On the Structural Controllability of Compartmental Systems

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Abstract—**In this paper compartmental systems are modeled by a class of linearly parameterized matrix pairs and the controllability characteristics are expressed in terms of a compartmental graph. The result gives intuitive and physical information about the controllability and it extends the well-known result on the single-sink case to the multisink case. Furthermore, this result is applied to more special models, called undirected compartmental models, by which physical systems such as a class of liquid systems and of network systems, etc., can be concisely modeled.**

I. Introduction

CONSIDER the linear time-invariant system

$$\dot{x}(t)=A_p x(t)+B_p u(t) \tag{1}$$

where $x(t)\in R^n$ and $u(t)\in R^m$. The matrices A_p and B_p are parameterized by a set of physical parameters P to reflect *a priori* structural information of the physical system. In this framework it is sometimes of interest to know whether or not the matrix pair (A_p, B_p) is controllable for all but an exceptional set of values of P. Parameterized matrix pairs with this property are called structurally controllable. The concept was first introduced by Lin who in [1] developed graph-theoretic conditions for certain linearly and independently parameterized matrix (called structural matrix) pairs with single-input to be structurally controllable. Since then, the subject attracted considerable interest [2]–[8]. One obvious importance of the study lies in the fact that structural controllability is a property that is as useful as controllability and can be determined free from numerical computations. However, a more important and interesting point would be the fact that it is sometimes possible to know structural controllability directly from the information about physical systems (e.g., connections between certain parts of a system). Stressing the second viewpoint, in this paper we shall also discuss the structural controllability of linearly parameterized matrix pairs but of more restricted systems, called compartmental models.

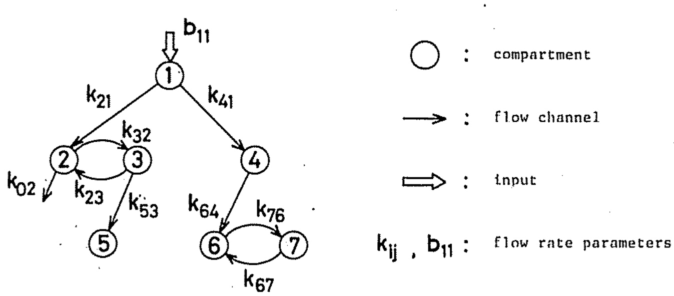

Fig. 1. A compartmental system.

Compartmental models have been widely used in studying biological, biomedical, ecological, chemical, and economical systems [9]–[11]. *A priori* structural information of this class of systems is usually given by a certain kind of digraph called a compartmental graph. For example, consider a compartmental model whose structure is represented by the compartmental graph in Fig. 1. Denoting by x_i the amount of the material in compartment i and taking $x_1, \cdots, x_7$ as a state variable, the matrices A_p and B_p in (1) can be represented as

$$A_p=\begin{pmatrix} -k_{21}-k_{41} & 0 & 0 & 0 & 0 & 0 & 0\\ k_{21} & -k_{02}-k_{32} & k_{23} & 0 & 0 & 0 & 0\\ 0 & k_{32} & -k_{23}-k_{53} & 0 & 0 & 0 & 0\\ k_{41} & 0 & 0 & -k_{64} & 0 & 0 & 0\\ 0 & 0 & k_{53} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & k_{64} & 0 & -k_{76} & k_{67}\\ 0 & 0 & 0 & 0 & 0 & k_{76} & -k_{67} \end{pmatrix}$$

and

$$B_p=\begin{pmatrix} b_{11}\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0 \end{pmatrix} \tag{2}$$

where

$$P=\{k_{21},k_{41},k_{02},k_{32},k_{23},k_{53},k_{64},k_{76},k_{67},b_{11}\}.$$

(For detailed explanations about the compartmental graph and the state equation (1) of the compartmental models, see Section III.) Clearly the matrix pair (A_p, B_p) in (2) can easily be shown to belong to the class of linearly parameterized matrix pairs defined in [7]. Thus, applying the result of [7] the conditions for the

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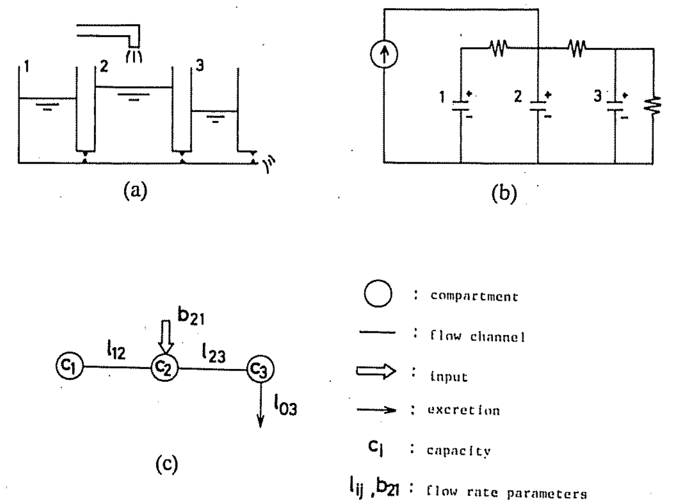


Fig. 2. Examples of undirected compartmental systems. (a) A liquid system. (b) A network system. (c) An undirected compartmental graph.

structural controllability can be represented by making use of a certain graph (see Section II) and by some matrix rank conditions. Unfortunately, however, the graph used in [7] and the compartmental graph are not the same.

To obtain intuitive and physical information about the controllability, it is of course desirable to represent conditions in terms of the original graph which directly reflects *a priori* information of the physical systems. In this paper it will be shown that this is possible for compartmental models. The result extends the one of [13] to the multisink case. The result will be applied to more special models, called undirected compartmental models, by which physical systems such as liquid system in Fig. 2(a) and a network system in Fig. 2(b), etc., can be concisely modeled.

II. Preliminaries

Before considering the structural controllability of compartmental models, we will summarize the concepts that are developed in [7] and some lemmas that will be used subsequently.

