Remark V.1: Theorem V.1 shows that the technique in [13] can be applied to the hierarchical problem considered in this paper (strict optimality in the primary objective). The bound on L in (26) implies that the resulting sequence of finite-dimensional convex optimization problems are of low order [see (25)] which makes our technique implementable, in contrast to the ϵ-suboptimal approach in the primary objective outlined earlier. This means that if a suboptimal approach is taken with respect to the *secondary objective*, the results in [13] (adapted to the present case) imply that the minimization can be carried out over a finite-dimensional subspace of fixed dimension.

VI. CONCLUSION

This paper outlined a solution to a hierarchical optimal control problem which combines an $\mathcal{H}_\infty$ cost in the primary problem with an $\mathcal{H}_\infty$- or an $\mathcal{H}_2$-type secondary objective. Using an allpass dilation technique and results from superoptimal interpolation theory, the problem was formulated as a multidisk minimization in terms of a free parameter of reduced dimension which can be tackled using standard convex programming algorithms. Although the results assume a single vector of external disturbances w (see Fig. 1), an extension to the more general case of two independent external inputs is straightforward.

The results presented in this paper can deal with (strict) optimality in the primary objective, in contrast to existing multiple-objective techniques. These methods could be applied to the problem considered here, by relaxing the upper bound from its optimal level. Compared to this approach, our solution offers a number of advantages: 1) It is known that near optimality the suboptimal $\mathcal{H}_\infty$ problem is numerically ill-conditioned [7]. Optimal approaches on the other hand, circumvent this problem by reducing the dimensionality of the optimal generator; 2) The present method makes the available degrees of freedom at optimality more transparent and leads to a dimensionality reduction in the convex optimization problem associated with the secondary objective [see (15) and (19)]; and 3) If a suboptimal approach is taken with respect to the secondary objective, our method reduces to a convex optimization over a finite-dimensional subspace of fixed dimension (see Remark V.1).

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LMI Approach to an H_∞-Control Problem with Time-Domain Constraints over a Finite Horizon

S. Hosoe

***Abstract*—The discrete time H_∞-control problem is considered with time-domain constraints over a finite horizon. It is shown that a solution of this problem can be obtained from a system of linear matrix inequalities (LMI's).**

***Index Terms*—H_∞ control, LMI, time-domain constraints.**

I. INTRODUCTION

In the last few years, problems of designing controllers capable of achieving both time- and frequency-domain performance specifications have attracted the attention of many authors. For instance, a convex optimization approach for solving such problems has been proposed in [1]–[3] (see also the references therein) by using the well-known Youla parameterization of stabilizing controllers. This approach has been further developed in [4]–[7]. There the time-domain specifications over a finite horizon were described by affine constraints on the first Markov parameters of the Youla parameter $Q(z)$ of the controller. Then, the exact fulfillment of the frequency-domain specification in [4]–[7] is accomplished by using the remaining freedom in the expression of $Q(z)$ and reducing the problem into a standard H_∞-control problem [10].

The objective of the present paper is to combine the idea of [4]–[7] with the linear matrix inequality (LMI) approach [8] to the standard H_∞ control. It is shown that both the time- and the frequency-domain constraints can be treated entirely within the framework of LMI. As a result, the convexity structure of the problem becomes much clearer. Furthermore, computational effort to get the solution can be considerably reduced.

II. PROBLEM FORMULATION

We consider the standard feedback configuration illustrated in Fig. 1. Here P and K represent the plant and the controller, which are discrete time-invariant linear systems. As usual, it is assumed

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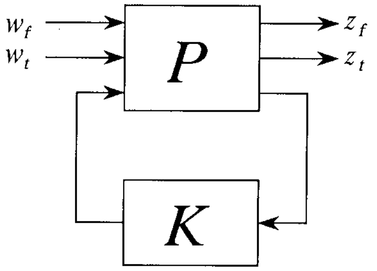

Fig. 1. Canonical control system configuration.

that P is stabilizable and detectable from u and y, respectively. The problem is to find K such that:

- the closed-loop system is stable;
- the H_∞ norm of $T_{z_f w_f}$ (the transfer function from w_f to z_f) is strictly less than γ. Without loss of generality, we will assume that $\gamma = 1$;
- the output $z_t(k)$ corresponding to a deterministic input $w_t(k)$ (e.g., steps, ramps etc.) lies within a prescribed envelope over a finite horizon $[0, N]$.

