bundles due to Narasimhan and Seshadri [11] —they establish a correspondence between stable bundles and certain types of irreducible unitary representations of a Fuchsian group.

We also discuss the H-stability of Takemoto from two points of view. In §3, we prove that over an abelian surface, a ruled surface or a hyperelliptic surface, H-stable bundle of rank 2 with $c_1^2-4c_2 \geq 0$, $c_1 > 0$, $c_2 > 0$ is positive in the sense of Nakano. We ask in general : Is a stable bundle of rank 2 over a surface with positive Chern class ample? This is the analogue of the lemma that we mentioned above. But this is false unless $c_1^2-4c_2 \geq 0$, even over

an abelian surface. Hence H-stability is not very comfortable in this case.

In §4, we deal with vanishing theorems. The first theorem is well known as the index theorem. In fact an algebraic proof is known. The second theorem (4.2) is a generalization of the Kodaira vanishing theorem and Mumford's result [8]. We also remark that a vanishing theorem of Griffiths is proved easily by the standard argument using cohomological dimension and the spectral sequence of de Rham cohomology.

In §5 we study $H(E)$ and $\mathcal{G}(E)$ for a vector bundle E. In general $H(E)$ and $\mathcal{G}(E)$ are too small. So we have to find a good family of vector bundles E such that $H(E)$ and $\mathcal{G}(E)$ reflect properties of E. Over an elliptic curve, if E is stable, $H(E)$ and $\mathcal{G}(E)$ are nice and give the Heisenberg group. So it is quite natural to ask if $H(E)$ and $\mathcal{G}(E)$ give sufficient information concerning E for an H-stable bundle E over an abelian surface. Unfortunately the answer is no unless $c_1^2 - 4c_2 = 0$. Here again the H-stability with $c_1^2 - 4c_4 < 0$ is unpleasant.

§1 Preliminaries

(1.1) Let V be a non-singular projective algebraic variety of dimension n defined over the complex number field $\mathbb{C}$. Let E be a holomorphic vector bundle of rank r defined over V. Let $\{U_\alpha\}$ be an open covering of V such that E is trivial on each U_α. Let $g_{\beta\alpha}$ be the transition matrix of E i.e. two elements $x \times \xi_\alpha$ and $x \times \xi_\beta$ with $x \in U_\alpha \cap U_\beta$, $\xi_\alpha, \xi_\beta \in \mathbb{C}^r$ are identified if and only if $g_{\beta\alpha}(x)\xi_\alpha = \xi_\beta$.

A hermitian metric on E is, by definition, a set of C^∞-maps h_α from U_α to the space of positive definite hermitian matrices of degree r such that ${}^t\overline{g}_{\beta\alpha}(z)h_\alpha(z)g_{\beta\alpha}(z) = h_\beta(z)$ for any $z \in U_\alpha \cap U_\beta$.

A (C^∞-) connection on E is a set of 1-forms θ_α on U_α such that $\theta_\alpha = g_{\alpha\beta}(\theta_\beta - \omega_{\alpha\beta})g_{\alpha\beta}^{-1}$ on $U_\alpha \cap U_\beta$ where $\omega_{\alpha\beta} = g_{\alpha\beta}^{-1}dg_{\alpha\beta}$. It is easy to see that a connection defines a $\mathbb{C}$-linear map $D : E \to \Omega^1(E)$ by putting

$$D(\varphi_\alpha) = d\varphi_\alpha + \theta_\alpha \wedge \varphi_\alpha$$

for a local section φ_α on each U_α. Similarly we can define two operators :

$$D' : E \longrightarrow \Omega^1(E)$$

$$\varphi_\alpha \longmapsto d'\varphi_\alpha + \theta_\alpha \wedge \varphi_\alpha,$$

$$D'' : E \longrightarrow \Omega^1(E)$$

$$\varphi_\alpha \longmapsto d''\varphi_\alpha$$

so that $D = D' + D''$.

D induces an operator from $\mathcal{A}^p(E)$ to $\mathcal{A}^{p+1}(E)$ by the following formula

$$D(\lambda, \varphi) = d\lambda \cdot \varphi + (-1)^p \lambda \wedge D\varphi,$$

where λ is a local section of $\mathcal{A}^p$ and φ is a local section of E. We denote this operator also by D. D^2 is called the curvature form of the connection $\theta = \{\theta_\alpha\}$.

Let $h = \{h_\alpha\}$ be a hermitian metric on E. Then by an easy calculation, we see that $\{h_\alpha^{-1} d' h_\alpha\}$ defines a connection on E. Let $\{\Theta_\alpha\}$ be the curvature form of the connection $h^{-1}d'h$. More explicitely $\Theta_\alpha = d\theta_\alpha + \theta_\alpha \wedge \theta_\alpha = -h_\alpha^{-1} d'd''h_\alpha - h_\alpha^{-1} d''h_\alpha \wedge h_\alpha^{-1} d'h_\alpha$. We put

$$\Theta_\xi(\eta) = \sum_{\substack{1 \le \rho, \tau, \sigma \le r \\ 1 \le i, j \le n}} h^{\rho}_{\tau} \Theta^{\tau}_{\sigma i j} \xi^{\rho} \bar{\xi}^{\sigma} \eta^{i} \bar{\eta}^{j}$$

where $\xi = {}^t(\xi_1, \cdots, \xi_r) \in \mathbb{C}^r$, $\eta = {}^t(\eta_1, \cdots, \eta_n) \in \mathbb{C}^n$, $\Theta = \left(\sum_{i \le i, j \le n} \Theta^{\rho}_{\sigma i j} dz_i \wedge d\bar{z}_j \right)_{i \le \rho, \sigma \le r}$ and $z_1, \cdots, z_n$ is a local coordinate system on V (we drop the index α when no cofusion is possible).

We say that a vector bundle E is positive (resp. negative) and denote it by $E > 0$ (resp. $E < 0$) if there exists a hermitian metric h on E such that $\Theta_\xi(\ )$ is positive (resp. negative) definite any point P of V for any $\xi \neq 0$.

A vector bundle E is non-negative (resp. non-positive) and we denote it by $E \geq 0$ (resp. $E \leq 0$) if there exists a hermitian metric h such that $\Theta_\xi(\ )$ is non-negative (non-positive) at any point P of V for any $\xi \neq 0$.

Remark (1.2) Let P be a point of V. By choosing a frame, we may

assume $d'h(P) = 0$ and $h(P) = I$. Then the curvature form at P is equal to $-d'd''h$.

Remark (1.3) Let $\{h_\alpha\}$ be a metric on E. If we put $\varphi_\alpha = \Sigma h_{\alpha t}^{s} \xi^{s} \bar{\xi}^{t}$ with $\xi = {}^t(\xi_1, \cdots, \xi_r) \in \mathbb{C}^r$ on each $U_\alpha \times \mathbb{C}^r$, then φ_α is a well defined function. Let P be a point of U_α. We normalize h at P as in Remark (1.2). Then the Levi form of φ at $P \times \xi$ is equal to

$$\begin{pmatrix} \sum_{1\le s,t\le r} \frac{\partial^2 h_t^s}{\partial z_1 \partial \bar{z}_1} \xi^s \bar{\xi}^t \cdots & \sum_{1\le s,t\le r} \frac{\partial^2 h_t^s}{\partial z_1 \partial \bar{z}_n} \xi^s \bar{\xi}^t & \\ \vdots & \vdots & 0 \\ \sum_{1\le s,t\le r} \frac{\partial^2 h_t^s}{\partial z_n \partial \bar{z}_1} \xi^s \bar{\xi}^t \cdots & \sum_{1\le s,t\le r} \frac{\partial^2 h_t^s}{\partial z_n \partial \bar{z}_n} \xi^s \bar{\xi}^t & \\ 0 & & h \end{pmatrix}$$

Hence # negative eigenvalues of $\Theta_\xi(\ )$ at $P = n$ - # negative eigenvalues of the Levi form of φ at P

Definition (1.4) A vector bundle E is positive (resp. non-negative) in the sense of Nakano if there exists a hermitian metric h on E such that

$$\sum_{\substack{1\le \rho,\tau,\sigma\le r \\ 1\le i,j\le n}} h^{\rho}_{\tau}{}_{\sigma i j} \xi^{(\rho,i)} \overline{\xi^{(\sigma,j)}}$$

is positive (resp. non-negative) at any point for any non-zero vector $(\cdots \xi^{(\rho,i)} \cdots) \in \mathbb{C}^{nr}$.

Lemma (1.5) If a vector bundle is positive in the sense of Nakano, then it is positive. The converse holds if either $r = 1$ or $n = 1$.

Proof. Trivial from the definition.

Remark (1.6). A positive vector bundle is not positive in the sense of Nakano in general.

Lemma (1.7) Let E be a vector bundle on V. Let h be a hermitian metric on E. Then h induces a natural hermitain metric $\check{h}$ on $\check{E}$. We have $\Theta = -{}^t\check{\Theta}$ where Θ (resp. $\check{\Theta}$) is the curvature form of h (resp. $\check{h}$).

proof. Let $\{h_\alpha\}$ define a metric on E. Then $\{{}^t h_\alpha^{-1}\} = \{\check{h}_\alpha\}$ is a metric on $\check{E}$. In fact.

$$
\begin{aligned}
&{}^t\overline{g}_{\beta\alpha} h_\alpha g_{\beta\alpha} = h_\beta \\
&\overline{g}_{\alpha\beta}^{-1}\, {}^t h_\alpha^{-1}\, {}^t g_{\beta\alpha}^{-1} = {}^t h_\beta^{-1} \\
&{}^t\overline{({}^t g_{\beta\alpha}^{-1})}\, {}^t h_\alpha^{-1}\, {}^t g_{\beta\alpha}^{-1} = {}^t h_\beta^{-1}.
\end{aligned}
$$

Let $\check{\theta}$ be the connection of $\{{}^t h_\alpha^{-1}\}$. Then

$$
\begin{aligned}
\check{\theta}_\alpha &= ({}^t h^{-1})^{-1} d' {}^t h^{-1} \\
&= -d' {}^t h \cdot {}^t h^{-1} \\
&= -{}^t(h^{-1} d' h) \\
&= -{}^t\theta.
\end{aligned}
$$

Hence $\check{\Theta}_{\rho}^{\sigma} = -\Theta_{\sigma}^{\rho}$.

Exmaple (1.8) Projective space P^n. We put $U_i = \{(x_0, x_1, \cdots, x_n) \in P^n \mid x_i \neq 0\}$ for $0 \leq i \leq n$. The transition function g_{ji} of $O(-1)$ is x_i/x_j.

If we put $h_i = |x_0/x_i|^2+|x_1/x_i|^2+\cdots+|x_n/x_i|^2$, then $h_j = |x_i/x_j|^2 h_j = |g_{ji}|^2 h_i$. Hence $\{h_i\}$ defines a metric on $O(-1)$. If we put $z_1 = \frac{x_0}{x_i}$, $z_2 = \frac{x_1}{x_i}$, ..., $z_i = \frac{x_{i-1}}{x_i}$, $z_{i+1} = \frac{x_{i+1}}{x_i}$, ..., $z_n = \frac{x_n}{x_i}$, the connection and the curvature form on U_i are given by

$$h^{-1}d'h = \frac{\bar{z}_1 dz_1+\bar{z}_2 dz_2+\cdots+\bar{z}_n dz_n}{1+|z_1|^2+|z_2|^2+\cdots+|z_n|^2}$$

$$\Theta = \frac{-(1+|z_1|^2+\cdots+|z_n|^2)(dx_1\wedge d\bar{z}_1+\cdots+dz_n\wedge d\bar{z}_n)+}{1+|z_1|^2+\cdots+|z_n|^2}$$

$$+ (\bar{z}_1 dz_1+\cdots+\bar{z}_n dz_n)(z_1 d\bar{z}_1+\cdots+z_n d\bar{z}_n).$$

Hence $O(-1)$ is negative and $O(1)$ is positive.

