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What are singular values of nonlinear operators?

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***Abstract*— This paper is devoted to a novel characterization of singular values of nonlinear operators. Although eigenvalue and spectrum analysis for nonlinear operators has been studied by many researchers in mathematics literature, singular value analysis has not been investigated so much. In this paper, a novel framework of singular value analysis is proposed which is closely related to the operator gain. The proposed singular value analysis is based on the eigenvalue analysis of a special class of nonlinear operators called differentially self-adjoint. Some properties of those operators are clarified which are natural generalization of the linear case results. Furthermore, a sufficient condition for the existence of singular values is provided. The proposed analysis tools are expected to play an important role in nonlinear control systems theory as in the linear case.**

I. INTRODUCTION

Eigenvalue analysis with the related techniques is one of the most beneficial tools in many scientific research fields. In particular, eigenvalue and singular value analysis plays a crucial role in linear control systems theory. It is quite natural to consider how to generalize these tools for nonlinear operators, whereas they are originally used for linear operators. In fact, there are several papers on eigenvalue and spectrum analysis for *nonlinear* operators in mathematics literature [1], [2], [3], [4].

Let us consider a Banach space X with a field $\mathbb{K}$ and a linear operator $A : X \to X$. Its *eigenvalue* λ and the corresponding *eigenvector* x are obtained by solving

$$Ax = \lambda x, \quad \lambda \in \mathbb{K}, \quad x (\neq 0) \in X.$$

Here λ rendering $A - \lambda I$ non-invertible is called a *spectrum* of A. The nonlinear version of this eigenvalue problem is formulated in a similar way as follows. Consider a nonlinear operator $f : X_0 \to X$ with $X_0 \subset X$. Its *eigenvalue* λ and the corresponding *eigenvector* x are obtained by solving

$$f(x) = \lambda x, \quad \lambda \in \mathbb{K}, \quad x (\neq 0) \in X.$$

Here λ rendering $f - \lambda I$ non-invertible is called a *spectrum* of f. The above nonlinear eigenvalue problem is a natural generalization of the linear case.

On the other hand, nonlinear versions of singular value problems were not investigated so much. This is because the definition of a nonlinear version of *adjoint* operators are not clear. In the linear case, singular vectors x's of a linear operator A are characterized by the eigenvectors of A^*A with A^* the adjoint of A, and the corresponding singular values are given by square roots of the eigenvalues of A^*A. See e.g. [5]. Although there are some research on adjoints of nonlinear operators [6], [7], [8], [9], its direct application does not derive any framework for singular value analysis so profitable as that in the linear case.

The objective of this paper is to provide a natural definition of singular values of nonlinear operators and to clarify some of their properties. First of all, recall that singular values in the linear case has a close relationship to the operator gain. A new definition of singular values of nonlinear operators is proposed based on their gain analysis. Then it is shown that thus defined singular values can be calculated by solving a special class of nonlinear eigenvalue problems with respect to a *differentially self-adjoint* operator which is a nonlinear counterpart of a self-adjoint operator in the linear case. Furthermore, some of their properties related to the existence of singular values are clarified.

In this paper, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ denotes a field where $\mathbb{R}$ and $\mathbb{C}$ denote the space of real numbers and that of complex numbers, respectively. The symbol $\mathbb{N}$ denotes the space of natural numbers. The operators $\mathrm{d}(\cdot)$ and $\mathbf{d}(\cdot)$ denote Fréchet derivative (for conventional operators) and exterior derivative (for differential forms), respectively, and the word 'differentiable' stands for 'Fréchet differentiable'. The symbols $\mathrm{Re}(x)$ and $\mathrm{Im}(x)$ with a complex number $x \in \mathbb{C}$ denote its real part and the imaginary part, respectively. The product $\langle \cdot, \cdot \rangle$ denotes the inner product for the corresponding Hilbert space with the field $\mathbb{K}$. The symbols S_r and D_r denote a sphere $S_r(X) := \{x \in X \mid \|x\| = r\}$ and a disk $D_r(X) := \{x \in X \mid \|x\| \leq r\}$, respectively. The symbol $\mathrm{T}_x M$ denotes the tangent space of M at x.