Notice that the first two conditions are the ordinary requirements in the standard H_∞ control and the last one imposes the time-domain constraint.

Let us denote the state-space representation of $P(z)$ by

$$P(z) = \begin{array}{cc} \begin{array}{cccc} \;n_P\; & \;m_1\; & \;m_{1t}\; & \;m_2\; \end{array} & \\ \left[\begin{array}{c|ccc} A_P & B_{Pf} & B_{Pt} & B_{P2} \\ \hline C_{Pf} & D_{ff} & D_{ft} & D_{f2} \\ C_{Pt} & D_{tf} & D_{tt} & D_{t2} \\ C_{P2} & D_{2f} & D_{2t} & D_{P22} \end{array}\right] & \begin{array}{c} n_P \\ p_1 \\ p_{1t} \\ p_2. \end{array} \end{array} \tag{1}$$

The Youla parameterization of the stabilizing controllers is given by

$$K(z) = F_l(J, Q) := J_{12}Q(I - J_{22}Q)J_{21} + J_{11} \tag{2}$$

where $Q(z) \in RH_\infty$ denotes a free parameter and $J(z)$ is given by

$$J(z) = \left[\begin{array}{c|cc} A_P + B_{P2}F + LD_{P22}F & -L & B_{P2} + LD_{P22} \\ \hline F & 0 & I \\ -C_{P2} - D_{P22}F & I & -D_{P22} \end{array}\right] \tag{3}$$

with F and L selected such that $A + B_{P2}F$ and $A + LC_{P2}$ are stable. From (2) and (3), the closed-loop transfer functions $T_{z_f w_f}$ and $T_{z_t w_t}$ (from w_t to z_t) are given by

$$T_{z_f w_f} = \mathcal{F}_\ell(T_f, Q) = T_{11} + T_{12}QT_{21} \tag{4}$$

$$T_{z_t w_t} = \mathcal{F}_\ell(T_t, Q) = T_{t11} + T_{t12}QT_{t21} \tag{5}$$

where

$$T_f = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & O \end{bmatrix} = \begin{array}{cc} \begin{array}{ccc} \;2n_P\; & \;m_1\; & \;m_2\; \end{array} & \\ \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & O \end{array}\right] & \begin{array}{c} 2n_P \\ p_1 \\ p_2 \end{array} \end{array} \tag{6}$$

$$T_t = \begin{bmatrix} T_{t11} & T_{t12} \\ T_{t21} & O \end{bmatrix} = \begin{array}{cc} \begin{array}{ccc} \;2n_P\; & \;m_{1t}\; & \;m_2\; \end{array} & \\ \left[\begin{array}{c|cc} A & B_{1t} & B_2 \\ \hline C_{1t} & D_{tt} & D_{t2} \\ C_2 & D_{2t} & O \end{array}\right] & \begin{array}{c} 2n_P \\ p_{1t} \\ p_2 \end{array} \end{array} \tag{7}$$

and

$$A = \begin{bmatrix} A_P + B_{P2}F & -B_{P2}F \\ 0 & A_P + LC_{P2} \end{bmatrix}, \qquad B_1 = \begin{bmatrix} B_{Pf} \\ B_{Pf} + LD_{2f} \end{bmatrix}$$

$$B_{1t} = \begin{bmatrix} B_{Pt} \\ B_{Pt} + LD_{2t} \end{bmatrix}, \qquad B_2 = \begin{bmatrix} B_{P2} \\ 0 \end{bmatrix}$$

$$C_1 = [C_{Pf} + D_{f2}F \;\; -D_{f2}F], \qquad C_{1t} = [C_{Pt} + D_{t2}F \;\; -D_{t2}F]$$

$$C_2 = [0 \quad C_{P2}], \qquad D_{11} = D_{ff}, \qquad D_{12} = D_{f2}, \qquad D_{21} = D_{2f}.$$