Example (1.9) An abelian variety. Let $A = \mathbb{C}^n/\Gamma$ be an abelian variety where Γ is a lattice in $\mathbb{C}^n$. Since $\mathbb{C}^n$ is stein and simply connected, every line bundle is defined by an element of $H^1(\Gamma, H^0(\mathbb{C}^n, O^*_{\mathbb{C}^n}))$ i.e. by a cocycle $u \longmapsto e_u(z)$ for $\Gamma\Sigma$ with coefficients in $H^0(\mathbb{C}^n, O^*_{\mathbb{C}^n})$:

$$e_{u+u'}(z) = e_u(z+u')\cdot e_{u'}(z).$$

Let H be a hermitian form on $\mathbb{C}^n$ such that $E = \mathrm{Im}H$ is integral on Γ. Let $\alpha: \Gamma \to \{\xi \in \mathbb{C} \mid |z| = 1\}$ be a map such that $\alpha(u_1+u_2)=e^{i\pi E(u_1,u_2)}\cdot \alpha(u_1)\cdot\alpha(u_2)$, $u_i \in \Gamma$. Then $u \longmapsto e_u(z) = \alpha(u)e^{\pi H(z,u)+\frac{1}{2}\pi H(u,u)}$ is a cocycle for Γ with coefficients in $H^0(\mathbb{C}^n, O^*_{\mathbb{C}^n})$. Hence $\{e_u(z)\}$ determines a line bundle $L(H, \alpha)$. The theorem of Appell-Humbert says that any line

bundle on A is uniquely determined by a pair, (H, α), satisfying the condition above.

Let $L = L(H, \alpha)$ be a line bundle on A. If we put $\varphi(z) = e^{-\pi H(z,z)}$, $z \in \mathbb{C}^n$, then $\varphi(z)$ is a metric on the trivial line bundle $\mathbb{C}^n \times \mathbb{C}$ on $\mathbb{C}^n$. Let $e_u(z)$ be the cocycle defined by (H, α), then we have

$$\varphi(z+u)\left|e^{\pi H(z,u)+\frac{\pi}{2}H(u,u)}\right|^2 = \varphi(z).$$

In fact, $\varphi(z+u)\left|e^{\pi H(z,u)+\frac{\pi}{2}H(u,u)}\right|^2$

$$= e^{-\pi H(z+u,z+u)}\left|e^{\pi H(z,u)+\frac{\pi}{2}H(u,u)}\right|^2$$

$$= e^{-\pi H(z,z)-\pi H(z,u)-\pi H(u,z)-\pi H(u,u)}\left|e^{\pi H(z,u)+\frac{\pi}{2}H(u,u)}\right|^2$$

$$= e^{-\pi H(z,z)}\cdot e^{2\mathrm{Re}(-\pi H(z,u))-\pi H(u,u)}\left|e^{\pi H(z,u)+\frac{\pi}{2}H(u,u)}\right|^2$$

$$= e^{-\pi H(z,z)}$$

$$= \varphi(z).$$

Hence $\varphi(z) = |e_u(z)|^2\varphi(z+u)$ i.e. $\varphi(z)$ defines a metric on $L(H, \alpha)$. We calculate the curvature.

$$d'\varphi = -\pi\Big(\sum_{1\le i,j\le n} h_{ij}\bar{z}_j dz_i\Big)\varphi$$

$$d''\varphi = -\pi\Big(\sum_{1\le i,j\le n} h_{ij} z_i d\bar{z}_j\Big)\varphi$$

$$d'd''\varphi = -d'\varphi\wedge\pi\Big(\sum_{1\le i,j\le n} h_{ij} z_i d\bar{z}_j\Big)$$

$$= -\pi\cdot\pi\sum_{1\le i,j\le n} h_{ij} dz_i\wedge d\bar{z}_j$$

$$\Theta = \varphi^{-2}(-\varphi d'd''\varphi + d'\varphi \wedge d''\varphi)$$

$$= \pi \sum_{1 \le i,j \le n} h_{ij} dz_i \wedge d\bar{z}_j$$

where $H = (h_{ij})$.

Proposition (1.10) Let E be a vector bundle. If E is positive (resp. non-negative, ~~positive in the sense of Nakano, non-negative in the sense of Nakano~~), then so is any quotient bundle F of E.

proof. We prove the dual assertion. Let E be negative (resp. non-positive) and F be a sub-vector bundle, then F is negative (resp. non-positive) by considering the induced metric. The proof is similar for Nakano positive bundles and for Nakano non-negative bundles. For the details see Griffiths [4] p. 197.

Proposition (1.11) Let E and F be vector bundles

(i) $E > 0$ and $F > 0$ if and only if $E \oplus F > 0$

(i)' E and F are positive in the sense of Nakano if and only if $E \oplus F$ is positive in the sense of Nakano.

(ii) If $E > 0$ and $F \geq 0$, then $E \otimes F > 0$.

(ii)' If E is positive in the sense of Nakano and F is non-negative in the sense of Nakano, then $E \otimes F$ is positive in the sense of Nakano.

proof. (i) ~~and (i)' are~~ is an easy consequeces of Proposition (1.10) and the definition#. The proof of (i)' is similar

Let h_E and h_F be metrics on E and on F, respectively. The pairing

$$(E \otimes F) \times (E \otimes F) \xrightarrow{h_E \otimes h_F} \mathbb{C} \underset{\mathbb{C}}{\otimes} \mathbb{C} = \mathbb{C}$$

$$(a \otimes b, c \otimes d) \longmapsto h_E(a, c)h_F(b, d)$$

defines a metric on $E \otimes F$. Calculation shows that the curvature of $h_{E\otimes F}$ is $\Theta_E \otimes I_s + I_r \otimes \Theta_F$ where r(resp. s) is the rank of E (resp. F). (ii) and (ii)' follow from what we have shown (See Griffiths [4] p.209).

Definition (1.12) A vector bundle E is negative in the sense of Grauert if there exists a relatively compact and strongly pseudoconvex neighbourhood of the zero-section of E. A vector bundle is positive in the sense of Grauert if $\check{E}$ is negative in the sense of Grauert.

A vector bundle E over V is said to be ample if, for any coherent sheaf F on V, we have

$$H^i(V, S^n(E) \otimes F) = 0 \quad \text{for sufficiently large } n \text{ and } i > 0.$$

Proposition (1.13). A vector bundle E is ample if and only if the tautological bundle is ample.

proof. See Hartshorne [5] p.69.

Proposition (1.14). A vector bundle E is positive in the sense of Grauert if and only if E is ample.

proof If E is positive in the sense of Grauert, then E is ample by Grauert [3] p.344 Hilfssatz 1.

If E is ample, then E is positive in the sense of Grauert by Hartshorne [5] p.72 Proposition (3.5).

We recall a well known

Proposition (1.15) A line bundle is positive if and only if it is ample.

Proposition (1.16) A positive vector bundle E is ample.

proof We deduce the Proposition from Proposition (1.15) and from the direct calculation of the curvature form of the metric on $O_{P(E)}(1)$ induced by the metric on E (See Griffiths [4]).

Another proof. We shall show that E is negative in the sense of Grauert. Consider the function φ on $\check{E}$ defined as in Remark (1.3). Then $\{Q \in \check{E} | \varphi(Q) < 1\}$ is a relatively compact strongly pseudoconvex neighbourhood of the zero-section. Hence E is negative in the sense of Grauert. It follows that E is ample by Proposition (1.14).

Theorem (1.17) (Andreotti and Grauert [1], p.257)

Let E be a vector bundle. If $\mathcal{H}_\xi(\ )$ is non-degenerate at any point P of V for any $\xi \neq 0$, then the number i of the negative eigen values of $\mathcal{H}_\xi(\ )$ is independent of P and ξ if $\xi \neq 0$ and we have

$$H^q(V, S^m(E)) = 0 \quad \text{for sufficiently large } m \quad \text{if } i \neq q.$$

Sketch of the proof. First, Andreotti and Grauert show that there exists a filtration in $H^q(E, O_E)$ such that the associated graded module $GH^q(E, O_E)$ is isomorphic to $\oplus_{n\geq 0} H^q(V, S^n(E))$. Secondely, by Remark (1.3), considering the functions φ and $e^{-c\varphi}$ with $c >> 0$, we deduce that E is strongly $(q+1)$-pseudoconvex and strongly $(n+r-q)$-pseudoconcave. $H^q(E, O_E)$ is finite dimensional if $q \neq i$. Now the theorem follows from what we have seen. For the details see Andreotti and Grauert [1]. (cf. Theorem (4.1.1), Theorem (4.2)

and Theorem (4.3.1))

(1.18) Let $\varphi = \{\varphi_\alpha\}$ and $\psi = \{\psi_\alpha\}$ be E valued differential forms. We define the inner product of φ and ψ by

$$(\varphi, \psi) = \int_V \sum_{\rho,\sigma} h^{\rho}_{\alpha\sigma}\, a_{\alpha\rho}\, \wedge^* \overline{\psi}_{\alpha\sigma}.$$

This defines a positive definite hermitian metric on the space of E valued differential forms. The adjoint operators of D and d" exist

$$D \longleftrightarrow \delta', \qquad d'' \longleftrightarrow \vartheta$$

where $\vartheta\varphi_\alpha = \delta''\varphi_\alpha - *\theta_\alpha \wedge^* \varphi_\alpha$, $\delta' = -*d''*$ and $\delta'' = -*d'*$. Then, $(D''D'+D'D'')\varphi_\alpha = \Theta_\alpha \varphi_\alpha$. We set $\square = D''\vartheta + \vartheta D''$. An E valued form φ is called harmonic if $\square\varphi = 0$ or equivalently $D''\varphi = 0$, $\vartheta\varphi = 0$.

$$H^q(V, \Omega^p(E)) \simeq \{E \text{ valued harmonic } (p, q) \text{ forms}\}.$$

Lemma (1.19) (Nakano). Let φ be an E valued harmonic form. Then

$$-\frac{\sqrt{-1}}{2}(\Lambda\Theta\wedge\varphi, \varphi) \leq 0.$$

proof

$$-\frac{\sqrt{-1}}{2}(\Lambda\Theta\wedge\varphi, \varphi) = -\frac{\sqrt{-1}}{2}(\Lambda(D''D'+D'D'')\varphi, \varphi)$$

$$= -\frac{\sqrt{-1}}{2}(\Lambda D''D'\varphi, \varphi)$$

$$= -\frac{\sqrt{-1}}{2}((-\sqrt{-1}\delta' + d''\Lambda)D'\varphi, \varphi),$$

since $\Lambda d'' - d''\Lambda = -\sqrt{-1}\delta'$ (See Weil [19]).

$$= -\frac{\sqrt{-1}}{2}\left((-\sqrt{-1}\delta' D'\varphi, \varphi) + (d'' \wedge D'\varphi, \varphi)\right)$$

$$= -\frac{\sqrt{-1}}{2}\left((-\sqrt{-1}\delta' D\varphi, \varphi) + (D'' \wedge D'\varphi, \varphi)\right)$$

$$= -\frac{\sqrt{-1}}{2}\left((-\sqrt{-1}\delta' D\varphi, \varphi) + (\wedge D'\varphi, \varphi)\right)$$

$$= -\frac{\sqrt{-1}}{2}(-\sqrt{-1}\delta' D\varphi, \varphi)$$

$$= -\frac{1}{2}(\delta' D\varphi, \varphi)$$

$$= -\frac{1}{2}(D\varphi, D\varphi) \leq 0$$

q. e. d.