II. SINGULAR VALUES OF NONLINEAR OPERATORS

First of all, recall the definition of singular values in the linear case in order to show the line of thinking in the nonlinear case. In the linear case, singular values and singular vectors of a linear (compact) operator $A : X \to Y$ with Hilbert spaces X and Y are characterized by the eigenvalue problem of $A^*A : X \to X$

$$A^*Ax = \lambda x, \quad \lambda \in \mathbb{K}, \ x (\neq 0) \in X. \tag{1}$$

Here the solution x is called a (right) *singular vector* of A. The eigenvalue λ is always *real* and non-negative because A^*A is *self-adjoint* and positive semi-definite. So the corresponding *singular value* can be defined by

$$\sigma = \sqrt{\lambda} \quad \left(= \frac{\|Ax\|}{\|x\|} \right). \tag{2}$$

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Singular values are important because it characterizes the operator gain by

$$\|A\| := \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \frac{\|Ax\|}{\|x\|} = \sup\ \sigma. \qquad (3)$$

How can we define singular values for nonlinear operators? Recall also that, when x is a singular vector corresponding to a nonzero singular value, then it is a critical point of the square of the input-output ratio $\|Ax\|^2/\|x\|^2$ under the constraint $x \in S_1(X)$ [5], that is,

$$\mathrm{d}\left(\frac{\|Ax\|^2}{\|x\|^2}\right)(\mathrm{d}x) = 0, \qquad \forall \mathrm{d}x \ \ \text{s.t.} \ \ (x, \mathrm{d}x) \in \mathrm{T}_x S_1(X).$$

Here we adopt this relationship as the starting point to define singular vectors of nonlinear operators. Consider a nonlinear operator $g : X_0 \to Y$ with an open set $X_0 \subset X$ containing 0. Take an arbitrary positive constant $r > 0$ and consider a similar problem finding critical points of the square of the input-output ratio under a constraint $x \in S_r(X) \cap X_0$.

$$\mathrm{d}\left(\frac{\|g(x)\|^2}{\|x\|^2}\right)(\mathrm{d}x) = 0$$
$$\forall \mathrm{d}x \ \ \text{s.t.} \ \ (x, \mathrm{d}x) \in \mathrm{T}_x(S_r(X) \cap X_0). \qquad (4)$$

Here an additional parameter r is introduced because the input-output ratio of a nonlinear operator varies according to the input magnitude r, differently from the linear case.

In the nonlinear case, the singular vectors satisfying (4) is characterized by the following nonlinear eigenvalue problem.

Theorem 1: Consider Hilbert spaces X and Y with a field $\mathbb{K}$, and a bounded nonlinear operator $g : X_0 \to Y$ with an open set X_0 satisfying $0 \in X_0 \subset X$. Suppose that g is differentiable. Then x is a solution of (4) if and only if it satisfies

$$(\mathrm{d}g(x))^* g(x) = \lambda\ x, \quad \lambda \in \mathbb{R},\ x(\neq 0) \in X_0. \qquad (5)$$

Proof: The problem (4) is to find a critical point of $\|g(x)\|^2/\|x\|^2$ under a constraint $\|x\| = r$. We start from

$$\begin{aligned} &\mathrm{d}\left(\frac{\|g(x)\|^2}{\|x\|^2}\right)(\mathrm{d}x) \\ &= \frac{\langle g(x), \mathrm{d}g(x)(\mathrm{d}x)\rangle + \langle \mathrm{d}g(x)(\mathrm{d}x), g(x)\rangle}{r^2} \\ &= \frac{2}{r^2}\mathrm{Re}\langle (\mathrm{d}g(x))^* g(x), \mathrm{d}x\rangle = 0. \end{aligned}$$