Notice that the structures of the matrices imply

$$C_2 A^j B_2 = 0, \qquad \forall j = 0, 1, \cdots, 2n_P \tag{8}$$

and by (5) the z-transform of z_t is obtained as

$$Z_t(z) = (T_{t11} + T_{t12}QT_{t21})W_t(z) \tag{9}$$

where $W_t(z)$ is the z-transform of w_t.

Now let us represent an arbitrary $Q(z) \in RH_\infty$ by using the first N Markov parameters Q_i and a new free parameter $\hat{Q}(z) \in RH_\infty$ as

$$Q(z) = Q_0 + Q_1 z^{-1} + Q_2 z^{-2} + \cdots + Q_{N-1} z^{-N+1} + \hat{Q}(z) z^{-N}. \tag{10}$$

Substituting this expression of $Q(z)$ and similar expressions of T_{t11}, T_{t12}, T_{t21}, $W_t(z)$ into (9) and expanding the result as a power series of z^{-1}, an arbitrary element of the output vector z_t at time k $(0 \le k \le N)$ can be expressed in the following convolution form:

$$a_k - \sum_{r=0}^{k} \sum_{s=0}^{r} b_{r-s}^T Q_s c_{k-r}$$

where a_k, b_k, and c_k are the constant scalars and vectors that are determined by T_{t11}, T_{t12}, T_{t21}, and $W_t(z)$.

Hence the time-domain specification in our problem can be reduced to

$$lb_k \le a_k - \sum_{r=0}^{k} \sum_{s=0}^{r} b_{r-s}^T Q_s c_{k-r} \le ub_k \qquad k = 0, 1, \cdots, N-1 \tag{11}$$

where lb_k and ub_k are the upper and the lower bound of the envelope, respectively.

Meanwhile, the frequency-domain specification is given by

$$\|T_{11} - T_{12}QT_{21}\|_\infty < 1. \tag{12}$$

Notice that with (10) the transfer function on the left-hand side of (12) can be represented as the closed-loop transfer function from w_f to z_f in Fig. 2. Thus by defining the augmented generalized plant $\hat{T}$ as in Fig. 2, (12) can be rewritten as

$$\|\hat{T}_{11} - \hat{T}_{12}\hat{Q}\hat{T}_{21}\|_\infty < 1. \tag{13}$$

Therefore, our problem can be restated as follows: Find stabilizing $Q(z) \in RH_\infty$ or equivalently Q_i's and $\hat{Q}(z) \in RH_\infty$ such that both (11) and (13) are simultaneously satisfied.

Note that the requirement $\hat{Q}(z) \in RH_\infty$ imposes the stability condition of the "stabilizing controller $\hat{Q}(z)$" for the generalized plant $\hat{T}$. This is not a usual condition in the standard H_∞-control problem and seems to cause some difficulties in determining $\hat{Q}(z)$. Fortunately, this is not the case. Indeed, the transfer function $\hat{T}_{22}$ from $\hat{u}$ to $\hat{y}$ equals zero, and therefore it is easily verified that the internal stability of the closed-loop system in Fig. 2 automatically implies the stability of $\hat{Q}(z)$.