A system (A_p, B_p) in (1) is said to be linearly parameterized by P, if the functional relationship between P and (A_p, B_p) can be expressed as

$$A_p = A_0 + \sum_{i=1}^{k} B_i P_i C_i$$
$$B_p = B_0 + \sum_{i=1}^{k} B_i P_i D_i \tag{3}$$

where the matrices A_0, B_0, B_i, C_i, and D_i are fixed and independent of P, and the elements of $P_1, \cdots, P_k$ are assumed to be algebraically independent parameters. Associated with this parameterization $\Pi \equiv \{A_0, B_0, B_i, C_i, D_i, k\}$, the transfer function H_{ij} from node j to node i is defined as

$$H_{ij}(\lambda) = \begin{cases} C_i(\lambda I_n - A_0)^{-1} B_j, & i, j \in \boldsymbol{k} \\ C_i(\lambda I_n - A_0)^{-1} B_0 + D_i, & i \in \boldsymbol{k}, \quad j = 0 \end{cases} \tag{4}$$

where $\boldsymbol{k} = \{1,2,\cdots,k\}$, and the graph of Π, denoted by G_Π, is defined to be the directed graph which has 1) the set $\boldsymbol{k_0}$ of $k+1$ nodes, labeled $0,1,2,\cdots,k$, and 2) edges $[j \rightarrow i]$ that correspond in a one-to-one way to the nonzero transfer functions.

If S is a nonempty subset of $\boldsymbol{k}$ with elements $i_1, i_2, \cdots, i_s$ ordered so that $i_1 < i_2 < \cdots < i_s$, then B_S, C_S, and D_S are defined so that

$$B_S = \left[B_{i_1}, B_{i_2}, \cdots, B_{i_s}\right], \quad C_S = \begin{pmatrix} C_{i_1} \\ C_{i_2} \\ \vdots \\ C_{i_s} \end{pmatrix}, \quad D_S = \begin{pmatrix} D_{i_1} \\ D_{i_2} \\ \vdots \\ D_{i_s} \end{pmatrix}. \tag{5}$$

If S is the empty set, then B_S, C_S, and D_S each are the 0×0 matrix. Furthermore, denote by $\boldsymbol{k} - S$ the complement of S in $\boldsymbol{k}$ and by $\boldsymbol{k}^*$ the power set of $\boldsymbol{k}$.

Then the result obtained by Corfmat and Morse [7] is the following.

Lemma 1 [7]: Let $\Pi \equiv \{A_0, B_0, B_i, C_i, D_i, k\}$ be a fixed parameterization of (A_p, B_p). Then (A_p, B_p) is structurally controllable if and only if:

1) for each $i \in \boldsymbol{k}$ there exists a path[1] in G_Π from node 0 to node i, and

2) for all λ in the (complex) spectrum of A_0;

$$\operatorname{rank}\begin{pmatrix} \lambda I_n - A_0 & B_0 & B_S \\ C_{k-S} & D_{k-S} & 0 \end{pmatrix} \geqslant n \qquad \text{for } \forall S \in \boldsymbol{k}^*. \tag{6}$$

To transform condition 2) of Lemma 1 to certain graphical conditions in the case of compartmental models we will need the following lemma, the first part of which was proven in [15] geometrically.

Lemma 2: Let $\mathscr{B}_1, \mathscr{B}_2, \cdots$, and $\mathscr{B}_k$ be linear subspaces in R^n. For a subset S of $\boldsymbol{k}$ let $\mathscr{B}_S$ be the subspace $\mathscr{B}_S = \Sigma_{i \in S} \mathscr{B}_i$ and let $\mathscr{B}_\varnothing = \{0\}$ where ϕ denotes the empty set. Then

i) if $k \leqslant n$,

$$\dim \mathscr{B}_S \geqslant |S| \qquad \text{for } \forall S \in \boldsymbol{k}^* \tag{7}$$

if and only if there exists $b_i \in \mathscr{B}_i$ for $i = 1,2,\cdots,k$ such that $b_1, \cdots, b_k$ are linearly independent, and

ii) if $k > n$,

$$\dim \mathscr{B}_S \geqslant |S| - k + n \qquad \text{for } \forall S \in \boldsymbol{k}^* \tag{8}$$

if and only if there exists $b_i \in \mathscr{B}_i$ for $i = 1,2,\cdots,k$ such that $b_1, \cdots, b_k$ span R^n, where $|\cdot|$ denotes the number of elements in the indicated set.

Proof of ii): The assertion ii) is an easy consequence of i). Indeed, let $\tilde{\mathscr{B}}_i = \mathscr{B}_i \oplus R^{k-n}$ in $R^n \oplus R^{k-n}$ where $\oplus$ denotes an external direct sum.

Then (8) is equivalent to the following:

$$\dim \tilde{\mathscr{B}}_S \geqslant |S| \qquad \text{for } \forall S \in \boldsymbol{k}^* \tag{9}$$

for $\dim \tilde{\mathscr{B}}_S = \dim \mathscr{B}_S + k - n$. Therefore, by i) of Lemma 2, (9) holds if and only if there exist $\tilde{b}_i \in \tilde{\mathscr{B}}_i$ for $i = 1, \cdots, k$ such that $\tilde{b}_1, \cdots, \tilde{b}_k$ are linear independent, and from the definition of $\tilde{\mathscr{B}}_i$'s this is equivalent to that there exists $b_i \in \mathscr{B}_i$ for $i = 1, \cdots, k$ such that $b_1, \cdots, b_k$ span R^n. Q.E.D.

Remark: Although we omit the detail here, we want to point out that the above lemma is a direct result of the König–Egerváry theorem in matroid theory [12].

III. Compartmental Models

Compartmental models are typically used to represent systems which can be viewed as a finite number of compartments interconnected by flow channels where each compartment contains some uniformly distributed material of interest. The rate of flow of material from compartment j to compartment i is assumed

[1]A path in G_Π from node i_1 to node i_f can be interpreted as a sequence of nodes $i_1, i_2, \cdots, i_f$ such that there exist edges $[i_{q-1} \rightarrow i_q]$ for $q = 2,3,\cdots,f$.

proportional to the amount of material in compartment j. There may be flow called excretion from some compartments to the model's environment. Flows from the environment to the model are assumed to be known or under the control of the model designer, and thus can be regarded as inputs to the model.