§2 Positive vector bundles over a compact Riemann surface

Lemma (2.1) Let V be a manifold or an algebraic variety defined over an algebraically closed field k. Let E_1 and E_2 be vector bundles on V. Let E_ξ be the extension of E_2 by E_1;

$$0 \to E_1 \to E_\xi \to E_2 \to 0$$

determined by an element $\xi \in H^1(V, \mathrm{Hom}(E_2, E_1))$. Then $E_\xi \simeq E_{\lambda\xi}$ for any $0 \neq \lambda \in k$.

proof Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & E_1 & \to & E_\xi & \to & E_2 & \to & 0 \\ & & \uparrow & & \uparrow & & \| \mathrm{id} & & \\ 0 & \to & E_1 & \to & E_{\lambda\xi} & \to & E_2 & \to & 0 \end{array}$$

where the vertical arrow on the left is the multiplication by λ.

Lemma (2.2) Let V be a non-singular projective algebraic variety defined over $\mathbb{C}$. Let E_1 and E_2 be vector bundles on V. If E_1 and E_2 are positive in the sense of Nakano, then an extension of E_2 by E_1 is positive in the sense of Nakano.

proof Let E be defined by $v \in H^1(V, \mathrm{Hom}(E_2, E_1))$. Take a sufficiently fine open covering $\{U_\alpha\}$ of V so that the extension E_v is given by patching $E_1|_{U_\alpha} \oplus E_2|_{U_\alpha}$ and $E_1|_{U_\beta} \oplus E_2|_{U_\beta}$ on $U_\alpha \cap U_\beta$ by $\begin{pmatrix} I & a_{\beta\alpha} \\ 0 & I \end{pmatrix}$

where $a_{\beta\alpha} \in \Gamma(U_\alpha \cap U_\beta, \mathrm{Hom}(E_2, E_1))$ and $(x_\alpha, y_\alpha) \in E_1|_{U_\alpha} \oplus E_2|_{U_\alpha}$ and $(x_\beta, y_\beta) \in E_1|_{U_\beta} \oplus E_2|_{U_\beta}$ are identified if $(x_\alpha + a_{\beta\alpha}y_\alpha, y_\alpha) = (x_\beta, y_\beta)$. Then the extension $E_{\lambda v}$ is defined by replacing $\begin{pmatrix} I & a_{\beta\alpha} \\ 0 & I \end{pmatrix}$ by $\begin{pmatrix} I & \lambda a_{\beta\alpha} \\ 0 & I \end{pmatrix}$. Since an

extension is differentiablly trivial, there exists a C^∞ homomorphism b_α : $E_{2|U_\alpha} \to E_{1|U_\alpha}$ for each α such that $a_{\beta\alpha} = b_\alpha - b_\beta$ on $U_\alpha \cap U_\beta$;

$$\begin{array}{ccc} E_{1|U_\alpha} \oplus E_{2|U_\alpha} & \xrightarrow{\begin{pmatrix} I & \lambda a_{\beta\alpha} \\ 0 & I \end{pmatrix}} & E_{1|U_\beta} \oplus E_{2|U_\beta} \\ \downarrow \begin{pmatrix} I & \lambda b_\alpha \\ 0 & I \end{pmatrix} & & \downarrow \begin{pmatrix} I & \lambda b_\beta \\ 0 & I \end{pmatrix} \\ E_{1|U_\alpha} \oplus E_{2|U_\alpha} & \xrightarrow[\text{id}]{\sim} & E_{1|U_\beta} \oplus E_{2|U_\beta} \end{array}$$

Let h_{E_1} and h_{E_2} be hermitian metrics on E_1 and E_2 such that their curvature forms,

$$\sum_{\substack{1\le\rho,\tau,\sigma\le r \\ 1\le i,j\le n}} h_{E_1}{}^{\rho}{}_{\tau} \textcircled{H} E_1{}^{\tau}{}_{\sigma ij}\, \xi_1^{(\rho,i)}\overline{\xi_1^{(\sigma,j)}}$$

and

$$\sum_{\substack{1\le\rho,\tau,\sigma\le s \\ 1\le i,j\le n}} h_{E_2}{}^{\rho}{}_{\tau} \textcircled{H} E_2{}^{\tau}{}_{\sigma ij}\, \xi_2^{(\rho,i)}\overline{\xi_2^{(\sigma,j)}}$$

are positive definite. We set

$$h_{E_{\lambda\nu}\alpha} = {}^t\begin{pmatrix} I & -\lambda b_\alpha \\ 0 & I \end{pmatrix} \begin{pmatrix} h_{1\alpha} & 0 \\ 0 & h_{2\alpha} \end{pmatrix} \begin{pmatrix} I & -\lambda b_\alpha \\ 0 & I \end{pmatrix},$$

$\{h_{E_{\lambda\nu}\alpha}\}$ defines a hermitian metric on $E_{\lambda\nu}$. If we set $H_\lambda(\xi) = \sum h_{E_{\lambda\nu}}{}^{\rho}{}_{\tau} \textcircled{H} E_{\lambda\nu}{}^{\tau}{}_{\sigma ij}\, \xi^{(\rho,i)}\overline{\xi^{(\sigma,j)}}$, then $H_\lambda(\ )$ is a hermitian form at each point P of V. We fix a point P. Then there exists a number $c_p > 0$ and an open neighbourhood U_p of P such that H_λ is positive definite at any point $Q \in U_p$ if $|\lambda| < c_p$, since H_0 is positive definite at P. Since

V is compact, there exists a number $c > 0$ such that H_λ is positive definite at any point if $|\lambda| < c$. Now the lemma follows from lemma (2.1).

Lemma (2.3) Let V be a non-singular projective algebraic variety defined over $\mathbb{C}$. Let E be a vecror bundle of rank r on V. Let $\{U_\alpha\}$ be an open covering of V such that $E_{|U_\alpha}$ is trivial. Assume that the transition matrices $g_{\beta\alpha}$ can be written in the form $g_{\beta\alpha} = f_{\beta\alpha} \cdot U_{\beta\alpha}$ where $f_{\beta\alpha}$ is a scalor function and $U_{\beta\alpha}$ is a unitary matrix on $U_\alpha \cap U_\beta$. If $\det E$ is positive, then E is positive in the sense of Nakano. In particular E is positive and ample.

proof Let $\{h_\alpha\}$ be a hermitian metric on $\det E$ such that its curvature form is positive definite. From the definition $|\det g_{\beta\alpha}|^2 h_\alpha = h_\beta$ on $U_\alpha \cap U_\beta$ i.e $|f_{\beta\alpha}|^{2r} \cdot h_\alpha = h_\beta$. Hence $|f_{\beta\alpha}|^2 h_\alpha^{\frac{1}{r}} = h_\beta^{\frac{1}{r}}$. Consider the matrix $h_\alpha^{\frac{1}{r}} I_r$. This is a positive definite hermitian metric on $E|U_\alpha$. On the other hand, we have

$$\begin{aligned} {}^t\overline{g}_{\beta\alpha} h_\alpha^{\frac{1}{r}} I_r g_{\beta\alpha} &= \overline{f}_{\beta\alpha} \cdot {}^t\overline{U}_{\beta\alpha} \cdot h_\alpha^{\frac{1}{r}} I_r \cdot f_{\beta\alpha} \cdot U_{\beta\alpha} \\ &= |f_{\beta\alpha}|^2 h_\alpha^{\frac{1}{r}} {}^t\overline{U}_{\beta\alpha} U_{\beta\alpha} \\ &= |f_{\beta\alpha}|^2 h_\alpha^{\frac{1}{r}} I_r \\ &= h_\beta^{\frac{1}{r}} I_r. \end{aligned}$$

Hence $\{h_\alpha^{\frac{1}{r}} I_r\}$ defines a hermitian metric on E. Let Θ be the curvature form of $\{h_\alpha\}$. Then the curvature form of $\{h_\alpha^{\frac{1}{r}} I_\alpha\}$ is given by $\frac{1}{r}\Theta I_r$. Hence E is positive in the sense of Nakano. q. e. d.

(2.4) We need some results of Narasimhan and Seshadri [11].

Let S be a compact Riemann surface of genus $g \geq 2$. Let π be a discrete group acting effectively, properly and holomorphically on the unit disc T such that $T/\pi \simeq S$ and such that the projection $P : T \to S$ is unramified except at only one point x_0 and ramified with order n at x_0. Such a group π always exists. Let $\rho : \pi \to GL(n ; \mathbb{C})$ be a representation. Then π operates on the trivial bundle $T \times \mathbb{C}^n$ by $(y, v) \mapsto (y, \rho(\gamma)v)$, $y \in T$, $v \in \mathbb{C}^n$, $\gamma \in \pi$. We denote by $E_\pi(\rho)$ this vector bundle carrying the action of π. We denote by $P_*^\pi(E(\rho))$ the subsheaf of $P_*(E(\rho))$ consisting of elements invariant under the action of π. Then $P_*^\pi(E(\rho))$ is a vector bundle of rank n on S. We call $E_\pi(\rho)$ the -bundle associated to ρ. The vector bundle $P_*^\pi(E(\rho))$ is called the vector bundle arising from the representation ρ of π.

Let $y_0 \in p^{-1}(x_0)$ and π_{y_0} be the isotropy group of π at y_0. Let z be a coordinate system around y_0 such that the action of π_{y_0} is multiplication by ζ^k where ζ is a primitive n^{th} root of unity. Let γ_0 be the generator of π_{y_0} corresponding to multiplication by ζ. Let τ be a character of π_{y_0}. If $\tau(\gamma_0) = \zeta^s$, $0 \leq s < n$, then the integer s is independent of s and z. The integer s is called the associated integer to τ.

A homomorphism $\rho : \pi \to U(n, \mathbb{C})$ is a representation of type τ, by definition, if for every $\gamma \in \pi_{y_0}$, we have $\rho(\gamma) = \tau(\gamma) I_n$.

A vector bundle E of rank r over a compact Riemann surface is said to be stable if

$$\frac{\text{degree } E}{\text{rank } E} < \frac{\text{degree } F}{\text{rank } F}$$

for any quotient bundle F of E.

Theorem (2.4.1) (Narasimhan and Seshadri [11]) A vector bundle F of rank n and degree $-n < q \leq 0$ over a compact Riemann surface of genus ≥ 2 is stable if and only if F is isomorphic to $P^{\pi}_{*}(E(\rho))$ where ρ is an irreducible unitary representation of type τ and the associated integer to τ is q.

Corollary (2.4.2). Let E be a stable vector bundle of rank r over a compact Riemann surface R of genus ≥ 2, then there exists an open covering $\{U_\alpha\}$ of R such that $E_{|U_\alpha}$ is trivial for each α and such that the transition matrices can be written in the form ;

Scalor function x unitary matrix.

proof of the Corollary If the transition matrices are of the desired form, so are the transition matrices of $E \otimes L$ for any line bundle L on R. Hence we may assume $-n < \deg E \leq 0$. In this case the corollary is an easy consequence of the Theorem and Narasimhan and Seshadri [11] Remark 6.2. p.550.

Lemma (2.5) A stable bundle of positive degree over a compact Riemann surface of genus ≥ 2 is positive in the sense of Nakano.

proof. The Lemma is an easy consequence of Lemma (2.2) and Corollary (2.4.2).

Theorem (2.6) Let R be a compact Riemann surface of genus g. Let E be a vector bundle of rank r over R. Then the following are equivalent.

(i) E is positive in the sense of Nakano.

(ii) E is positive.

(iii) E is ample.