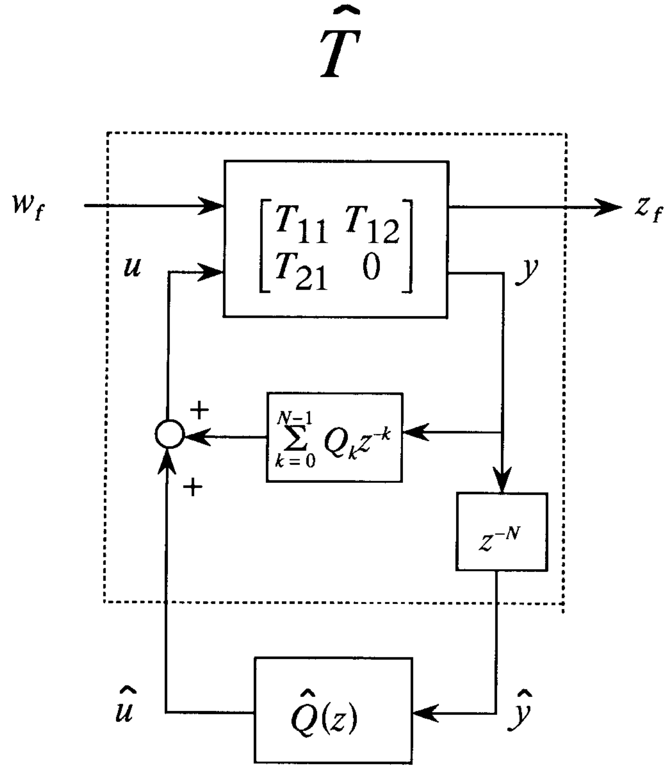

Fig. 2. Augmented plant.

In the subsequent discussion, we will use the following state-space representations:

$$\begin{bmatrix} \sum_{k=0}^{N-1} Q_k z^{-k} \\ z^{-N} \end{bmatrix} = \begin{array}{c} \begin{array}{cc} n_f & p_2 \end{array} \\ \left[\begin{array}{c|c} A_f & B_f \\ \hline C_{f1} & Q_0 \\ C_{f2} & 0 \end{array}\right] \end{array} \tag{14}$$

where $n_f = Np_2$ and

$$A_f = \begin{bmatrix} O & I_{p_2} & O & \cdots & O \\ O & O & I_{p_2} & \cdots & O \\ \vdots & & & \ddots & \\ O & O & & \cdots & I_{p_2} \\ O & O & \cdots & \cdots & O \end{bmatrix}, \qquad B_f = \begin{bmatrix} O_{p_2} \\ O_{p_2} \\ \vdots \\ O \\ I_{p_2} \end{bmatrix}$$

$$\begin{bmatrix} C_{f1} \\ C_{f2} \end{bmatrix} = \begin{bmatrix} O & Q_{N-1} & Q_{N-2} & \cdots & Q_1 \\ I_{p_2} & O & O & \cdots & O \end{bmatrix}. \tag{15}$$

From (8) and (14) the augmented generalized plant $\hat{T}$ thus has the representation

$$\begin{aligned} &\quad |\leftarrow \quad n = 2n_P + n_f \quad \rightarrow| \\ \hat{T} &= \left[\begin{array}{cc|cc} A_f & B_f C_2 & B_f D_{21} & O \\ B_2 C_{f1} & A + B_2 Q_0 C_2 & B_2 Q_0 D_{21} + B_1 & B_2 \\ \hline D_{12} C_{f1} & C_1 + D_{12} Q_0 C_2 & D_{11} + D_{12} Q_0 D_{21} & D_{12} \\ C_{f2} & O & 0 & 0 \end{array}\right] \\ &=: \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & O & O \end{array}\right]. \end{aligned} \tag{16}$$

III. LMI FORMULATION

Now our plant is given by (16). To this plant, we shall use the LMI result for linear discrete time H_∞ control [8]. Notice that $(\hat{A}, \hat{B}_2, \hat{C}_2)$ is stabilizable and detectable. This can be easily verified since A is stable by the selection of F and L. Thus the necessary conditions for applying the LMI result are already met.