Now consider a compartmental system which is composed of n compartments, and give these n compartments a numbering from 1 to n in an arbitrary but fixed way. For the easiness of explanation, the environment will be conveniently called compartment 0.

Now let x_j be the state variable which denotes the amount of material in compartment j; k_{ij} be the proportionality parameter which characterizes the rate of flow from compartment j to compartment i; and k_{0j} be the proportionality parameter which characterizes the rate of flow from compartment j to compartment 0 (the environment). Then the compartmental system can be modeled by $\dot{x}=A_px+B_pu$ where, denoting (i,j) entry of A_p (resp. B_p) by a_{ij} (resp. b_{ij}),

$$a_{ii}=-\sum_{\substack{j=0\\ j\neq i}}^{n} k_{ji} \qquad (i=1,\cdots,n) \tag{10-1}$$

$$a_{ij}=k_{ij} \qquad (i,j=1,\cdots,n;\ i\neq j) \tag{10-2}$$

and b_{ij} is the flow rate parameter from the jth input to compartment i. It is assumed that a number of the model parameters k_{ij} are fixed zeros and the rest are indeterminate and unrelated. Note that if $k_{0i}=0$, the sum of all the entries in the ith column of A_p is zero. Furthermore, without loss of generality we restrict our consideration to a compartmental system (A_p, B_p) where each row of B_p contains at most one nonfixed element and each column of B contains exactly one nonfixed element [13]. For such a system, its structure is usually characterized by the compartmental graph [13] denoted by G_C. For the terminology about the compartmental graph, please refer to [13].

Corresponding to the flow rates k_{ij} $(i\neq 0)$, k_{0j}, and the weight b_{kl} of input edge, that are regarded as independent parameters, define n-dimensional vectors e_{ij}, e_{0j}, e_k as follows:

$$k_{ij}:\begin{pmatrix}0\\0\\-1\\ \vdots\\0\\1\\0\\ \vdots\\0\end{pmatrix}\begin{matrix}\\ \\ j\text{th}\\ \\ \\ i\text{th}\\ \\ \\ \\ \end{matrix}\quad (:=e_{ij}),\qquad k_{0j}:\begin{pmatrix}0\\ \vdots\\0\\-1\\0\\ \vdots\\0\\ \vdots\\0\end{pmatrix}\begin{matrix}\\ \\ \\ j\text{th}\\ \\ \\ \\ \\ \\ \end{matrix}\quad (:=e_{0j}) \tag{11}$$

and

$$b_{kl}:\begin{pmatrix}0\\ \vdots\\0\\1\\0\\ \vdots\\0\end{pmatrix}\begin{matrix}\\ \\ \\ k\text{th}\\ \\ \\ \\ \end{matrix}\quad (:=e_k)\qquad (l=1,\cdots,m).$$

Furthermore, denote by K_j the set of all the flow rates on outgoing edges from compartment j $(j=1,2,\cdots,n)$ and define matrices B_j's and vectors P_j's as follows. If K_j is a nonempty set, say

$$K_j=\{k_{i_1j},\cdots,k_{i_sj}\},$$

then

$$B_j=\left[e_{i_1j},\cdots,e_{i_sj}\right],\qquad P_j=\begin{pmatrix}k_{i_1j}\\ \vdots\\ k_{i_sj}\end{pmatrix}. \tag{12-1}$$

If K_j is the empty set, then

$$B_j=\begin{pmatrix}0\\ \vdots\\0\end{pmatrix},\qquad P_j:\ \text{scalar parameter}. \tag{12-2}$$

Finally, let the set of all the excited compartments be $E=\{l_1,\cdots,l_m\}$ and define B_{n+j} and P_{n+j} for $j=1,2,\cdots,m$ as

$$B_{n+j}=e_{l_j},\qquad P_{n+j}=b_{l_jj}. \tag{12-3}$$

Example: Consider the compartmental system in Fig. 1, whose state space equation is given by (2). Then

$$K_1=\{k_{21},k_{41}\},\quad K_2=\{k_{02},k_{32}\},\quad K_3=\{k_{23},k_{53}\},$$
$$K_4=\{k_{64}\}\quad K_5=\varnothing,\quad K_6=\{k_{76}\},\text{ and } K_7=\{k_{67}\}.$$
$$\text{Furthermore } E=\{1\}.$$

Thus, B_j's and P_j's are, respectively, given by

$$B_1=\begin{pmatrix}-1&-1\\1&0\\0&0\\0&1\\0&0\\0&0\\0&0\end{pmatrix},\quad B_2=\begin{pmatrix}0&0\\-1&-1\\0&1\\0&0\\0&0\\0&0\\0&0\end{pmatrix},\quad B_3=\begin{pmatrix}0&0\\1&0\\-1&-1\\0&0\\0&1\\0&0\\0&0\end{pmatrix},$$

$$B_4=\begin{pmatrix}0\\0\\0\\-1\\0\\1\\0\end{pmatrix},\quad B_5=\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\end{pmatrix},\quad B_6=\begin{pmatrix}0\\0\\0\\0\\0\\-1\\1\end{pmatrix},\quad B_7=\begin{pmatrix}0\\0\\0\\0\\0\\1\\-1\end{pmatrix},$$

$$P_1=\begin{pmatrix}k_{21}\\k_{41}\end{pmatrix},\quad P_2=\begin{pmatrix}k_{02}\\k_{32}\end{pmatrix},\quad P_3=\begin{pmatrix}k_{23}\\k_{53}\end{pmatrix},$$
$$P_4=(k_{64}),\quad P_5,\quad P_6=(k_{76}),\quad P_7=(k_{67}).$$