(iv) The degree of every quotient bundle of E (including E itself) is positive.

proof. The equivalence of (i) and (ii) follows from Lemma (1.5). (ii) $\Longrightarrow$ (iii) follows from Proposition (1.16). Since any quotient bundle of an ample vector bundle is ample, (iii) $\Longrightarrow$ (iv). Hence it is sufficient to show that (iv) $\Longrightarrow$ (i). Let E be a vector bundle on R such that the degree of any quotient bundle is positive. If $g = 0$, E is the direct sum of line bundles. Hence the assertion follows from Proposition (1.11). If $g = 1$. The assertion was proved in Umemura [18]. We may assume $g \geq 2$. We proceed by induction on the rank r of E.

If $r = 1$, the assertion (iv) $\Longrightarrow$ (i) follows from Proposition (1.15) and the Riemann-Roch theorem.

Now we assume that the assertion (iv) $\Longrightarrow$ (i) is proved for rank strictly less than r. Suppose that E contains a subbundle E_1 which is positive in the sense of Nakano :

$$0 \to E_1 \to E \to E_2 \to 0.$$

E_2 is positive by the inductive hypothesis since every quotient bundle of E_2 has positive degree. Hence by Lemma (2.2), E is positive in the sense of Nakano. If E does not contain a subbundle which is positive in the sense of Nakano, then E is stable. In fact, let F be a subbundle of E. We prove that the degree F is ≤ 0. We use induction on the rank s of F.

If $s = 1$, then the degree $F \leq 0$ since otherwise F would be positive in the sense of Nakano. Now we suppose that the degree of a subbundle is ≤ 0 if its rank is less than s. Let F be a subbundle of rank s. By the inductive hypothesis every subbundle of F has degree ≤ 0. Hence if the degree of F were positive, every quotient bundle of F would be positive. By the inductive hypothesis F would be positive in the sense of Nakano. Hence we may assume E to be stable. Since the degree of E is positive, E is positive in the sense of Nakano by Lemma (3.5).

Remark (2.7). Hartshorne proved the equivalence of (iii) and (iv) (cf. Hartshorne [6]).

§3 Some positive vector bundles of rank 2 over an algebraic surface.

(3.1) We recall the results of Takemoto [14] and [15]. Let S be a non-singular projective surface defined over $\mathbb{C}$. Let E be a vector bundle of rank 2 over S. Let H be an ample line bundle over S.

Definition (3.1.1) E is said to be H-stable (resp H-semi-stable) if for any succesive blowing ups $\pi : S' \to S$ and for any sub line bundle F of $\pi^* E$, we have

$$(\det E, H)/2 > (F, \pi^*H)$$

$$(\text{resp. } \geq)$$

where $(\ ,\ )$ denotes the intersection number.

Proposition (3.1.2) An H-stable bundle is simple i.e. $H^0(S, \mathrm{End}(E)) = \mathbb{C}$.

Theorem (3.1.3) The set of all H-stable vector bundles of rank 2 with fixed numerical Chern class is bounded i.e. there exists a scheme T of finite type over $\mathbb{C}$ and a vector bundle E on $T \times S$ such that, for any H-stable vector bundle F of rank 2 with the fixed numerical Chern class, there exists a closed point $t \in T$ with $F \simeq E_{|t \times S}$.

Theorem (3.1.4) Let H_1, H_2 be ample line bundles on S. Assume that S is relatively minimal and $N(E) = c_1^2 - 4c_2 \geq 0$ where c_i is i-th Chern class of E. Then E is H_1-stable if and only if E is H_2-stable.

Remark (3.1.5) If $N(E) < 0$, the H-stability depends on the choice of an ample line bundle H. Hence the H-stability does not seem very reasonable unless $N(E) \geq 0$ (See Takemoto [14]. See also Example (3.3)). By the Riemann-Roch theorem, $N(E) \leq 0$ on an abelian surface and $N(E) \leq -2$ on P^2.

Proposition (3.1.6) Let G be a finite solvable group. Let G operate on S holomorphically so that the projection $\pi : S \to S/G$ is unramified. Let H be an ample line bundle on S/G. Let E be an H-semi-stable bundle on S/G. Then π^*E is π^*H-semi-stable.

Definition (3.1.7) A non-sigular projective surface S is said to be hyperelliptic if the first betti number of S is 2 and if there exist an elliptic curve Δ and a smooth morphism $\pi : S \to \Delta$ such that every fibre is an elliptic curve.

Theorem (3.2) Let S be either an abelian variety of dimension 2, a geometrically ruled surface or a hyperelliptic surface. Let E be a vector bundle of rank 2 over S. Let H be an ample line bundle on S. If E is H-stable with $N(E) \geq 0$ and $c_1 > 0$, then E is positive in the sense of Nakano.

proof Case I. $S = A$ is an abelian variety. In this case , we prove a slightly more general assertion : If E is H-semi-stable with $N(E) \geq 0$, and $c_1 > 0$, then E is positive in the sense of Nakano. If E is simple, then by Oda [12], there exists an isogeny $p : A' \to A$ of degree 2 and a line bundle L on A' such that $p_*L = E$. We have $p^*p_*L = \bigoplus_{x \in \mathrm{Kerp}} T_x^*L$

be Oda [12] where T_x denotes translation by x. Since p is finite and $\det p_*L$ is ample by the Nakai criterion and the hypothesis, $\det(p^*p_*L)$ ample. Since L and T_x^*L are numerically equivalent, $\det(p^*p_*L)$ is numerically equivalent to $L^{\otimes 2}$. It follows that $L^{\otimes 2}$ is ample by the Nakai criterion hence L is ample or equivalently positive in the sense of Nakano. The direct image p_*L is positive in the sense of Nakano.

If E is not simple, then, by Takemoto [14], E is written in the form $E' \otimes M$ where E' is an extension :

$$0 \to 0 \to E' \to 0 \to 0$$

and M a line bundle on A.

Hence $c(E) = (1 + Mt)(1 + Mt)$

$$= 1 + M^{\otimes 2}t + (M^2)t^2.$$

From our hypothesis $M^{\otimes 2}$ is positive, hence ample by the Nakai criterion. It follows that M is ample. Since M is a line bundle, M is positive in the sense of Nakano by Proposition (1.15). By Lemma (2.2) the extension $0 \to M \to E' \otimes M \to M \to 0$ is positive in the sense of Nakano.

Case 11 Geometrically ruled surfaces. Let $p : S = P(V) \to C$ be a geometrically ruled surface over a curve C of genus g. We know from the hypothesis and Takemoto [14] that there exist a stable vector bundle F of rank 2 over C and a line bundle L on $P(V)$ such that $E = L \otimes p^*F$.

If $g = 0$, we have nothing to prove, since there is no stable bundle of rank 2 on C.

If $g = 1$, there exist an isogeny $\pi : C' \to C$ of elliptic curves and a line bundle L' on C' such that $\pi_* L' = F$ by Oda [12]. Consider the diagram :

$$\begin{array}{ccc} W = P(V) \underset{C}{\times} C' & \xrightarrow{\bar{\pi}} & P(V), \\ \downarrow \bar{p} & & \downarrow p \\ C' & \xrightarrow{\pi} & C \end{array} .$$

Since π is affine, we have $p^*F = \bar{\pi}_* \bar{p}^* L'$. Hence $\bar{\pi}_*(\bar{p}^* L' \otimes \bar{\pi}^* L) = E$. Let N be the kernel of π. Then W is a Galois covering of $P(V)$ with Galois group N. By Takemoto [15], we have $\bar{\pi}^* E = \bigoplus_{g \in N} T_g^*(\bar{p}^* L' \otimes \bar{\pi}^* L)$ where T_g denotes the operation of g on W. As in case I, we know $\det(\bar{p}^* L' \otimes \bar{\pi}^* L \oplus T_g^*(\bar{p}^* L' \otimes \bar{\pi}^* L))$ is ample with $e \neq g \in N$. On the other hand, $T_g^*(\bar{p}^* L' \otimes \bar{\pi}^* L) = T_g^*(\bar{p}^* L') \otimes T_g^*(\bar{\pi}^* L) = \bar{p}^*(T_p^* L') \otimes \bar{\pi}^* L$. Since $T_g^* L'$ and L' are numerically equivalent, we proved that $\det(\bar{p}^* L' \otimes \bar{\pi}^* L \oplus T_g^*(\bar{p}^* L' \otimes \bar{\pi}^* L))$ is numerically equivalent to $(\bar{p}^* L' \otimes \bar{\pi}^* L)^{\otimes 2}$. Hence $\bar{p}^* L' \otimes \bar{\pi}^* L$ is ample by the Nakai criterion and the direct image $\bar{\pi}_*(\bar{p}^* L' \otimes \bar{\pi}^* L) = E$ is positive in the sense of Nakano.

If $g \geq 2$, then by Corollary (2.4.2) the transition matrices of F can be written in the form :

Scalar function x unitary matrix.

Hence $\pi^* F$ and $\pi^* F \otimes L$ have the same property. By Lemma (2.3), the Theorem is proved for geometrically ruled surfaces.

Case III Hyperelliptic surface. In this case, there exists an abelian variety A and s finite abelian group G such that G operates on A, A/G is isomorphic to S and the projection $\pi: A/G \to S$ is unramified. By Proposition (3.1.6), π^*E is π^*H-semi-stable with $N(\pi^*E) = 0$. By what we have proved in case I_1 π^*E is positive in the sense of Nakano. Hence $\pi_*\pi^*E$ is positive in the sense of Nakano. Since E is a direct summand of $\pi_*\pi^*E$, E is positive in the sense of Nakano by Proposition (1.11).

q. e. d.

Example (3.3) Let E_1, E_2 be elliptic curves. Let M'_i be a line bundle of degree 1 on E_i, $i = 1,2$. We put $A = E_1 \times E_2$ and $p_i^*M'_i = M_i$ for $i = 1,2$ where p_i is the projection $p_i : A = E_1 \times E_2 \to E_i$. We set $L_1 = M_1^{\otimes n_1} \otimes M_2^{\otimes n_2}$, $L_2 = M_1^{\otimes m_1} \otimes M_2^{\otimes m_2}$. We define a vector bundle E of rank 2 by the exact sequence

$$0 \to L_1 \to E \to L_2 \to 0.$$

Then

$$c_1(E) = M_1^{\otimes(n_1+m_1)} \otimes M_2^{\otimes(n_1+m_2)}$$

$$c_2(E) = n_2m_1 + n_1m_2.$$

If $n_1 - m_1$ and $n_2 - m_2$ are coprime, $n_1 > m_1$, $n_2 < m_2$ and if we take a non-trivial extension, then, by Takemoto [14] p.41, E is H-stable for a certain ample line bundle H on A. But there exists an ample line bundle H' such that E is not H'-stable. In fact we have $c_1^2 - 4c_2 < 0$. It

is easy to see that $c_1 > 0$, $c_2 > 0$, $c_1^2 - c_2 > 0$ if and only if $n_1 + m_1 > 0$, $n_2 + m_2 > 0$, $n_2m_1 + n_1m_2 > 0$, $2n_1n_2 + n_1m_2 + m_1n_2 + 2m_1m_2 > 0$. Hence if we take $n_1 = 1000$, $n_2 = 0$, $m_1 = -1$, $m_2 = 1$, then E is H-stable with $c_1 > 0$, $c_2 > 0$, $c_1^2 - c_2 > 0$ and $c_1^2 - 4c_2 < 0$. Since L_2 is not ample, E is not ample. In view of this example, it seems agreable to add $c_1^2 - 4c_2 \geq 0$ to the definition of a stable vector bundle (see Remark (3.1.5)).

Remark (3.4) Theorem (3.2) is an analogue of Lemma (2.5). It is natural to ask :

Problem (3.4.1) Let S be a non-singular projective surface defined over $\mathbb{C}$. Let H be an ample line bundle. Let E be a vector bundle of rank 2 on S. Assume that E is H-stable with $N(E) = c_1^2 - 4c_2 \geq 0$, $c_1 > 0$, $c_2 > 0$. Then is E ample?