By differentiating the constraint $\|x\| = r$ of $S_r(X)$, we obtain

$$\mathrm{Re}\langle x, \mathrm{d}x\rangle = 0.$$

Hence Equation (4) is equivalent to

$$\mathrm{Re}\langle (\mathrm{d}g(x))^* g(x), \xi\rangle = 0, \qquad \forall \xi \in X \ \ \text{s.t.} \ \ \mathrm{Re}\langle x, \xi\rangle = 0$$

Sufficiency of the theorem is proved first. If $\lambda \in \mathbb{R}$ and $x \in X_0$ satisfy Equation (5), then

$$\begin{aligned} \mathrm{Re}\langle (\mathrm{d}g(x))^* g(x), \xi\rangle &= \mathrm{Re}\langle \lambda x, \xi\rangle = \mathrm{Re}\,(\lambda \cdot \langle x, \xi\rangle) \\ &= \mathrm{Re}\lambda \cdot \mathrm{Re}\langle x, \xi\rangle - \mathrm{Im}\lambda \cdot \mathrm{Im}\langle x, \xi\rangle \\ &\equiv 0. \end{aligned}$$

This proves the sufficiency.

Necessity is proved then. Note that X can be decomposed into two orthogonal subspaces

$$\begin{aligned} X &= X_{\parallel x} \oplus X_{\perp x} \\ X_{\parallel x} &:= \{\nu x \mid \nu \in \mathbb{K}\} \\ X_{\perp x} &:= \{\xi \in X \mid \langle x, \xi\rangle = 0\}. \end{aligned}$$

Further, using an expression $\nu = \lambda + i\mu$, $X_{\parallel x}$ can be divided into two sets

$$\begin{aligned} X_{\parallel x} &= X_{\mathbb{R}x} \cup X_{i\mathbb{R}x} \\ \{0\} &= X_{\mathbb{R}x} \cap X_{i\mathbb{R}x} \\ X_{\mathbb{R}x} &:= \{\lambda x \mid \lambda \in \mathbb{R}\} \\ X_{i\mathbb{R}x} &:= \{i\mu x \mid \mu \in \mathbb{R}\} \end{aligned}$$

Then we obtain a unique decomposition

$$\begin{aligned} (\mathrm{d}g(x))^* g(x) &= \lambda x + i\mu x + x_{\perp x}, \\ &\quad \lambda x \in X_{\mathbb{R}x},\ i\mu x \in X_{i\mathbb{R}x},\ x_{\perp x} \in X_{\perp x}. \end{aligned}$$

This yields

$$\begin{aligned} &\mathrm{Re}\langle (\mathrm{d}g(x))^* g(x), \xi\rangle \\ &= \mathrm{Re}\langle \lambda x + i\mu x + x_{\perp x}, \xi\rangle \\ &= \mathrm{Re}\langle \lambda x, \xi\rangle - i\mathrm{Im}\langle \mu x, \xi\rangle + \mathrm{Re}\langle x_{\perp x}, \xi\rangle \\ &= -\mu \cdot \mathrm{Im}\langle x, \xi\rangle + \mathrm{Re}\langle x_{\perp x}, \xi\rangle \\ &\equiv 0, \qquad \forall \xi \ \ \text{s.t.} \ \ \mathrm{Re}\langle x, \xi\rangle = 0. \end{aligned}$$

Since $\mathrm{Im}\langle x, \xi\rangle$ can be an arbitrary (non-zero) value, $\mu = 0$. Moreover, we have

$$x_{\perp x} \in X_{\perp x} \subset \{\xi \in X \mid \mathrm{Re}\langle x, \xi\rangle = 0\},$$

that is, $x_{\perp x}$ can be taken as ξ. Therefore $x_{\perp x} = 0$ has to hold, which proves the necessity. This completes the proof. ■

This property motivates us to characterize singular values and singular vectors of nonlinear operators as follows.