From [8], it follows that, for any fixed Q_i's, there exists a stabilizing (and therefore stable) $\hat{Q}(z)$ such that (13) is satisfied if and only if there are symmetric matrices R, S solving the following LMI system:

$$\begin{aligned} &\begin{bmatrix} N_R & O \\ O & I_{m_1} \end{bmatrix}^T \begin{bmatrix} \hat{A} R \hat{A}^T - R & \hat{A} R \hat{C}_1^T & \hat{B}_1 \\ \hat{C}_1 R \hat{A}^T & -I_{p_1} + \hat{C}_1 R \hat{C}_1^T & \hat{D}_{11} \\ \hat{B}_1^T & \hat{D}_{11}^T & -I_{m_1} \end{bmatrix} \\ &\quad \times \begin{bmatrix} N_R & O \\ O & I_{m_1} \end{bmatrix} < O \end{aligned} \tag{17}$$

$$\begin{aligned} &\begin{bmatrix} N_S & O \\ O & I_{p_1} \end{bmatrix}^T \begin{bmatrix} \hat{A}^T S \hat{A} - S & \hat{A}^T S \hat{B}_1 & \hat{C}_1^T \\ \hat{B}_1^T S \hat{A} & -I_{m_1} + \hat{B}_1^T S \hat{B}_1 & \hat{D}_{11}^T \\ \hat{C}_1 & \hat{D}_{11} & -I_{p_1} \end{bmatrix} \\ &\quad \times \begin{bmatrix} N_S & O \\ O & I_{p_1} \end{bmatrix} < O \end{aligned} \tag{18}$$

$$\begin{bmatrix} R & I_{n+n_f} \\ I_{n+n_f} & S \end{bmatrix} \geq O. \tag{19}$$

Here N_R and N_S denote bases of the null spaces of $(\hat{B}_2^T, \hat{D}_{12}^T)$ and $(\hat{C}_2, \hat{D}_{21})$, respectively. In our problem, we have

$$N_R = \begin{bmatrix} I_{n+n_f} & O \\ O & N_{R1} \\ O & N_{R2} \end{bmatrix} \quad \text{and} \quad N_S = \begin{bmatrix} N_{S2} & O & O \\ O & I_n & O \\ O & O & I_{m_1} \end{bmatrix} \tag{20}$$

where $[\begin{smallmatrix} N_{R1} \\ N_{R2} \end{smallmatrix}]$ and N_{S2} are bases of the null spaces of (B_2^T, D_{12}^T) and C_{f2}, respectively.

Hence

$$N_{R1}^T B_2 + N_{R2}^T D_{12} = O$$

$$N_{S2} = \begin{bmatrix} O & \cdots & O \\ I_{p_2} & \cdots & O \\ & \ddots & \\ O & \cdots & I_{p_2} \end{bmatrix}. \tag{21}$$

Thus, considering the constraints for the time-domain together, our problem becomes determining $R, S,$ and Q_i's so that (11) and (17)–(19) are satisfied. Notice here that Q_i are not fixed but are variables and that matrices C_{f1}, and therefore $\hat{A}$, contain Q_i's. So (17) and (18) contain products of the unknown matrices R, S and Q_i's. This implies that the problem does not, in general, remain in the realm of LMI.

Assumption: D_{21} $(= D_{2f})$ is of row full rank. Its right inverse is denoted by D_{21}^+. Thus, $D_{21} D_{21}^+ = I$.

We can obtain the following main result of this paper.