(3.4.2) The answer may be negative in characteristic $p > 0$, because in characteristic $p > 0$, there exists a curve C of genus $g \geq 2$ such that there exists a stable bundle of positive degree on C which is not ample (cf. Hartshorne [6]).

§4 vanishing theorems

(4.1) The following theorem is well known. We prove it by our own methods.

Theorem (4.1.1) Let A be an abelian variety of dimension n defined over $\mathbb{C}$. Let $L = L(H, \alpha)$ be a non-degenerate line bundle over A. Then,

$$H^i(A, L) = 0 \quad \text{if} \quad i \neq \#\{\text{negative eigen values of } H\}.$$

proof. Let r be the number of negative eigen values of H. We choose a coordinate system $(z_1, \cdots, z_n)$ of the universal covering space $\mathbb{C}^n$ of A so that H is written in the following form :

$$\begin{pmatrix} -I_r & 0 \\ 0 & \frac{1}{n-r+1} I_{n-r} \end{pmatrix}.$$

We use the metric $\varphi(z)$ defined in Example (1.9). Then the curvature form Θ is given by

$$\Theta = -\pi(dz_1 \wedge d\bar{z}_1 + \cdots + dz_r \wedge d\bar{z}_r).$$

$$+ \frac{\pi}{n-r+1}(dz_{r+1} \wedge d\bar{z}_{r+1} + \cdots\cdots + dz_n \wedge d\bar{z}_n).$$

We define a kähler metric on $\mathbb{C}^n$ hence on A by $\omega = \frac{\sqrt{-1}}{2} \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i$. Now the theory of harmonic integrals (1.18) is applied.

Lemma (4.1.2) If $\varphi = \varphi_0 \, d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_q}$ is a $(0,q)$-form with coefficients in L, then

$$-\frac{\sqrt{-1}}{2}(\Lambda\Theta\wedge\varphi,\varphi) \geq \pi(r-q-\frac{n-r}{n-r+1})(\varphi,\varphi).$$

proof of the lemma We compare integrands. We may assume that $\varphi = d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{i_q}$ since the question is local.

$$-\Theta\wedge\varphi = \pi((dz_1\wedge d\bar{z}_1 + \cdots + dz_r\wedge d\bar{z}_r) - \frac{1}{n-r+1}(dz_{r+1}\wedge d\bar{z}_{r+1}+\cdots+dz_n\wedge d\bar{z}_n))$$

$$\wedge\, d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{i_q}.$$

We use the following notation :

$$I = \{i_1,\cdots, i_q\},\quad N = \{1,\cdots, n\},\quad \omega_\ell = dz_\ell\wedge d\bar{z}_\ell.$$

$$-\Theta\wedge\varphi = \pi \sum_{\ell\in N-\{r+1,\cdots,n\}-I} d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{i_q}\wedge\omega_\ell - \frac{\pi}{n-r+1}\sum_{\ell\in N-\{1,\cdots,r\}-I} d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{i_q}\wedge\omega_\ell.$$

Hence

$$-\frac{\sqrt{-1}}{2}\Lambda\Theta\wedge\psi = \pi \sum_{\ell\in N-\{r+1,\cdots,n\}-I} d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{i_q} - \frac{\pi}{n-r+1}\sum_{\ell\in N-\{1,\cdots,r\}-I} d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{i_q}.$$

The lemma follows from the inequality

$$\#\{N-\{r+1,\cdots,n\}-I\} - \frac{1}{n-r+1}\times\#\{N-\{1,\cdots,r\} - I\}$$

$$\geq \{n-(n-r)-q\} - \frac{1}{n-r+1}(n-r) = r-q-\frac{n-r}{n-r+1}.$$

q. e. d.

Let φ be a harmonic form of type (0,q) with coefficients in L. Let

$\varphi = \sum_I \varphi_I$ where $I = \{i_1, \cdots, i_q\} \subset N$ and $\varphi_I = \psi_I d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_q}$ and the summation is taken for all such I's.

$$0 \geq -\frac{\sqrt{-1}}{2}(\Lambda\Theta\wedge\varphi, \varphi) \quad \text{by Lemma (1.19)}$$

$$= -\frac{\sqrt{-1}}{2}(\Lambda\Theta\wedge\sum_I \varphi_I, \sum_I \varphi_I)$$

since $(d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{i_q})\wedge\overline{*(d\bar{z}_{j_1}\wedge\cdots\wedge d\bar{z}_{j_q})} = 0$

if $\{i_1, \cdots, i_q\} \neq \{j_1, \cdots, j_q\}$,

$$= \sum_I -\frac{\sqrt{-1}}{2}(\Lambda\Theta\wedge\varphi_I, \varphi_I)$$

by lemma (4.1.2)

$$\geq \pi\sum_I((r-q-\frac{n-r}{n-r+1})\varphi_I, \varphi_I)$$

$$= \pi(\sum_I(r-q-\frac{n-r}{n-r+1})\varphi_I, \sum_I \varphi_I)$$

$$= \pi(r-q-\frac{n-r}{n-r+1})(\varphi, \varphi).$$

Hence if $r-q-\frac{n-r}{n-r+1} > 0$ i.e $r > q$, we have $H^q(A, E) = 0$. The line bundle, $\check{L}$, dual to $L(H, \alpha)$ is given by $L(-H, \alpha^{-1})$ (cf. Mumford [9]). Hence by Serre duality we have $H^q(A, L) = 0$ for $q > r$. This completes the proof of the theorem. q. e. d.

Theorem (4.2) Let V be a projective non-singular variety defined over $\mathbb{C}$. Let L be a line bundle on V such that $L^{\otimes m}$ is generated by global sections for large m. Let $\varphi: V \to W \subset P^N$ be a morphism defined by $L^{\otimes m}$.

If $\dim W = q$, then $H^i(V, L) = 0$ for $i \leq q - 1$.

proof Let H be a hyperplane section of P^N. Then, $L^{\otimes m} = \varphi^* H$. Let h' be a metric on H. Then $h = h' \circ \varphi$ is a metric on $L^{\otimes m}$. The curvature form Θ_h of h is, by an easy calculation, given by

$$\Theta_h = J \Theta_{h'} {}^t\overline{J}$$

where J is the Jacobian of φ and $\Theta_{h'}$ is the curvature form of h'. Hence $h^{\frac{1}{\ell}}$ defines a metric on L and its curvature form is $\frac{1}{\ell}\Theta_h$. Let $\tilde{\Theta}$ be the curvature form of L^{-1}. We fix a kähler metric ω on V. From what we have seen, at a point $Q \in V$, we may assume that

$$\tilde{\Theta}(Q) = \begin{pmatrix} -\lambda_1 & & & & 0 \\ & \ddots & & & \\ & & -\lambda_q & & \\ & & & 0 & \\ 0 & & & & \ddots 0 \end{pmatrix}, \quad \omega(Q) = I_n.$$

and that there exists a point Q such that $\lambda_i > 0$, $1 \leq i \leq q$. i.e. a point where J attains its maximal rank q. Let φ be a non-zero harmonic $(0,p)$-form with coefficients in L^{-1}. Then φ is non-zero on a dense open set in V. As in the proof of Lemma (4.1.2), the integrand of $-\frac{1}{2}(\Lambda\Theta\wedge\varphi, \varphi)$ is non-negative at any point Q and there exists a point Q where the integrand is positive if $p < q$. Hence if $p < q$, we have $-\frac{\sqrt{-1}}{2}(\Lambda\Theta\wedge\varphi, \varphi) > 0$. On the other hand $-\frac{\sqrt{-1}}{2}(\Lambda\Theta\wedge\varphi, \varphi) \leq 0$ by Lemma (1.19). This is a contradition. Hence when φ is non-zero, $p \geq q$. q. e. d.

(4.3) Let V be a non-singular projective variety of dimension n defined

over $\mathbb{C}$. Let E be a vector bundle of rank $r \leq n$. Let s be a section of E. s is said to be a regular section if $S = \{z \in V \mid s(z) = 0\}$ is non-singular and of condimension r. Griffiths [4] proved the following vanishing theorem.

Theorem (4.3.1) If E is positive, $r = 2$ and if E has a regular section, then $H^i(V, \check{E}) = 0$ for $i \leq n-2$.

His proof depends on the generalized Lefschetz theorem and the Hodge decomposition. In fact he compared the cohomology group of V and S with coefficients in $\mathbb{Z}$ by using Morse theory. But to abtain the vanishing theorem, we need only the generalized Lefsehetz theorem with coefficients in $\mathbb{C}$. We remark here that the generalized Lefschetz theorem with coefficients in $\mathbb{C}$ is proved by a stantdard technique using cohomological dimension and de Rham cohomology.

Lemma (4.3.2) Let V be a non-singular projective variety of dimension n defined over $\mathbb{C}$. Let E be a vector bundle of rank $r \leq n$. Let S be a zero locus of a regular section s. If E is positive, then $V - S$ is r-complete. In particular $\operatorname{ancd}(V-S) \leq r - 1$ where $\operatorname{ancd}(V-S)$ denotes the analytic cohomological dimension of $V - S$ (cf. Umemura [16]).

proof. Let h be a metric on E. Consider the function $\varphi(z) = {}^t\overline{s(z)}h(z)s(z)$ on $V - S$. Since $\varphi(z) > 0$ on $V - S$, we can put $\psi(z) = -\log \varphi(z)$. We shall calculate the Levi form of $\psi(z)$ at a fixed point 0 in $V - S$. We may assume that $h(0) = I$, $d'h(0) = 0$, hence we have $\Theta(0) = -d'd''h(0)$. Then, by a direct calculation,

$$\left[\frac{\partial^2\psi}{\partial z_i \partial \bar{z}_j}\right]_0 = \left[\frac{1}{\varphi}\, {}^t\bar{s}(\Theta_{ij})s + \frac{1}{\varphi^2}\left[\left({}^t\bar{s}\frac{\partial s}{\partial z_i}\right)\left(\frac{\partial\, {}^t\bar{s}}{\partial \bar{z}_i}\, s\right) - \left({}^t\bar{s}\, s\right)\left(\frac{\partial\, {}^t\bar{s}}{\partial \bar{z}_j}\frac{\partial s}{\partial z_i}\right)\right]\right]_0$$

It is sufficient to show that the Levi form $\left[\frac{\partial^2\psi}{\partial z_i \partial \bar{z}_j}\right]_0$ is positive definite on an $n-r+1$ dimensional subspace. Since $\sum_{1\le i,j\le n} \frac{1}{\varphi}\, {}^t\bar{s}\Theta_{ij}s\eta^i\bar{\eta}^j$ is positive definite, it suffices to show that the form in $\eta = (\eta^1, \cdots, \eta^n) \in \mathbb{C}^n$

$$(4.3.3)\qquad \sum_{1\le i,j\le n}\left[\left({}^t\bar{s}\frac{\partial s}{\partial z_i}\right)\left(\frac{\partial\, {}^t\bar{s}}{\partial \bar{z}_j}\, s\right) - {}^t\bar{s}s\left(\frac{\partial\, {}^t\bar{s}}{\partial \bar{z}_j}\frac{\partial s}{\partial z_i}\right)\right]_0 \eta^i\bar{\eta}^j$$

vanishes on an $n-r+1$ dimensional subspace. Let $s = {}^t(s^1, s^2, \cdots, s^r)$. Consider the equation

$$\begin{pmatrix} \frac{\partial s^1}{\partial z_1} & \frac{\partial s^1}{\partial z_2} & \cdots & \frac{\partial s^1}{\partial z_n} \\ \frac{\partial s^2}{\partial z_1} & & & \\ \vdots & & & \vdots \\ \frac{\partial s^r}{\partial z_1} & \cdots\cdots & & \frac{\partial s^r}{\partial z_n} \end{pmatrix}_0 \begin{pmatrix} \eta^1 \\ \eta^2 \\ \vdots \\ \eta^n \end{pmatrix} = \lambda \begin{pmatrix} s_1 \\ \vdots \\ s^r \end{pmatrix}_0$$

for some $\lambda \in \mathbb{C}$.