Definition 1: Consider Hilbert spaces X and Y with a field $\mathbb{K}$, and a differentiable bounded nonlinear operator $g : X_0 \to Y$ with an open set X_0 satisfying $0 \in X_0 \subset X$. An eigenvector of the operator $x \mapsto (\mathrm{d}g(x))^* g(x)$ corresponding to a *real* eigenvalue, that is, $x \in X_0$ satisfying (5) is called a *singular vector* of g and the corresponding input-output ratio defined by

$$\sigma = \frac{\|g(x)\|}{\|x\|}$$

with the singular vector x is called a *singular value* of g.

Note that Equation (5) is a natural nonlinear generalization of the singular value problem in the linear case (1). The reason why we adopt the second equation of (2) as the definition of singular values of nonlinear operators instead of the first one, is because λ in (5) can be negative. Furthermore, this definition yields the property

$$\|g\| := \sup_{x \in X_0} \frac{\|g(x)\|}{\|x\|} = \sup\ \sigma$$

as in the linear case (3), because the input maximizing the input-output ratio $\arg\sup(\|g(x)\|/\|x\|)$ has to satisfy Equation (4). Namely, nonlinear singular values are also closely related to the operator gain.

Remark 1: The author has provided a similar definition of singular values for nonlinear Hankel operators in [10]. This definition works quite nicely for Hankel operators, and nonlinear balanced realization and model reduction procedure are obtained consequently [11], [12]. Another important example of the proposed singular values can be found in L_2 gain analysis [13]. In fact, investigating the singular values for L_2 stable nonlinear input-output systems is equivalent to analyzing the solution of the corresponding Hamiltonian extension giving the solution of L_2 gain analysis of the original operator [12].

III. DIFFERENTIALLY SELF-ADJOINT OPERATORS

In order to investigate the solution of (5), we need to characterize a nonlinear version of a *self-adjoint* operator, since the eigenstructure of such operators play an important role in investigating singular values of linear operators. Let us define *differentially self-adjoint* operators as follows.

Definition 2: Consider a Hilbert space X with a field $\mathbb{K}$, and a bounded nonlinear operator $f : X_0 \to X$ with an open set X_0 satisfying $0 \in X_0 \subset X$. The operator f is said to be *differentially self-adjoint* if it is differentiable and if $\mathrm{d}f(x) : X \to X$ is self-adjoint for all $x \in X_0$.

An intuitive motivation of this definition is explained by the following lemma and corollary.

Lemma 1: Consider a Hilbert space X with a field $\mathbb{K}$, and a bounded nonlinear operator $h : X_0 \to \mathbb{R}$ with an open set X_0 satisfying $0 \in X_0 \subset X$. Suppose that h is continuously differentiable and that there there exists an operator $f : X_0 \to X$ satisfying

$$\mathrm{d}h(x)(\mathrm{d}x) = \mathrm{Re}\langle f(x), \mathrm{d}x\rangle. \tag{6}$$

Then the operator f is differentially self-adjoint.

Proof: First of all, the second order derivative of h is given by

$$\begin{aligned}
&\mathrm{d}^2h(x)(\mathrm{d}x)(\mathrm{d}x)\\
&= \mathrm{d}(\mathrm{d}h(x)(\mathrm{d}x))(\mathrm{d}x)\\
&= \mathrm{d}(\mathrm{Re}\langle f(x), \mathrm{d}x\rangle)(\mathrm{d}x)\\
&= \frac{1}{2}\mathrm{d}(\langle f(x), \mathrm{d}x\rangle + \langle \mathrm{d}x, f(x)\rangle)(\mathrm{d}x)\\
&= \frac{1}{2}(\langle \mathrm{d}f(x)(\mathrm{d}x), \mathrm{d}x\rangle + \langle \mathrm{d}x, \mathrm{d}f(x)(\mathrm{d}x)\rangle)\\
&= \mathrm{Re}\langle \mathrm{d}f(x)(\mathrm{d}x), \mathrm{d}x\rangle.
\end{aligned}$$