Now it is clearly seen that the vector B_iP_i (resp. $B_{n+j}P_{n+j}$) coincides with the ith (resp. jth) column of A_p (resp. B_p) for $i=1,\cdots,n$, (resp. $j=1,\cdots,m$), and thus the compartmental system (A_p, B_p) can be expressed in the following linearly parameterized form:

$$A_p=\sum_{i=1}^{n+m} B_iP_iC_i$$
$$B_p=\sum_{i=1}^{n+m} B_iP_iD_i \tag{13}$$

where

$$C_i=\begin{cases}(0,\cdots,0,\overset{i\text{th}}{\overset{\downarrow}{1}},0,\cdots,0); & 1\times n\quad \text{for } i=1,2,\cdots,n\\ (0,\cdots\ \cdots\ \cdots\ \cdots,0); & 1\times n\quad \text{for } i=n+1,\cdots,n+m\end{cases} \tag{14-1}$$

and

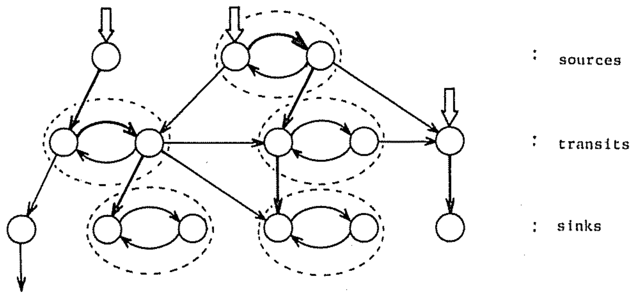


Fig. 3. An example of structurally controllable compartmental systems.

$$D_i = \begin{cases} (0,\cdots\ \cdots\ \cdots\ \cdots,0); & 1\times m \quad \text{for } i=1,\cdots,n \\ (0,\cdots,0,\overset{(i-n)\text{th}}{\overset{\downarrow}{1}},0,\cdots,0); & 1\times m \quad \text{for } i=n+1,\cdots,n+m. \end{cases} \tag{14-2}$$

Therefore, by applying Lemmas 1 and 2 in Section II we can obtain the following theorem.

Theorem 1: Without loss of generality assume that the compartmental graph does not split into two or more disjoint subgraphs. A compartmental system (A_p, B_p) is structurally controllable if and only if in its corresponding compartmental graph G_c:

1) a compartment from each source is excited, and

2) sinks with no excretions are connected with some excited compartments by mutually disjoint paths (Fig. 3).

If the graph G_c is strongly connected, it should be regarded as a source in the condition 1) while as a sink in the condition 2).

Remark: This theorem extends Zazworsky and Knudsen's results [13] on the single-sink case to the multisink case.

Proof of Theorem 1: A compartmental system (A_p, B_p) has the parameterization $\Pi \equiv \{0,0,B_i,C_i,D_i,n+m\}$ with B_i and C_i, D_i being given, respectively, by (12) and (14). Since $D_j \neq 0$ $(j=n+1,\cdots,n+m)$, G_Π has edges $[0\to j]$ by the definition of the graph G_Π. Furthermore, since $C_i(\lambda I_n)^{-1}B_jP_j = k_{ij}/\lambda$ $(1\leqslant i, j\leqslant n,\ i\neq j)$, G_Π has edge $[j\to i]$ if and only if G_c does. Therefore, it is obvious that the condition 1) of Lemma 1 is equivalent to the condition 1) of Theorem 1. It remains to prove that the condition 2) of Lemma 1 is equivalent to the condition 2) of Theorem 1.

To show this we need the following remarks. First, observe that for the parameterization $\Pi \equiv \{0,0,B_i,C_i,D_i,n+m\}$ with B_i and C_i, D_i being given by (12) and (14), the condition 2) of Lemma 1 is equivalent to

$$\operatorname{rank} B_S \geqslant |S| - m \qquad \text{for} \quad \forall S \in (\boldsymbol{n}+\boldsymbol{m})^*.$$

Thus, it follows from ii) of Lemma 2 that the condition 2) of Lemma 1 is equivalent to the following. Denote by $\mathscr{T}$ the collection of $n\times(n+m)$ matrices whose ith column is a column vector of B_i, i.e.,

$$\mathscr{T} = \left\{ T_{k_1k_2\cdots k_{n+m}} = \left(b_{k_1}^{(1)}, b_{k_2}^{(2)}, \cdots, b_{k_{n+m}}^{(n+m)}\right) \Big| b_{k_i}^{(i)}: \text{the } k_i\text{th column of } B_i \right\}. \tag{15}$$

Then there exists a $T\in\mathscr{T}$ such that $\operatorname{rank} T = n$.

Secondly, recall that the matrices B_j's are composed of the column vectors of the form e_{ij}, e_{0j}, or e_j, and there are one-to-one correspondences between the edges $[j\to i]$, the flow rate k_{ij}, b_{ij}, and the vectors e_{ij}, e_{0j}, e_j. From this fact and the definition of $\mathscr{T}$ it is clearly seen that each matrix $T\in\mathscr{T}$ corresponds also in a one-to-one way to a subgraph of G_c, each compartment of which has at most one outgoing edge.

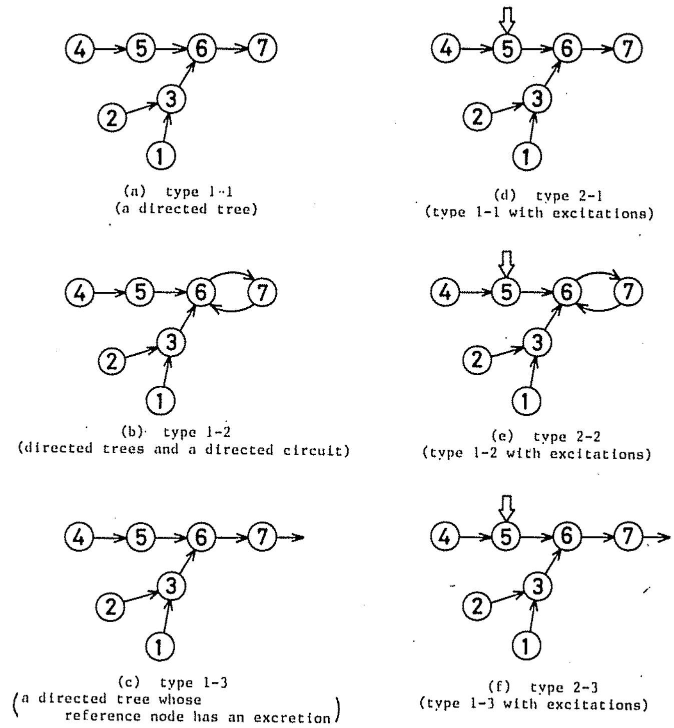


Fig. 4. Examples of type i-j subgraphs.