Theorem: The H_∞-control problem with time-domain constraints is solvable if and only if the LMI system consisting of (11) and

$$\begin{bmatrix} F_{R1} R F_{R1}^T - F_{R2} R F_{R2}^T - F_{R3} F_{R3}^T & G_R \\ G_R^T & -I_{m_1} \end{bmatrix} < O \tag{22}$$

$$\begin{bmatrix} F_{S1}^T \hat{S} F_{S1} - F_{S2}^T \hat{S} F_{S2} - F_{S3}^T F_{S3} & G_S^T \\ G_S & -I_{p_1} \end{bmatrix} < O \tag{23}$$

$$\begin{bmatrix} R & W_1^{-1} \\ W_1^{-T} & \hat{S} \end{bmatrix} \geq O \tag{24}$$

is feasible, i.e., it has solution $(R, \hat{S}, Q_0, Q_1, \cdots, Q_{N-1})$ satisfying (11) and (22)–(24) with

$$F_{R1} = \begin{bmatrix} A_f & B_f C_2 \\ O & N_{R1}^T A + N_{R2}^T C_1 \end{bmatrix}, \qquad F_{R2} = \begin{bmatrix} I_{Np_2} & O \\ O & N_{R1}^T \end{bmatrix}$$

$$F_{R3} = \begin{bmatrix} O \\ N_{R2}^T \end{bmatrix}, \qquad G_R = \begin{bmatrix} B_f D_{21} \\ N_{R1}^T B_1 + N_{R2}^T D_{11} \end{bmatrix}$$

and

$$\begin{aligned}
F_{S1} &= \begin{bmatrix} A_f N_{S2} & O & B_f D_{21} \\ O & A - B_1 D_{21}^{+} C_2 & B_1 \end{bmatrix} \\
F_{S2} &= \begin{bmatrix} N_{S2} & O_{Np_2 * n} & O_{Np_2 * m_1} \\ O_{n*(N-1)p_2} & I_n & O_{n*m_1} \end{bmatrix} \\
F_{S3} &= [O_{mi*(N-1)p_2} \quad -D_{21}^{+} C_2 \quad I_{m1}], \\
G_S &= [G_{S1} \quad G_{S2} \quad G_{S3}], \\
G_{S1} &= D_{12}[Q_{N-1} \quad \cdots \quad Q_1] - C_1 [T_{N-2} \quad \cdots \quad T_0] \\
G_{S2} &= C_1 - D_{11} D_{12}^{+} C_2 \\
G_{S3} &= D_{11} + D_{12} Q_0 D_{21}
\end{aligned} \tag{25}$$

$$W_1^{-1} = \begin{bmatrix} I_{p_2} & O & O & \cdots & O \\ O & I_{p_{21}} & O & \cdots & O \\ & & \ddots & & \\ O & O & \cdots & I_{p_2} & O \\ -T_{N-1} & -T_{N-2} & \cdots & -T_0 & I_n \end{bmatrix} \tag{26}$$

where

$$\begin{aligned}
T_0 &= -B_2 Q_0 \\
T_k &= -\sum_{s=0}^{k} (A - B_1 D_{21}^{+} C_2)^{k-s} B_2 Q_s, \quad k = 1, 2, \cdots, N-1.
\end{aligned} \tag{27}$$

Notice that when the problem is feasible, the computation of the $\hat{Q}(z)$ can be performed exactly in the same way as stated in [8], using Q_i's, R, and S, whose relation with $\hat{S}$ is given by (28) below in the proof of the theorem. Then substituting $\hat{Q}(z)$ into (10), the controller to the original problem is given by (2).

One of the immediate but interesting consequences from the above result is the following.

Corollary: Suppose that $Q^a(z)$ and $Q^b(z)$ are the two solutions to the H_∞-control problem with time-domain constraints (11) and that $z_{ta}(k)$ and $z_{tb}(k)$, $\{k = 1, 2, \cdots, N-1\}$ are the corresponding outputs to a given input $w_t(k)$, respectively. Then there exists a controller $Q(z)$ giving the output $\alpha z_{ta}(k) + (1 - \alpha) z_{tb}(k)$, $(0 \le \alpha \le 1)$ that satisfies the same norm condition.