The definition of Fréchet derivative implies that the second order derivative $\mathrm{d}^2h(x) : X \times X \to \mathbb{R}$ is symmetric with respect to the two variables, that is,

$$\mathrm{d}^2h(x)(\xi)(\eta) = \mathrm{d}^2h(x)(\eta)(\xi), \quad x \in X_0,\ \xi, \eta \in X.$$

Therefore, we obtain

$$\begin{aligned}
\mathrm{Re}\langle \mathrm{d}f(x)(\eta), \xi\rangle &= \mathrm{Re}\langle \mathrm{d}f(x)(\xi), \eta\rangle = \mathrm{Re}\overline{\langle \eta, \mathrm{d}f(x)(\xi)\rangle}\\
&= \mathrm{Re}\langle \eta, \mathrm{d}f(x)(\xi)\rangle, \quad x \in X_0,\ \xi, \eta \in X
\end{aligned}$$

because $\mathrm{d}^2h(x)(\xi)(\eta) \in \mathbb{R}$. Further, substituting $\eta = i\zeta$, $\zeta \in X$,

$$\mathrm{Im}\langle \mathrm{d}f(x)(\zeta), \xi\rangle = \mathrm{Im}\langle \zeta, \mathrm{d}f(x)(\xi)\rangle.$$

Namely, $\mathrm{d}f(x)$ is self-adjoint for all $x \in X_0$. This proves the lemma. ■

Corollary 1: Consider Hilbert spaces X and Y with a field $\mathbb{K}$, and a bounded nonlinear operator $g : X_0 \to Y$ with an open set X_0 satisfying $0 \in X_0 \subset X$. Suppose that g is continuously differentiable. Then the operator $f : X_0 \to X$ defined by

$$f(x) := (\mathrm{d}g(x))^* g(x) \tag{7}$$

is differentially self-adjoint.

Proof: Corollary follows directly from Lemma 1 by defining $h : X_0 \to \mathbb{R}$ as

$$h(x) := \frac{1}{2}\|g(x)\|^2$$

because

$$\begin{aligned}
\mathrm{d}h(x)(\mathrm{d}x) &= \frac{1}{2}(\langle g(x), \mathrm{d}g(x)(\mathrm{d}x)\rangle + \langle \mathrm{d}g(x)(\mathrm{d}x), g(x)\rangle)\\
&= \mathrm{Re}\langle g(x), \mathrm{d}g(x)(\mathrm{d}x)\rangle\\
&= \mathrm{Re}\langle (\mathrm{d}g(x))^* g(x), \mathrm{d}x\rangle
\end{aligned}$$

which implies (6). ■

Therefore, any singular value problem reduces down to an eigenvalue problem with respect to a special class of operators called differentially self-adjoint.

The final objective of this section is to provide a converse result of of Lemma 1. To this end, let us state the following lemma.

Lemma 2: Consider a Hilbert space X with a field $\mathbb{K}$, and a bounded nonlinear operator $f : X_0 \to X$ with any simply connected open set X_0 satisfying $0 \in X_0 \subset X$. Suppose that f is differentially self-adjoint. Then

$$\langle f(x), x\rangle \in \mathbb{R}, \quad \forall x \in X_0.$$

Proof: Suppose that X_0 is convex without loss of generality, since a simply connected set is homeomorphic to a convex one. Hence we can define an operator $F : X_0 \times X \to X$ by

$$F(x)(\xi) := \int_0^1 \mathrm{d}f(tx)(\xi)\ \mathrm{d}t.$$

$F(x) : X \to X$ is linear and self-adjoint for all $x \in X_0$, because

$$\begin{aligned}
\langle \eta, F(x)(\xi)\rangle &= \int_0^1 \langle \mathrm{d}f(tx)(\xi), \eta\rangle \mathrm{d}t\\
&= \int_0^1 \langle \xi, \mathrm{d}f(tx)(\eta)\rangle \mathrm{d}t\\
&= \langle \xi, F(x)(\eta)\rangle.
\end{aligned}$$