Lastly, let us notice that in a subgraph of G_c if every compartment has at most one outgoing edge, then the subgraph consists of mutually disjoint subgraphs which can be classified into at most six types. They are type 1-1, 1-2, 1-3, 2-1, 2-2, and 2-3 subgraphs whose definitions are as follows (Fig. 4). Type 1-1 is a subgraph which has directed tree structure [14]. Type 1-2 is a subgraph which consists of directed trees and a directed circuit [14], in which the reference nodes of directed trees are nodes of the directed circuit. If type 1-1 has an excretion at its reference node, it is called type 1-3. If type 1-1, type 1-2, and type 1-3 have at least one excitation, they are called type 2-1, type 2-2, and type 2-3, respectively.

Now assume first that the condition 2) of Theorem 1 is satisfied. We assert that under the condition 2) of Theorem 1, G_c is spaned[2] by a subgraph consisting of mutually disjoint subgraphs of type 1-3, 2-1, 2-2, 2-3. For this note that sinks are strongly connected components of G_c by the definition, and therefore if a sink is composed of more than one compartment, it has at least one directed circuit.

Let us assume that G_c has α simple sinks[3] $S_{11},\cdots,S_{1\alpha}$ without excretions, β nonsimple sinks $S_{21},\cdots,S_{2\beta}$ without excretions, and γ sinks $S_{31},\cdots,S_{3\gamma}$ with excretions. Denote by H_{2j} any one of the directed circuits contained in S_{2j} $(j=1,\cdots,\beta)$ and by H_{3k} any one of the compartments in S_{3k} that have excretions $(k=1,\cdots,\gamma)$. Finally, set $H_{1l}=S_{1l}$ $(l=1,\cdots,\alpha)$. Now the assumption implies that there are disjoint paths $P_{11},\cdots,P_{1\alpha}$, $P_{21},\cdots,P_{2\beta}$ leading from some excited compartments to $H_{11},\cdots,H_{1\alpha}$, $H_{21},\cdots,H_{2\beta}$. Define

[2]If the compartments set of a subgraph H of G_c coincides with the compartments set of G_c, G_c is said to be spaned by H.

[3]A simple sink means a sink which is composed of only one compartment.

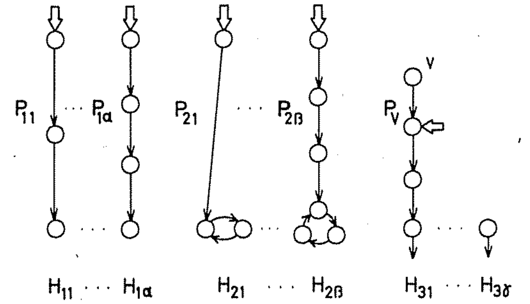


Fig. 5. A structure of the subgraph H_2.

$$H_1=(H_{11}\cup P_{11})\dot{\cup}\cdots\dot{\cup}(H_{1\alpha}\cup P_{1\alpha})\dot{\cup}(H_{21}\cup P_{21})\dot{\cup}\cdots\dot{\cup}(H_{2\beta}\cup P_{2\beta})\dot{\cup}H_{31}\dot{\cup}\cdots\dot{\cup}H_{3\gamma}.$$

Note that H_1 is composed of type 2-1, 2-2, and 1-3 subgraphs. Therefore, if G_c is spaned by H_1, then H_1 is the desired subgraph. On the other hand, assume that $V_{G_C}\supsetneqq V_{H_1}$ where V_{G_C} and V_{H_1}, respectively, denote the compartments set of G_C and H_1, and let v be an arbitrary compartment contained in $V_{G_C}-V_{H_1}$. Since any compartment of G_C is connected with some sink through a path, it can be seen that a path P_v exists such that the initial compartment of P_v is the compartment v and P_v has exactly one compartment in common with H_1. If the common belongs, for instance, to H_{31}, then write

$$H_2=(H_{11}\cup P_{11})\dot{\cup}\cdots\dot{\cup}(H_{1\alpha}\cup P_{1\alpha})\dot{\cup}(H_{21}\cup P_{21})\dot{\cup}\cdots\dot{\cup}(H_{2\beta}\cup P_{2\beta})\dot{\cup}(H_{31}\cup P_v)\dot{\cup}H_{32}\dot{\cup}\cdots\dot{\cup}H_{3\gamma}.$$

Note that this is composed of type 2-1, 2-2, 2-3, 1-3 subgraphs (see Fig. 5). Clearly the same procedure can be applied successively until we obtain a subgraph H_ν such that $V_{G_C}=V_{H_\nu}$. Therefore, the assertion was verified.