Then the dimension of the vector space $W = \{\eta \in \mathbb{C}^n \mid \left(\frac{\partial s^i}{\partial z_j}\right)_0 \eta = \lambda \cdot s(0)$ for some $\lambda \in \mathbb{C}\} \ge n-r+1$. We show (4.3.3) vanishes on W. In fact we have $\sum_{1\le i,j\le n}\left({}^t\bar{s}\cdot\frac{\partial s}{\partial z_i}\right)_0\left(\frac{\partial\, {}^t\bar{s}}{\partial \bar{z}_j}\cdot s\right)_0 \eta^i\bar{\eta}^j = |\lambda|^2\, ({}^t\bar{s}(0)s(0))^2$ and $\sum_{1\le i,j\le n}({}^t\bar{s}(0)\, s(0)\left(\frac{\partial\, {}^t\bar{s}}{\partial \bar{z}_j}\frac{\partial s}{\partial z_i}\right)\eta^i\bar{\eta}^j = |\lambda|^2({}^t\bar{s}(0)s(0))^2$. Hence (4.3.3) $= 0$ on W.

The last assertion of the lemma follows from Andreotti and Grauert [1].

q. e. d.

Theorem (4.3.4) Using the notation of lemma (4.3.2), the homomorphism $H^i(V, \mathbb{C}) \to H^i(S, \mathbb{C})$ is bijective for $i \leq n-r-1$ and injective for $i = n-r$.

proof. Consider the sepectral sequence of de Rham cohomology :

$$I_1^{pq} = H^q(V - S, \Omega^p) \Rightarrow H^*_{DR}(V - S) = H^*(V - S, \mathbb{C}).$$

Since $\operatorname{ancd}(V - S) \leq r-1$, we have :

$$I_1^{pq} = 0 \quad \text{if} \quad q \geq r \quad \text{or} \quad p \geq n+1.$$

Hence $H^i(V-S, \mathbb{C}) = 0$ for $i \geq n+r$ and consequently $H^j(V, S ; \mathbb{C}) = 0$ for $j \leq n-r$. The theorem follows from the exact sequence of cohomology

$$\to H^{j-1}(S, \mathbb{C}) \to H^j(V, S ; \mathbb{C}) \to H^j(V, \mathbb{C}) \to H^j(S, \mathbb{C}) \to H^{j+1}(V, S ; \mathbb{C}) \to .$$

proof of Theorem (4.3.1). The following argument is due to Griffiths [4].

Let I be the ideal sheaf of S :

$$0 \to I \to \mathcal{O}_V \to \mathcal{O}_S \to 0.$$

We have the commutative diagram

$$\begin{array}{ccc} H^i(V, \mathbb{C}) & \longrightarrow & H^i(S, \mathbb{C}) \\ \uparrow & & \uparrow \\ H^i(V, \mathcal{O}_V) & \longrightarrow & H^i(S, \mathcal{O}_S) \\ \uparrow & & \uparrow \\ 0 & & 0 \end{array}$$

Since the Hodge decomposition is functional.

By Theorem (4.3.4), we have $H^i(V, I) = 0$ for $i \leq n - r$. From now on we suppose that the rank r of E is equal to 2. Since s is regular, we can construct an exact Koszul complex by using s ;

$$0 \to \Lambda^2 \check{E} \to \check{E} \to I \to 0$$

By Kodaira vanishing theorem or Theorem (4.2), $H^i(V, \Lambda^2 \check{E}) = 0$ for $i \leq n - 1$. Hence $H^i(V, E) = 0$ for $i \leq n - r$. q. e. d.

§5 $\mathcal{G}(E)$ and $H(E)$

(5.1) Let H be a finite abelian group. Let k be an algebraicaly closed field of characteristic p. We consider a central extension,

$$0 \to k^* \to G \to H \to 0.$$

Let x,y be elements of H. We put $e(x, y) = \tilde{x}^{-1}\tilde{y}^{-1}\tilde{x}\tilde{y}$ where $\tilde{x}$, $\tilde{y} \in G$ lie over x, y. $e(x,y)$ is an elememt of k^* and is independ of the choice of $\tilde{x}$ and $\tilde{y}$. Then $e(x, y)$ is a skew-symmetric bilinear pairing from H to k^*. A subgroup $\tilde{K}$ of G is said to be a level subgroup if $\tilde{K} \cap k^* = \{0\}$ i.e. $\tilde{K}$ is isomorphic to its image in H. Let K be a subgroup of H such that the pairing $e(x, y)$ is trivial on K, then there exists a level subgroup $\tilde{K}$ lying over K. In fact in this case the extension is commutative over K and in the category of commutative group schemes over k, an extension of a finite group by k^* is trivial. If the pairing is degenerate, there exists a subgroup K such that the pairing e is trivial on K and such that $|K|^2 > |H|$. Hence there exits a level subgroup $\tilde{K}$ of order $> |H|^{\frac{1}{2}}$.

Lemma (5.1.1) Let H be a finite abelian group. Let $0 \to k^* \to G \xrightarrow{\pi} H \to 0$ be a central extension of H. If G has a representation of degree 1 on which k^* operates as the natural character, then G is isomorphic to $k^* \times H$.

proof. Let V be a representation of G of degree 1 on which k^* operates

as the natural character. We denote by U_x the operation of $x \in G$ on V. Let $\tilde{K}$ be maximal level subgroup. Then there exists a character $\chi_0 \in \mathrm{Hom}(\tilde{K}, k^*)$ such that $U_x \cdot s = \chi_0(x) \cdot s$ for any $x \in \tilde{K}$ and any $s \in V$. Let $y \in G$. Setting $x^{-1}y^{-1}xy = \chi^y(x)$, $x \in K$, $\chi^y(x)$ is an element of k^*. $\chi^y : \tilde{K} \to k^*$ is a character. If $\pi(y) \notin \pi(\tilde{K})$, then χ^y is not trivial since K is maximal. It is sufficient to show that $\pi(\tilde{K}) = H$. Hence we have to show that χ^y is trivial for any $y \in G$. In fact, if $x \in \tilde{K}$, we have

$$\begin{aligned}
\chi_0(x)U_y \cdot s &= U_x U_y s \\
&= U_{xy} \cdot s \\
&= U_{\chi^y(x)yx} \cdot s \\
&= \chi^y(x) U_y U_x \cdot s \\
&= \chi^y(x) U_y \chi_0(x) \cdot s \\
&= \chi^y(x)\chi_0(x) U_y \cdot s
\end{aligned}$$

Hence $\chi_0(x) = \chi^y(x)\chi_0(x)$ and $\chi^y(x) = 1$. χ^y is trivial. q. e. d.

(5.2) Let A be an abelian varietry of dimension g defined over an algebraicaly closed field k of characteristic p. Let E be a vector bundle on A. We put

$$H(E) = \{a \in A \mid E \xrightarrow{\sim} T_a{}^*E \text{ where } T_a : A \to A,\ x \mapsto x+a\}$$

$$\mathcal{G}(E) = \{(a,\varphi) \mid a \in H(E) \text{ and } \varphi : E \xrightarrow{\sim} T_a{}^*E\}.$$

Let $(a, \varphi), (b, \psi) \in \mathcal{G}(E)$. Consider the compositon $T_a^*\psi \circ \varphi : E \to T_a^*E \xrightarrow{T_b^*\psi} T_a^*T_b^*E = T_{a+b}^*E$. If we define $(b, \psi)\circ(a, \varphi) = (a+b, T_a^*\psi \circ \varphi)$, then $\mathcal{G}(E)$ is a group. We have an exact suquence

$$1 \to \text{Aut } E \to \mathcal{G}(E) \to H(E) \to 0.$$

Let $x = (a, \varphi) \in \mathcal{G}(E)$. Then $U_x : H^0(A, E) \to H^0(A, E)$ defined by $U_x s = T_{-a}^*(\varphi(s))$ is a representation of $\mathcal{G}(E)$.

(5.2.1) If $E = L$ is of rank 1 and ample, the structure of $\mathcal{G}(L)$ is well known and $\mathcal{G}(L)$ plays a very important role in the algebraic theory of theta functions. We recall some basic properties of $\mathcal{G}(L)$ (For the details see Mumford [7]). From now on, for simplicity we assume that ch k=p is zero. However all the results hold in positive characteristic if we avoid inseparable isogenies.

$H(L)$ is a finite subgroup of A. Since Aut $L \simeq k^*$, we have the exact suquence studied in (5.1) and (5.1.1) :

$$0 \to k^* \to \mathcal{G}(L) \to H(L) \to 0$$

By the Riemann-Roch theorem, there exists an integer d such that $\dim H^0(X, L^{\otimes n}) = d \cdot n^g$ for all $n \geq 1$. Let $\hat{A}$ be the dual of A. We define $\Lambda(L) : A \to \hat{A}$ by sending x to $T_x^*L \otimes L^{-1}$. Then we have

$$d^2 = |\chi(L)|^2 = \text{degree } \Lambda(L) = |H(L)|.$$

Given a level subgroup $\tilde{K} \subset \mathcal{G}(L)$, L descends to A/K i.e. there

exists an invertible sheaf L' on A/K such that $p^*L' \simeq L$ where p is the projective $p : A \to A/K$. Conversely let K be a finite subgroup of A, the descent date associated to L is given by a level subgroup lying over K.

Proposition (5.2.2) There exist a subgroup K of $H(L)$ and an invertible sheaf L' on A/K such that $\chi(L') = |H(L')| = 1$ and $p^*L' \simeq L$ where p is the projection $A \to A/K$.

Theorem (5.2.3) $H^0(A, L)$ is the unique irreducibel representation of $\mathcal{G}(L)$ in which k^* operates by its natural character.

What is $\mathcal{G}(E)$ for a vector bundle E on A?

(5.2.4) Assume A is of dimension 1, an elliptic curve. Let E be an ample irreducible vector bundle of rank r and of degree d, in Atiyah's notation, $E \in E(r, d)$. First we assume r, d are coprime. Then E is stable and in paticular, simple : $H^0(A, \mathrm{End}(E)) \simeq k$. Hence we get the exact sequence $0 \to k^* \to \mathcal{G}(E) \to H(E) \to 0$ and a level subgroup of $\mathcal{G}(E)$ corresponds to a descent date for E. By Oda [12], there exists an isogeny $p : A' \to A$ of degree r and an ample line bundle L of degree d on A' such that E is isomorphic to the direct image p_*L and the intersection of Ker p and Ker $\Lambda(L)$ is just 0. Moreover, $d = \dim H^0(A, E) = \dim H^0(A', L)$. Since the intersection of $H(L)$ and Ker p is 0, a non-trivial translation by an element of $H(L)$ induces a non-zero element of $H(E)$. Hence $H(L)$ is a subgroup of $H(E)$. We have $|H(L)|^2 = d^2$, hence $|H(E)| \geq d^2$. There exists a level subgroup of order $\geq d$. If we had

$|H(E)| > d^2$, then there would exists a level subgroup of order $d' > d$. Hence there would be an isogeny $\varphi : A \to A''$ of degree d' and a vector bundle E' on A'' such that $\varphi^*E' \simeq E$. But we have $d = \chi(A, E)$ $= d'\chi(A', E')$. This is a contradiction. Hence $H(E) = H(L)$ and $\mathcal{G}(E)$ is nothing but $\mathcal{G}(L)$ and the unique representation of $\mathcal{G}(E)$ is given by $H^0(A, E)$. There exists an isogeny $\psi : A \to B$ and a vector bundle E'' on B such that $\chi(E'') = 1$ and E is isomorphic to the inverse image ψ^*E''.