Then we have

$$\begin{aligned}
f(x) &= \int_0^1 \frac{\mathrm{d}f(tx)}{\mathrm{d}t}\mathrm{d}t + f(0) = \int_0^1 \mathrm{d}f(tx)(x)\mathrm{d}t\\
&= F(x)(x).
\end{aligned}$$

Here we use the relation $f(0) = 0$, which is implied by boundedness of f. Since $F(x)$ is self-adjoint,

$$\begin{aligned}\langle f(x), x\rangle &= \langle F(x)(x), x\rangle = \langle x, F(x)(x)\rangle \\ &= \langle x, f(x)\rangle = \overline{\langle f(x), x\rangle}\end{aligned}$$

holds for all $x \in X_0$. That is, $\langle f(x), x\rangle$ is real for all $x \in X_0$ which is the claim of the lemma. ■

Using this lemma, we can prove a converse result of Lemma 1, which is a variation of Stokes's theorem.

Theorem 2: Consider a Hilbert space X with a field $\mathbb{K}$, and a bounded nonlinear operator $f : X_0 \to X$ with a simply connected open set X_0 satisfying $0 \in X_0 \subset X$. Suppose that f is differentiable. Then f is differentially self-adjoint if and only if there exists an operator $h : X_0 \to \mathbb{R}$ satisfying

$$\mathrm{d}h(x)(\mathrm{d}x) = \mathrm{Re}\langle f(x), \mathrm{d}x\rangle. \tag{8}$$

Proof: Since sufficiency is proved in Lemma 1, only necessity is proved here. Consider a curve C in X_0 connecting 0 and x. Define an operator $h_C : X \to \mathbb{R}$ by

$$h_C(x) := \int_C \mathrm{Re}\langle f(x), \mathrm{d}x\rangle \tag{9}$$

which depends on the choice of the curve C. Let $\{e_k\}$, $k \in \mathbb{N}$ denote a set of orthonormal basis of the space X. Here let us define a sequence of sets $\{K_l\}$, $l \in \mathbb{N}$ as follows

$$\begin{aligned}K_1 &= \{1\} \\ K_2 &= \{2, \ldots, k_2\} \\ K_3 &= \{k_2 + 1, \ldots, k_3\} \\ &\vdots\end{aligned}$$

such that each set K_l contains finite elements and $\cup_{l=1}^{\infty} K_l = \mathbb{N}$. Take an arbitrary curve $C = \{c(t) \mid t \in [0, 1]\}$ in X_0 connecting 0 and x

$$c(t) = \sum_{k}^{\infty} c_k(t) e_k.$$

Here we suppose that $e_1 = x/\|x\|$ holds without loss of generality. Then the following relations hold.

$$\begin{aligned}c_k(0) &= 0, \quad k \in \mathbb{N} \\ c_1(1) &= 1 \\ c_k(1) &= 0, \quad k(\neq 1) \in \mathbb{N}\end{aligned}$$

Define a set of curves $C_l = \{c^l(t) \mid t \in [0, 1]\}$,

$$c^l(t) := \sum_{k \in K_l} c_k(t) e_k.$$

Then we have a relation

$$\int_C \mathrm{Re}\langle f(x), \mathrm{d}x\rangle = \sum_{l=1}^{\infty} \int_{C_l} \mathrm{Re}\langle f(x), \mathrm{d}x\rangle$$

because C_l's are contained in X_0 due to its convexity (which can be assumed without loss of generality as in the proof of Lemma 2). Let us represent $f(x)$ using the basis $\{e_k\}$ as