Now observe that every compartment of subgraph H_ν has at most one outgoing edge, and thus the subgraph corresponds in a one-to-one way to a matrix in $\mathscr{T}$. By renumbering the compartments and inputs appropriately, it will be easy to see that the matrix, denoted by T_{H_ν}, can be put into the following form:

$$T_{H_\nu}=\left(\begin{array}{cccccccccc:cccccccc}
T_1^{13} & & & & & & & & & & 0 & & & & & & & \\
 & \ddots & & & & & & & & & \vdots & & & & & & & \\
 & & T_{\gamma_1}^{13} & & & & & \bigcirc & & & 0 & & & & \bigcirc & & & \\
 & & & T_1^{21} & & & & & & & U_1^{21} & & & & & & & \\
 & & & & \ddots & & & & & & & \ddots & & & & & & \\
 & & & & & T_\alpha^{21} & & & & & & & U_\alpha^{21} & & & & & \\
 & & & & & & T_1^{22} & & & & & & & U_1^{22} & & & & \\
 & & & & & & & \ddots & & & & & & & \ddots & & & \\
 & \bigcirc & & & & & & T_\beta^{22} & & & & \bigcirc & & & U_\beta^{22} & & & \\
 & & & & & & & & T_{\gamma_1+1}^{23} & & & & & & & U_{\gamma_1+1}^{23} & & \\
 & & & & & & & & & \ddots & & & & & & & \ddots & \\
 & & & & & & & & & T_\gamma^{23} & & & & & & & & U_\gamma^{23}
\end{array}\right) \tag{16}$$

where T^{13}'s and $[T_k^{ij}\ \ U_k^{ij}]$'s $[(i,j)\neq(1,3)]$ represent, respectively, the matrices which are uniquely determined corresponding to the subgraphs type 1-3 and type i-j $[(i,j)\neq(1,3)]$ of H_ν (regarding the subgraphs as the compartmental graphs by themselves). For instance, the matrices corresponding to the graphs in Fig. 4 are given by

$$T_k^{13}=\begin{pmatrix}-1&0&0&0&0&0&0\\0&-1&0&0&0&0&0\\1&1&-1&0&0&0&0\\0&0&0&-1&0&0&0\\0&0&0&1&-1&0&0\\0&0&1&0&1&-1&0\\0&0&0&0&0&1&-1\end{pmatrix}$$

$$[T_k^{21}\ \ U_k^{21}]=\left(\begin{array}{ccccccc:c}-1&0&0&0&0&0&0&0\\0&-1&0&0&0&0&0&0\\1&1&-1&0&0&0&0&0\\0&0&0&-1&0&0&0&0\\0&0&0&1&-1&0&0&1\\0&0&1&0&1&-1&0&0\\0&0&0&0&0&1&0&0\end{array}\right)$$

$$[T_k^{22}\ \ U_k^{22}]=\left(\begin{array}{ccccccc:c}-1&0&0&0&0&0&0&0\\0&-1&0&0&0&0&0&0\\1&1&-1&0&0&0&0&0\\0&0&0&-1&0&0&0&0\\0&0&0&1&-1&0&0&1\\0&0&1&0&1&-1&1&0\\0&0&0&0&0&1&-1&0\end{array}\right)$$

$$[T_k^{23}\ \ U_k^{23}]=\left(\begin{array}{ccccccc:c}-1&0&0&0&0&0&0&0\\0&-1&0&0&0&0&0&0\\1&1&-1&0&0&0&0&0\\0&0&0&-1&0&0&0&0\\0&0&0&1&-1&0&0&1\\0&0&1&0&1&-1&0&0\\0&0&0&0&0&1&-1&0\end{array}\right).$$

Now let us observe that each matrix in the above example is of full row rank. It is straightforward to verify that this is true in the general case. Therefore, the matrix T_{H_ν} given by (16) is also of full row rank, i.e., rank $T_{H_\nu}=n$. Thus, the condition 2) of Lemma 1 holds.

Conversely, assume that the condition 2) of Lemma 1 holds, or equivalently, a matrix $T\in\mathscr{T}$ exists such that rank $T=n$. Then we assert that the subgraph of G_C which corresponds to T, denoted by H_T, consists of mutually disjoint subgraphs of type 1-3, 2-1, 2-2, 2-3. For if H_T contains a subgraph of type 1-1 or 1-2, by renumbering the compartments and inputs appropriately, T is transformed to

$$\begin{pmatrix} T^{ij} & \vdots & 0 \\ \cdots & \cdot & \cdots \\ 0 & \vdots & \times \end{pmatrix} \qquad (i,j)=(1,1) \text{ or } (1,2) \tag{17}$$

where T^{ij} $[(i,j)=(1,1)$ or $(1,2)]$ represents the matrix which is uniquely determined corresponding to the subgraphs type 1-1 or 1-2. For instance, the matrix corresponding to the graphs in Fig. 4 is given by

$$T^{11}=\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$T^{12}=\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

The above matrices are not of full row rank and it is straightforward to verify that this is true in the general case. Thus, the matrix T given by (17) cannot be of full row rank. This is a contradiction. Therefore, the assertion was verified. Now since G_C is spaned by subgraphs type 1-3, 2-1, 2-2, and 2-3, the result follows. Q.E.D.

IV. Undirected Compartmental Models

In the real world, it is quite usual that compartmental models have bidirectional flows in the sense that if there is a flow from compartment j to compartment i, then there is also a flow of reversed direction. Such a compartmental model will be called an undirected compartmental model. This terminology was motivated by the fact that the behavior of such models can be expressed by an undirected graph.

Example 2: Consider a liquid system in Fig. 2(a) and a network system in Fig. 2(b). If one regards the tunk 1, 2, 3 and the capacitor 1, 2, 3, respectively; as compartments, and one regards water in the tunk and charge in the capacitor as the material of interest, these systems are undirected compartmental models. Furthermore, the behavior of these systems can be expressed by an undirected graph in Fig. 2(c).

This section aims at giving a criterion for undirected compartmental models to be structurally controllable in terms of undirected compartmental graph.

In undirected compartmental models each compartment i has a parameter c_i representing the capacity. If there is a flow between the two compartments i and j, the rate of flow of material from compartment i to compartment j (resp. from compartment j to compartment i) is assumed to be proportional to $x_i/c_i - x_j/c_j$ (resp. $x_j/c_j - x_i/c_i$) where x_i and x_j are, respectively, the amount of material in compartment i and j (In the case of the liquid system in Fig. 2(a), c_i is the area of tunk i and x_i/c_i is the water level in tunk i. Similarly, in the case of the network system in Fig. 2(b), c_i is the capacitance of capacitor i and x_i/c_i is the voltage in capacitor i.) Furthermore, like compartmental models in Section III, there may be flows called excretions from some compartments to the model's environment. The flows from the environment to the model can be regraded as inputs to the model.