Remark: Note that the existence condition expressed by (22)–(24) can also be used for H_∞-control problems with constraints other than time-domain constraints (11). A necessary property of such constraints is the convexity with respect to $Q_0, Q_1, \cdots, Q_{N-1}$. Among them, the H_2/H_∞ problem [7] would be an interesting application.

Proof of the Theorem: We will show that it is possible to transform (17)–(19) into the equivalent expressions, which are linear with respect to unknown variables, thus one can recover the LMI structure once again.

First, by substituting the expressions (16) and (20) for $\hat{A}$, $\hat{B}_1$, $\hat{C}_1$, N_R into (17), we get immediately (22). Thus (17) and (22) are equivalent.

Next, let us consider (18). Unfortunately, it is not an LMI as it stands, but it can be transformed to an LMI by the change of a variable.

Indeed, set

$$S = W_1^T \hat{S} W_1 \tag{28}$$

where

$$W_1 = \begin{bmatrix} I_{p_2} & O & O & \cdots & O \\ O & I_{p_2} & O & \cdots & O \\ & & \ddots & & \\ O & O & \cdots & I_{p_2} & O \\ T_{N-1} & T_{N-2} & \cdots & T_0 & I_n \end{bmatrix} \tag{29}$$

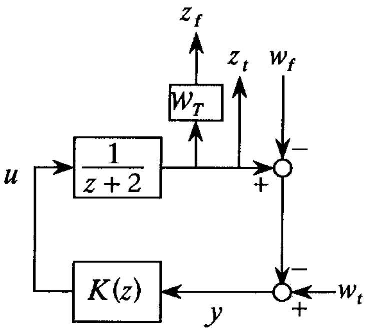

Fig. 3. Block diagram for the example.

and T_i are defined by (27). From the definition of T_i's we have

$$\begin{aligned}
&B_2[Q_{N-1} \quad Q_{N-2} \quad \cdots \quad Q_1] + [T_{N-1} \quad T_{N-2} \quad \cdots \quad T_1] \\
&\quad = (A - B_1 D_{21}^{+} C_2)[T_{N-2} \quad T_{N-1} \quad \cdots \quad T_0]
\end{aligned} \tag{30}$$

and by (8)

$$C_2 T_k = 0, \qquad k = 0, 1, 2, \cdots, N-1. \tag{31}$$

Furthermore, define

$$W_2 = \begin{bmatrix} I_{(N-1)p_2} & 0 & 0 & 0 \\ -[T_{N-2} \;\; T_{N-3} \;\; \cdots \;\; T_0] & I_n & 0 & 0 \\ 0 & -D_{21}^{+} C_2 & I_{m_1} & 0 \\ 0 & 0 & 0 & I_{p_1} \end{bmatrix}.$$

Now multiply both sides of (18) by W_2^T from the left and W_2 from the right and substitute (28). Then by making use of (30), (31) and with a little bit of tedious but straightforward computation, we can get (23).

Finally, by substitution of (27) into (18), we get

$$\begin{bmatrix} R & I \\ I & W_1^T \hat{S} W_1 \end{bmatrix} \ge O$$

which is obviously equivalent to (24). The inverse of W_1 is given by (26). This completes the proof.

IV. A COMPUTATIONAL EXAMPLE

To illustrate the result, let us consider the simple example depicted in Fig. 3. The plant has the unstable transfer function given by

$$G(z) = \frac{1}{z+2}.$$

We are interested in designing a controller $K(z)$ such that: 1) it stabilizes the closed loop; 2) the closed-loop transfer function $G_{z f w f}$ from w_f to z_f satisfies

$$\|w_T G_{z_f w_f}\|_\infty < 1 \tag{32}$$

to realize the robust stability property against the multiplicative perturbation, whose upper bound is assumed to be w_T; and 3) the output z_t to the unit impulsive input w_t satisfies

$$|z_t(k)| \le 1.7 \quad (k = 0, 1), \qquad |z_t(k)| \le 1 \quad (2 \le k \le 9).$$