(5.2.4.1) In other words the theory of $\mathcal{G}(E)$ for a stable undle over an elliptic curve is absorbed in the theory of the usual Heisenberg group and its representation.

If r, d are not coprime, then $H(E)$ is too small it is not useful to consider $\mathcal{G}(E)$. Say $r = d = 2$. Then by Atiyah [2], E is isomorhpic to $F_2 \otimes L$ where L is a line bundle of degree 1 and F_2 is the non-trivial extension uniquely determined up to isomorphism by the exact sequence $0 \to 0 \to F_2 \to 0 \to 0$ with $\Gamma(A, F_2) \neq 0$. F_2 is homogeneous : $F_2 \simeq T_a^*F_2$ for any $a \in A$. Hence if $x \in H(E)$, we have $F_2 \otimes L \simeq E \simeq T_x^*E \simeq T_x^*(F_2 \otimes L) \simeq T_x^*F_2 \otimes T_x^*L \simeq F_2 \otimes T_x^*L$. It follows that $L \simeq T_x^*L$. In fact consider the diagram

$$\begin{array}{ccccccccc} 0 & \to & L & \xrightarrow{i_1} & F_2 \otimes L & \xrightarrow{\pi_2} & L & \to & 0 \\ & & \downarrow & & \varphi \downarrow \wr & & & & \\ 0 & \to & T_x^*L & \xrightarrow{i_2} & F_2 \otimes T_x^*L & \xrightarrow{\pi_2} & T_x^*L & \to & 0. \end{array}$$

If L and T_x^*L were not isomorphic, then the composite $\pi_2 \circ \varphi \circ i_1$:

$L \to T_x^*L$ would be trivial since $H^0(L^{-1} \otimes T_x^*L) = 0$. Hence $\varphi \circ i_1$ would factor through T_x^*L. And $\varphi \circ i_1$ would be injective. But this is impossible since $H^0(L^{-1} \otimes T_x^*L) = 0$.

We proved that $H(E) = H(L)$. On the other hand by (5.2.1), we have $|H(L)|^2 = 1^2 = 1$. $H^0(A, E)$ is two dimensional. Hence $H(E)$ is too saml1.

This shows that the group $\mathcal{G}(E)$ does not give sufficient information concerning E for a general vector bundle E. We have to restrict ourselves to a certain appropriate family of vector bundles. Over an elliptic curve, as we have seen above, the family of stable vector bundles is nice. So it is natural to ask if the group $\mathcal{G}(E)$ has good properties for H-stable bundles over an abelian surface. Unortunately, the answer is no.

Example (5.2.5) Let A be an abelian surface. Let E be an ample vector bundle of rank 2 over A. Let H be an ample line bundle on A. We assume that E is H-stable.

(5.2.5.1) In the case $c_1^2 - 4c_2 \geq 0$ (This automatically is = 0), since E is simple, there exist an isogeny $A' \xrightarrow{\pi} A$ and an ample line bundle L on A' such that E is isomorphic to the direct image π_*L. By the same argument as in (5.2.4), we conclude that $\mathcal{G}(E)$ is nothing but the Heisenberg group $\mathcal{G}(L)$, $H^0(A, E)$ is the unique representation of $\mathcal{G}(E)$ and that E descends to E' with $\chi(E') = 1$. $\mathcal{G}(E)$ is considered to be reasonable.

Example (5.2.6) We use the notation of Example (3.3). We substitute

$n_1 = n_2 = 0$, $m_1 = 1$, $m_2 = -1$ and we consider a non-trivial extension :

$0 \to 0 \to E \to M_1 \otimes M_2^{-1} \to 0$. Then E is H-stable for a certain ample line bundle H. Let $L = M_1 \otimes M_2$. Let q be a prime number. Tensoring $L^{\otimes q^n}$ with the exact sequence above, we get an exact sequence

$$0 \to L^{\otimes q^n} \to E \otimes L^{\otimes q^n} \to M_1^{\otimes (q^n+1)} \otimes M_2^{\otimes (q^n-1)} \to 0.$$

If $n \geq 2$, then $E \otimes L^{\otimes q^n}$ is ample and $\dim H^0(E \otimes L^{\otimes q^n}) = q^{2n} + (q^n+1)(q^n-1) = 2 \cdot q^{2n} - 1$.

Let $x \in H(E \otimes L^{\otimes q^n})$ i.e. $E \otimes L^{\otimes q^n} \simeq T_x^*(E \otimes L^{\otimes q^n})$.

$$\begin{array}{ccccccccc} 0 & \to & L^{\otimes q^n} & \xrightarrow{i_1} & E \otimes L^{\otimes q^n} & \xrightarrow{\pi_1} & M_1^{\otimes (q^n+1)} \otimes M_2^{\otimes (q^n-1)} & \to & 0 \\ & & \uparrow\downarrow & & \varphi \downarrow \wr & & \searrow & & \\ 0 & \to & T_x^* L^{\otimes q^n} & \xrightarrow{i_2} & T_x^*(E \otimes L^{\otimes q^n}) & \xrightarrow{\pi_2} & T_x^*(M_1^{\otimes (q^n+2)} \otimes M_2^{\otimes (q^n-1)}) & \to & 0 \end{array}$$

Since the index of $L^{\otimes -q^n} \otimes T_x^*(M_1^{\otimes (q^n+1)} \otimes M_2^{\otimes (q^n-1)})$ is 1, we have $H^0(L^{\otimes -q^n} \otimes T_x^*(M_1^{\otimes (q^n+1)} \otimes M_2^{\otimes (q^n-1)})) = 0$. It follows that $\pi_2 \circ \varphi \circ i_1 = 0$. Hence $\varphi \circ i_1$ factors through $T_x^* L^{\otimes q^n}$. For the same reason, $\varphi^{-1} \circ i_2$ factors through $L^{\otimes q^n}$. Hence φ induces isomorphisms $L^{\otimes q^n} \xrightarrow{\sim} T_x^* L^{\otimes q^n}$ and $M_1^{\otimes (q^n+1)} \otimes M_2^{\otimes (q^n-1)} \xrightarrow{\sim} T_x^*(M_1^{\otimes (q^n+1)} \otimes M_2^{\otimes (q^n-1)})$. This proves $x \in H(L^{\otimes q^n})$ & $H(M_1^{\otimes (q^n+1)} \otimes M_2^{\otimes (q^n-1)})$ and so $H(E \otimes L^{\otimes q^n}) \subset H(L^{\otimes q^n}) \cap H(M_1^{\otimes (q^n+1)} \otimes M_2^{\otimes (q^n-1)})$. On the other hand,

$$|H(L^{\otimes q^n})| = h^0(L^{\otimes q^n})^2 = q^{4n}$$

$$|H(M_1^{\otimes (q^n+1)} \otimes M_2^{(q^n-1)})| = h^0(M_1^{\otimes (q^n+1)} \otimes M_2^{\otimes (q^n-1)})^2$$

$$= (q^{2n}-1)^2.$$

Hence $|(H(L^{\otimes q^n})|$ and $|H(M_1^{\otimes(q^n+1)} \otimes M_2^{\otimes(q^n-1)})|$ are coprime and consequently $H(L^{\otimes q^n}) \cap H(M_1^{\otimes(q^n+1)} \otimes M_2^{\otimes(q^n-1)}) = \{0\}$.

We proved $H^0(A, E \otimes L^{\otimes q^n}) = 2 \cdot q^{2n}-1$ and $H(E \otimes L^{q^n}) = \{0\}$ for $n \geq 2$. $\mathcal{Q}(E)$ is too small in this case.

Remark (5.2.7) In the example above $c_1^2(E \otimes L^{\otimes q^n}) - c_2(E \otimes L^{\otimes q^n}) < 0$ and the stability of E depends on the choice of an ample line bundle. Again, by the conclusion that we have deduced above, the notion of an H-stable vector bundle with $c_1^2 - 4c_2 < 0$ is not very agreable (See Example (3.3))

Remark (5.2.8). Let A be an abelian surface. Let E be a vector bundle of rank 2 on A. The following are equivalent.

(1) E is H-stable for any ample line bundle H on A and $c_1^2 - 4c_2 = 0$

(2) E is H-stable for an ample line bundle H on A and $c_1^2 - 4c_2 = 0$

(3) E is simple and $c_1^2 - 4c_2 = 0$

(4) There exist an isogeney $\pi : A' \to A$ of abelian surfaces and an ample line bundle L' on A' such that $\mathrm{Ker}\Lambda(L') \wedge \mathrm{Ker}\ \pi = \{0\}$ and E is isomorphic to the direct image $\pi_* L'$.

(5) E is simple. For any ample line bundle L on A and for sufficiently large n, we have

$$0 \to k^* \to \mathcal{Q}(E \otimes L^n) \to H(E \otimes L^{\otimes n}) \to 0$$

is a Heisenberg group : the pairing $e(x, y)$ is non-degenerate and

$$|H(E \otimes L^{\otimes n})| = \frac{1}{4} h^0(L^{\otimes 2n} \otimes \det E)^2.$$

(6) E is simple. The same assertion as in (5) holds for one ample line bundle L and for infinitely many $n > 0$.

proof The equivalence of (1), (2), (3) and (4) follows from Takemoto [14] and Oda [13] if we note that an extension of O_A by O_A is not a smiple vector bundle.

We prove that (4)$\Longleftrightarrow$(5). Let E be a simple vector bundle and let H be an ample line bundle. By (4) $E \otimes L^{\otimes n}$ is isomorphic to the direct image of $L' \otimes {}^*L^{\otimes n}$. Let $x \in \mathrm{Ker}\pi \wedge \mathrm{Ker}\, \Lambda(L' \otimes \pi^*L^{\otimes n})$. Then $L' \otimes \pi^*L^{\otimes n} \simeq T_x^*(L' \otimes \pi^*L^{\otimes n}) \simeq T_x^*L' \otimes T_x^*\pi^*L^{\otimes n} \simeq T_x^*L' \otimes \pi^*L^{\otimes n}$. Hence $L' \simeq T_x^*L' : x \in \mathrm{Ker}\, \pi \wedge \Lambda(L')$. It follows that $x = 0$. As in (5.2.4), we get

$$|H(L' \otimes \pi^*L^{\otimes n})| \leq |H(E \otimes L^{\otimes n})|.$$

By descent theory (cf. (5.2.4)), we conclude that $H(L' \otimes \pi^*L^{\otimes n}) \simeq H(E \otimes L^{\otimes n})$ and $|H(E \otimes L^{\otimes n})| = h^0(E \otimes L^{\otimes n})^2$ if $L' \otimes L^{\otimes n}$ is ample. By the Riemann-Roch theorem

$$h^0(E \otimes L^{\otimes n}) = \chi(E \otimes L^{\otimes n}) = \frac{1}{2} c_1^2(E \otimes L^{\otimes n}) - c_2(E \otimes L^{\otimes n})$$

$$= \frac{1}{2} c_1^2(E \otimes L^{\otimes n}) - \frac{1}{4} c_1^2(E \otimes L^{\otimes n}) = \frac{1}{4} c_1^2(E \otimes L^{\otimes n})$$

$$= \frac{1}{4} (L^{\otimes 2n} \otimes \det E)^2 = \frac{1}{2} h^0(L^{\otimes 2n} \otimes \det E).$$

Since (6) is a special case of (5), (5)$\Longrightarrow$(6) is trivial.

Now we show that (6)$\Longrightarrow$(3). Let E be a simple vector bundle of rank 2.