An undirected compartmental model is composed of a set V of compartments, a set Γ of undirected flow channels which interconnect compartments, a subset E of V representing excited compartments, and a subset F of V whose elements are the compartments with excretions. Consequently, a model can be characterized by the triple (D, E, F) where D is an undirected graph $D=(V,\Gamma)$. This triple (D, E, F) will be called an undirected compartmental graph [see Fig. 2(c)].

Now consider an undirected compartmental system which is composed of n compartments and give these n compartments a numbering from 1 to n in an arbitrary but fixed way. Let x_j be the state variable which denotes the amount of material in compartment j, l_{ij} be the proportionality parameter which characterizes the rate of flow between compartment i and j so that $l_{ij}=l_{ji}$, and l_{0j} be the proportionality parameter which characterizes the rate of flow from compartment j to compartment 0 (the environment). Then defining k_{ij} by

$$k_{ij}=l_{ij}/c_j \qquad \text{for } i=0,1,\cdots,n, \quad j=1,\cdots,n \tag{18}$$

the undirected compartmental system can be modeled by

$$\dot{x}=A_P x + B_P u \tag{19}$$

where $P=\{1/c_i, l_{ij}, b_{ij}\}$, each element of A_p is given by (10) and (18), and b_{ik} represents the flow rate parameter from the kth input to compartment i. It is assumed that a number of the model parameters l_{ij}, b_{ik} are fixed zeros, but the rest of them and all the c_j's are independent and unrelated. Furthermore, without loss of generality, we restrict our consideration to an undirected compartmental system (A_p, B_p) where each row of B_p contains at most one nonfixed element and each column of B_p contains exactly one nonfixed element. This is due to the same reasoning as in the case of compartmental systems in Section III.

Now from the above definitions the undirected compartmental system has the following features: F1) Nonfixed flow rates k_{ij}'s are not mutually independent since $l_{ij}=l_{ji}$ in (18), and F2) k_{ij} is not fixed zero if and only if k_{ji} is not fixed zero. Note that the former constraint does not allow us the direct use of Theorem 1 to determine whether or not undirected compartmental systems are structurally controllable.

Theorem 2: Without loss of generality assume that the undirected compartmental graph does not split into two or more disjoint subgraphs. An undirected compartmental system (A_p, B_p) is structurally controllable if and only if the corresponding undirected compartmental graph G_U contains at least one excited compartment.

Proof: The necessity is obviously true. Thus, we shall prove only the sufficiency part. Below it will be shown that it is possible to fix some of the parameters to zeros and the resulting system is still structurally controllable. For this recall that any undirected, connected graph can be spaned by a tree [14]. Suppose here that G_U is spaned by a tree G'_U. Now fix l_{ij} $(=l_{ji})$ to zero if edge $\langle i,j\rangle$ is not contained in G'_U, and set all the parameters b_{ij} but an arbitrary one to zeros. Denote the nonzero one by b. Denote by P' the subset of P which is composed of parameters in P not being fixed to zeros. Finally, by $[A'_p, B'_p]$ denote the matrix obtained from $[A_p, B_p]$ by fixing its parameters as above. Note that the undirected compartmental graph corresponding (A'_p, B'_p) is given by G'_U. We shall show that (A'_p, B'_p) is structurally controllable. From now on we shall drop prime from P' for notational convenience.

First, observe that if elements $k_{ij}=l_{ij}/c_j$ of A_p were regarded as mutually independent and if (19) were regarded as a usual compartmental system, then the assumptions of the theorem imply that the system would be structurally controllable (Theorem 1). This implies that there is a proper algebraic variety V in the parameter space $\{k_{ij}, b\}$ defined by a polynomial $\psi(k_{ij}, b)$ as

$$V=\left\{(k_{ij},b)\in R^{2n-1}|\psi(k_{ij},b)=0\right\} \tag{20}$$

such that for all $(k_{ij}, b)\in R^{2n-1}-V$, (A_P, B_P) is controllable. Next substitute $k_{ij}=l_{ij}/c_j$ into (20). Then it is possible to show that, under the assumption of G_U being a tree,

$$\psi(l_{ij}/c_j, b)=0 \tag{21}$$

defines an nontrivial algebraic variety W in the parameter space $\{1/c_1,\cdots,1/c_n,\cdots,l_{ij},\cdots,b\}$. This is the direct consequence of Lemma 3 below. Therefore, since (A_p, B_p) is controllable for all $(k_{ij}, b)\in R^{2n-1}-V$, it is controllable for all $(1/c_1,\cdots,1/c_n,\cdots,l_{ij},\cdots,b)\in R^{2n}-W$. These imply that (A_p, B_p) is structurally controllable.

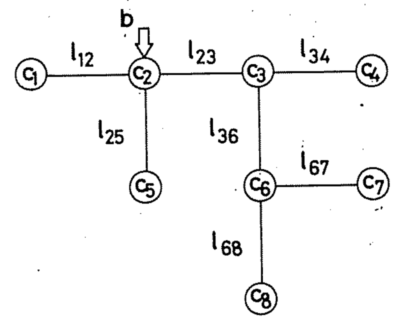
Therefore, the theorem was completely proven. Q.E.D.

Lemma 3: Suppose that edges $\langle 1, i_1\rangle$, $\langle 2, i_2\rangle,\cdots$, and $\langle n-1, i_{n-1}\rangle$ form a tree spaning G_U, and let l_{ti_t} be the proportionality parameter of edge $\langle t, i_t\rangle$ for $t=1,2,\cdots,n-1$; c_j be the capacity of compartment j for $j=1,2,\cdots,n$; b be the weight of input edge. Let φ: $R^{2n}\to R^{2n-1}$;

$$\left(1/c_1,\cdots,1/c_n,l_{1i_1},\cdots,l_{n-1i_{n-1}},b\right)$$
$$\to\left(l_{1i_1}/c_1,l_{1i_1}/c_{i_1},\cdots,l_{n-1i_{n-1}}/c_{n-1},l_{n-1i_{n-1}}/c_{i_{n-1}},b\right).$$

Then the inverse image W of any proper algebraic variety V in R^{2n-1}, i.e., $W=\varphi^{-1}(V)$, is a proper algebraic variety in R^{2n}.