We assume $w_T = 0.4$ (constant) for simplicity. The generalized plant corresponding to this example is given by

$$P(z) = \left[\begin{array}{c|ccc} -2 & 0 & 0 & 1 \\ \hline w_T & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \end{array}\right].$$

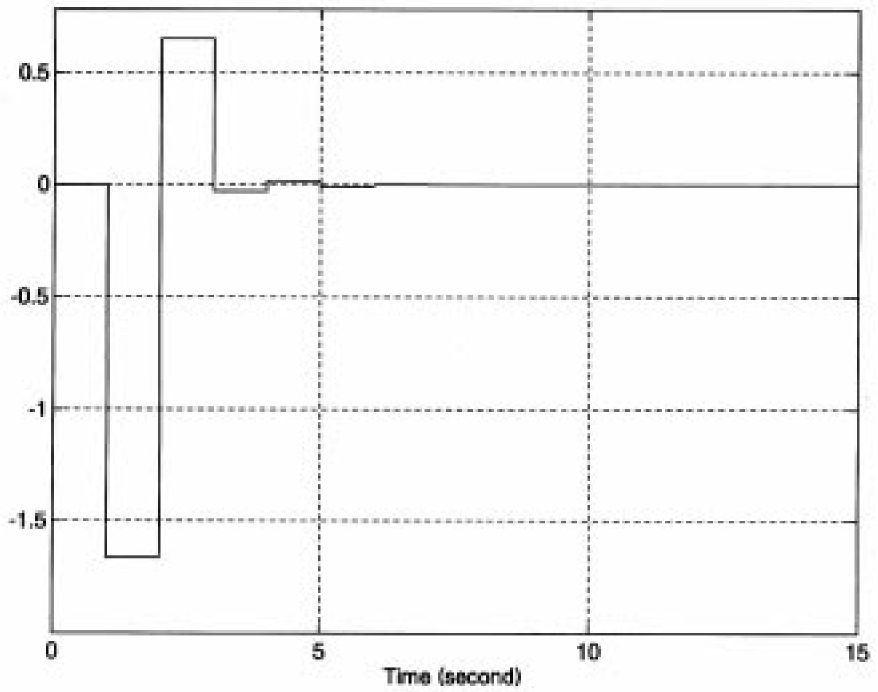

Fig. 4. Impulse response.

Applying the result in the previous section, a twenty-third order controller was obtained. Then using the balanced realization, it was reduced to the second-order controller

$$K(z) = \frac{-1.663z^2 + 0.3594z + 0.087\,72}{z^2 - 0.1598z - 0.040\,41}.$$

The simulation result is illustrated in Fig. 4. The norms $\|w_T G_{z_f w_f}\|_\infty$ attained by the full and the reduced-order controllers were, respectively, 0.942 and 0.946, both of which satisfy (32). All LMI-related computations were performed with MATLAB's LMI Control Toolbox [9].

V. CONCLUSION

The H_∞-control problem with time-domain constraints has been discussed. The problem has been formulated as a model-matching problem, to which an LMI technique has been employed. The dimension of the controllers to be constructed according to the present theory is $3n_P + 2p_2 N$, where n_P, p_2, N are, respectively, the degree of the plant $P(z)$, the number of the observed output, and the duration of the time-domain constraints. Generally this number is not small and its reduction is necessary. In our example we observed that, using the balanced realization, considerable reduction of the controller's order was possible (this point was also observed in [4]). However, the general discussion concerning the order reduction remains as one of the main topics of the future work. With respect to this, it would be interesting to examine some possibility of using freedom in the determination of $\hat{Q}(z)$. The extension to the infinite horizon case and also the problem of avoiding the possible asymptotic explosion of the response after N steps are also important works. For this we can use the technique developed in [4]–[6]

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