We assume that (6) holds. Then there exists an integer n and an ample line bundle H on A such that $H^0(A, E \otimes L^{\otimes n}) \neq 0$, $H^i(A, E \otimes L^{\otimes n}) = 0$ for $i = 1,2$, $0 \to k^* \to \mathcal{G}(E \otimes L^{\otimes n}) \to H(E \otimes L^{\otimes n}) \to 0$ is a Heisenberg group and $|H(E \otimes L^{\otimes n})| = \frac{1}{4} h^0(L^{\otimes 2n} \otimes \det E)^2$. Since $E \otimes L^{\otimes n}$ is simple, by the Riemann-Roch theorem, we have $c_1^2(E \otimes L^{\otimes n}) - 4c_2(E \otimes L^{\otimes n}) \leq 0$. Hence

$$(*) \quad 1 \leq h^0(E \otimes L^{\otimes n}) = \frac{1}{2} c_1^2(E \otimes L^{\otimes n}) - c_2(E \otimes L^{\otimes n}) \leq \frac{1}{4} c_1^2(E \otimes L^{\otimes n})$$

$$= \frac{1}{2} h^0(L^{\otimes 2n} \otimes \det E).$$

Since $\mathcal{G}(E \otimes L^{\otimes n})$ is a Heisenberg group and $h^0(E \otimes L^{\otimes n})$ is a representation of $\mathcal{G}(E \otimes L^{\otimes n})$ in which k^*_1 opertes by its natural character, $h^0(E \otimes L^{\otimes n})$ is divisible by $|H(E \otimes L^{\otimes n})|^{\frac{1}{2}} = \frac{1}{2} h^0(L^{\otimes 2n} \otimes \det E)$. By inequality (*), we get $h^0(E \otimes L^{\otimes n}) = \frac{1}{2} h^0(L^{\otimes 2n} \otimes \det E)$. Hence $\frac{1}{2} c_1^2(E \otimes L^{\otimes n}) - c_2(E \otimes L^{\otimes n})$ $= \frac{1}{4} c_1^2(E \otimes L^{\otimes n})$: $c_1^2(E \otimes L^{\otimes n}) - 4c_2(E \otimes L^{\otimes n}) = 0$. If follows that $c_1^2(E)$ $- 4c_2(E) = 0$. This completes the proof. q. e. d.

Example (5.2.9) Let A be an abelian surface. Let L be a very ample line bundle over A. Let $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in H^0(A, L)^{\oplus 3}$. We assume that $\varphi_1, \varphi_2, \varphi_3$ do not vanish simultaneously. This we can define the vector bundle $E(\varphi)$, frequelty referred to as Kleiman's example, by the exact sequence

$$(**) \qquad 0 \to 0 \to L^{\oplus 3} \to E(\varphi) \to 0.$$

$$1 \mapsto (\varphi_1, \varphi_2, \varphi_3)$$

Lemma (5.2.10) $H^2(A, E(\varphi) \otimes \check{L}) = 0$.

proof. $0 \to \check{E}(\varphi) \to \check{L}^{\otimes 3} \to 0 \to 0.$

$0 \to \check{E}(\varphi) \otimes L \to 0^{\oplus 3} \to L \to 0.$

Hence we get

$$0 \to H^0(A, \check{E}(\varphi) \otimes L) \to H^0(A, 0)^{\oplus 3} \to H^0(A, L) \to$$

The map $H^0(A, 0)^{\oplus 3} \to H^0(L)$ which sends $(1, 0, 0) \mapsto \varphi_1$, $(0, 1, 0) \mapsto \varphi_2$, $(0, 0, 1) \mapsto \varphi_3$ is injective. Hence $H^0(A, \check{E}(\varphi) \otimes L) = 0$. Applying Serre duality, we get the lemma. q. e. d.

Lemma (5.2.11) $E(\varphi)$ is simple.

proof Tensoring $E(\varphi)$ with the exact sequence (* *), we get the exact sequence,

$$0 \to E(\varphi) \otimes \check{E}(\varphi) \to (E(\varphi) \otimes \check{L})^{\oplus 3} \to E(\varphi) \to 0.$$

The exact sequence of cohomology is :

$$H^1(E(\varphi)) \to H^2(\mathrm{End}(E(\varphi)) \to H^2(E(\varphi) \otimes \check{L})^{\oplus 3}$$

$H^1(E(\varphi)) = H^2(O_A)$ is one dimesnional and $H^2(E(\varphi) \otimes \check{L}) = 0$ by Lemma (5.2.10). Hence $\dim H^2(\mathrm{End}(E(\varphi))) \leq 1$. By Serre duality, the lemma is proven.

Lemma (5.2.12) The vector bundles $E(\varphi)$ and $E(\varphi')$ are isomorphic if and only if the vector space generated by $\varphi_1, \varphi_2, \varphi_3$ is equal to the vector space generated by $\varphi_1', \varphi_2', \varphi_3'$ in $H^0(A, L)$.

proof Assume that the vector space generated by $\varphi_1, \varphi_2, \varphi_3$ = the vector space generated by $\varphi_1', \varphi_2', \varphi_3'$. We have

$$\begin{pmatrix} \varphi_1' \\ \varphi_2' \\ \varphi_3' \end{pmatrix} = B \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} \quad \text{with} \quad B \in GL(3, k).$$

B defines an automorphism of $L^{\otimes 3}$ which we denote by B. We note that the following diagram is commutative :

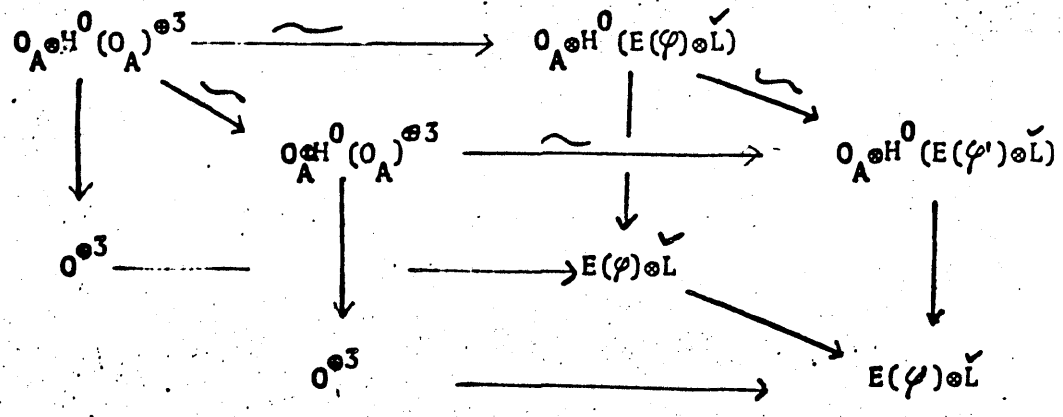
$$\begin{array}{ccccccccc} & & & & 1 & \longmapsto & \varphi & & \\ 0 & \longrightarrow & 0 & \longrightarrow & L^{\otimes 3} & \longrightarrow & E(\varphi) & \longrightarrow & 0 \\ & & \| & & \downarrow A & & & & \\ 0 & \longrightarrow & 0 & \longrightarrow & L^{\otimes 3} & \longrightarrow & E(\varphi') & \longrightarrow & 0. \\ & & & & 1 & \longmapsto & \varphi' & & \end{array}$$

This proves the if part of the lemma.

Now we prove only if part. Suppose that $E(\varphi)$ and $E(\varphi')$ are isomorphic. Tensoring $\overset{\vee}{L}$ with the exact sequense (**), we get the diagram :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overset{\vee}{L} & \longrightarrow & 0^{\otimes 3} & \longrightarrow & E(\varphi) \otimes \overset{\vee}{L} & \longrightarrow & 0 \\ & & & & & & \downarrow\uparrow \sigma & & \\ 0 & \longrightarrow & \overset{\vee}{L} & \longrightarrow & 0^{\otimes 3} & \longrightarrow & E(\varphi') \otimes \overset{\vee}{L} & \longrightarrow & 0 \end{array}$$

where σ is an isomorphism. It is sufficient to show that σ can be lifted to an automorphism of $0^{\otimes 3}$. Noting that $H^0(\overset{\vee}{L}) = H^1(\overset{\vee}{L}) = 0$, we get the commutative diagram :



which induces an automorphism of $O^{\otimes 3}$ making the diagram commutative. This completes the proof. q. e. d.

Since $(E(\varphi))$ has a representation $H^1(A, E)$ of degree 1, $\mathcal{G}(E(\varphi))$ is commutative by Lemma (5.1.1). We calculate $\mathcal{G}(E(\varphi))$ and its representation for a very special case. Let C_1, C_2 be elliptic curves. Let $A = C_1 \times C_2$. Let L_i be an ample line bundle of degree divisible by 3 on C_i, $i = 1,2$. Let L be the tensor product $p_1^*L_1 \otimes p_2^*L_2$ where p_i denotes the projection from $C_1 \times C_2$ onto the i-th factor. Let $\tilde{K}_1$ be a maximal level subgroup of $\mathcal{G}(L_1)$. Then there exists an element $\psi_1 \in H^0(C_1, L_1)$ and an element a of $\mathcal{G}(L_1)$ of order 3 such that ψ_1 is invariant under the operation of K_1 and ψ_1, $\psi_2 = a\psi_1$ and $\psi_3 = a^2\psi_1$ do not have a common zero. We do the same thing with L_2 on C_2 and we get ψ_1, $\psi_2' = b\psi_1'$, $\psi_3' = b^2\psi_1'$, $b \in \mathcal{G}(L_2)$ $b^3 = 1$. We put $\varphi_1 = \psi_1\psi_1'$, $\varphi_2 = \psi_2\psi_2'$, $\varphi_3 = \psi_3\psi_3'$ $\varphi = (\varphi_1, \varphi_2, \varphi_3)$. Since the operations $\tilde{K}_1 \times \tilde{K}_2$ and ab, a^2b^2 leave the vector space generated by φ_1, φ_2, φ_3 invariant, by Lemma (5.2.12) we have $|H(E(\varphi))| \geq 3 \times h^0(L_1) \times h^0(L_2)$ where $\tilde{K}_i$ is the maximal level subgroup chosen above. On the other hand $\chi(E) = 3 \times h^0(L_1) \times h^0(L_2)$. Since $\mathcal{G}(E)$ is commutative, $|H(E)|$ divides $\chi(E)$ by descent theory. Hence $|H(E)| = 3 \times h^0(L_1) \times h^0(L_2)$. It follows that

$$H(E) = \mathbb{Z}/3\mathbb{Z} \oplus \tilde{K}_1 \oplus \tilde{K}_2$$

$$\mathcal{G}(E) = k^* \oplus H(E).$$

Consider the exact sequece of representations of $\mathcal{G}(E(\varphi))$:

$$0 \longrightarrow \underset{\substack{\text{trivial}\\ \text{representation}}}{H^0(0)} \longrightarrow \underset{\substack{\text{regular}\\ \text{representation}\\ \text{of } \mathbb{Z}/3\mathbb{Z}\oplus\tilde{K}_1\oplus\tilde{K}_2}}{H^0(L)^{\oplus 3}} \longrightarrow H^0(E(\varphi)) \longrightarrow \underset{\substack{\text{trivial}\\ \text{representation}\\ \text{of degree 2}}}{H^1(0)} \longrightarrow 0$$

If follows that

$$H^0(E(\varphi)) \simeq \text{regular representation of } \mathbb{Z}/3\mathbb{Z} \oplus \tilde{K}_1 \oplus \tilde{K}_2 \oplus \text{trivial}$$

$$\text{representation of } \mathbb{Z}/3\mathbb{Z} \oplus \tilde{K}_1 \oplus \tilde{K}_2 \text{ of degree } 1.$$

I don't know if $E(\varphi)$ is H-stable for some ample line bundle H or not.

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