Proof: Clearly the inverse image of any algebraic variety in R^{2n-1} is an algebraic variety in R^{2n}. Thus, it is enough to show that the image of φ includes some open set in R^{2n-1}, or equivalently, Jacobi matrix $(\partial\varphi(p))/\partial p$ is generically of full row rank. We illustrate this by the following example:



In this case,

$$p=(1/c_1,1/c_2,1/c_3,1/c_4,1/c_5,1/c_6,1/c_7,1/c_8, l_{12},l_{23},l_{34},l_{25},l_{36},l_{67},l_{68},b)$$
$$\varphi(p)=(l_{12}/c_1,l_{12}/c_2,l_{23}/c_2,l_{23}/c_3,l_{34}/c_3, l_{34}/c_4,l_{25}/c_2,l_{25}/c_5,l_{36}/c_3,l_{36}/c_6, l_{67}/c_6,l_{67}/c_7,l_{68}/c_6,l_{68}/c_8,b)$$

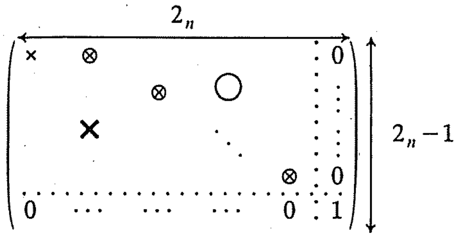
and

$$\frac{\partial\varphi(p)}{\partial p}=\left[\begin{array}{cccccccc|ccccccc|c}
l_{12}&0&0&0&0&0&0&0&1/c_1&0&0&0&0&0&0&0\\
0&l_{12}&0&0&0&0&0&0&1/c_2&0&0&0&0&0&0&0\\
0&l_{23}&0&0&0&0&0&0&0&1/c_2&0&0&0&0&0&0\\
0&0&l_{23}&0&0&0&0&0&0&1/c_3&0&0&0&0&0&0\\
0&0&l_{34}&0&0&0&0&0&0&0&1/c_3&0&0&0&0&0\\
0&0&0&l_{34}&0&0&0&0&0&0&1/c_4&0&0&0&0&0\\
0&l_{25}&0&0&0&0&0&0&0&0&0&1/c_2&0&0&0&0\\
0&0&0&0&l_{25}&0&0&0&0&0&0&1/c_5&0&0&0&0\\
0&0&l_{36}&0&0&0&0&0&0&0&0&0&1/c_3&0&0&0\\
0&0&0&0&0&l_{36}&0&0&0&0&0&0&1/c_6&0&0&0\\
0&0&0&0&0&l_{67}&0&0&0&0&0&0&0&1/c_6&0&0\\
0&0&0&0&0&0&l_{67}&0&0&0&0&0&0&1/c_7&0&0\\
0&0&0&0&0&l_{68}&0&0&0&0&0&0&0&0&1/c_6&0\\
0&0&0&0&0&0&0&l_{68}&0&0&0&0&0&0&1/c_8&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{array}\right].$$

Now introduce a permutation mapping Θ; $P\to q=(1/c_1, l_{12}, 1/c_2, l_{23}, 1/c_3, l_{34}, 1/c_4, l_{25}, 1/c_5, l_{36}, 1/c_6, l_{67}, 1/c_7, l_{68}, 1/c_8, b)$. Note that the order of elements in q corresponds to the tree structure spaning G_U. Then we obtain

$$\frac{\partial\varphi(p)}{\partial p}\cdot\frac{\partial\Theta^{-1}(q)}{\partial q}=\left[\begin{array}{ccccccccccccccc|c}
l_{12}&1/c_1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1/c_2&l_{12}&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&l_{23}&1/c_2&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1/c_3&l_{23}&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&l_{34}&1/c_3&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1/c_4&l_{34}&0&0&0&0&0&0&0&0&0\\
0&0&l_{25}&0&0&0&0&1/c_2&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1/c_5&l_{25}&0&0&0&0&0&0&0\\
0&0&0&0&l_{36}&0&0&0&0&1/c_3&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1/c_6&l_{36}&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&l_{67}&1/c_6&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1/c_7&l_{67}&0&0&0\\
0&0&0&0&0&0&0&0&0&0&l_{68}&0&0&1/c_6&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1/c_8&l_{68}&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{array}\right]$$

and thus it is clear that $(\partial\varphi(p))/\partial p$ is generically of full row rank. It is straightforward to verify that this is true in the general case. In fact, the essential point of the above derivation is that by a permutation mapping θ, the Jacobi matrix $(\partial\varphi(p))/\partial p$ can be transformed into the following form:



where $\otimes$'s are generically nonzeros.

Permutation mapping θ for the general case is also obtainable by making use of the tree structure spaning G_U and the way for the construction of θ would be clear from the above example.

Q.E.D.

V. Conclusions

In this paper, a criterion for compartmental models to be structurally controllable was obtained in terms of a compartmental graph. This criterion extends the well-known result on the single-sink case to the multisink case. Moreover, the above result was applied to undirected compartmental models and a criterion for this class of models to be structurally controllable was also obtained in terms of undirected compartmental graph. From this result we can obtain intuitive and physical information about the controllability of some liquid systems and some network systems.

There are some physical systems whose parameterizations are outside the class of linear parameterization. Further work is needed for such systems.

Acknowledgment

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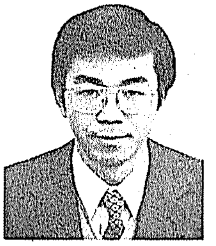

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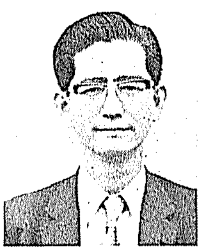

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