$$f(x) = \sum_{k=1}^{\infty} f_k(x) e_k$$

which yields

$$\int_{C_l} \mathrm{Re}\langle f(x), \mathrm{d}x\rangle = \int_{C_l} \mathrm{Re} \sum_{k \in K_l} f_k(x)\ \mathrm{d}x_k$$

where $x = \sum_{k=1}^{\infty} x_k e_k$. Since f is differentially self-adjoint, gradient of the vector $\{f_k(x)\}$, $k \in K_l$ with respect the variables $\{x_k\}$, $k \in K_l$ is symmetric, that is, the differential 1-form $\sum_{k \in K_l} f_k(x)\ \mathrm{d}x_k$ is closed

$$\mathbf{d}_{\{x_k\}_{k \in K_l}} \sum_{k \in K_l} f_k(x)\ \mathrm{d}x_k = 0$$

where $\mathbf{d}_{\{x_k\}_{k \in K_l}}$ denotes the exterior derivative for differential forms with respect to $\{x_k\}$, $k \in K_l$. Furthermore, for $l \geq 2$, C_l is a closed curve because each set K_l contains a finite number of the basis. Hence we can define compact connected sets $\tilde{C}_l$'s such that

$$\partial \tilde{C}_l = C_l$$

holds. Stokes's theorem implies that

$$\begin{aligned}&\int_{C_l} \mathrm{Re} \sum_{k \in K_l} f_k(x)\ \mathrm{d}x_k \\ &= \int_{\tilde{C}_l} \mathrm{Re}\ \mathbf{d}_{\{x_k\}_{k \in K_l}} \sum_{k \in K_l} f_k(x)\ \mathrm{d}x_k \\ &= 0\end{aligned}$$

for any $l \geq 2$. Finally we obtain

$$\int_C \mathrm{Re}\langle f(x), \mathrm{d}x\rangle = \int_{C_1} \mathrm{Re}\ f_1(x)\ \mathrm{d}x_1 \tag{10}$$

which implies that the integral h_C in (9) does not depend on the choice of the curve C. (It depends only on the terminal point x.) It is obvious that $h = h_C$ satisfies (8). This proves the theorem. ■

IV. Eigenvalue Analysis of Differentially Self-Adjoint Operators

This section investigates the solution structure of the eigenvalue problems of differentially self-adjoint operators based on the results derived in the previous sections, which is particularly useful for singular value analysis of nonlinear operators as explained in Theorem 1.

Theorem 3: Consider a Hilbert space X with a field $\mathbb{K}$, and a bounded nonlinear operator $f : X_0 \to X$ with a simply connected open set X_0 satisfying $0 \in X_0 \subset X$. Suppose that f is differentially self-adjoint. Then all eigenvalues of f are real. Furthermore, if f satisfies (7), then all eigenvectors of f are singular vectors of g.

Proof: Former part is proved first. If $\lambda \in \mathbb{K}$ is an eigenvalue of f, then Lemma 2 implies that

$$\lambda \|x\|^2 = \langle \lambda x, x\rangle = \langle f(x), x\rangle \in \mathbb{R}$$

holds with the corresponding eigenvector x. Since $x \neq 0$, the eigenvalue λ has to be real, which is the claim of the former part. The latter part follows readily from the former part and Definition 1. ■

This theorem allows us to concentrate on *real* eigenvalues when we treat differentially self-adjoint nonlinear operators, as in the linear case.

The final objective of this paper is to obtain a some conditions on the existence of singular values of nonlinear operators (i.e., eigenvectors of differentially self-adjoint operators).

Theorem 4: Consider the Hilbert space $X = \mathbb{R}^n$ with the field $\mathbb{K} = \mathbb{R}$, and a bounded nonlinear operator $f : X_0 \to X$ with a simply connected open set X_0 satisfying $0 \in X_0 \subset X$. Suppose that f is differentially self-adjoint, and that $\mathrm{d}f(0) : X \to X$ has n distinct eigenvalues. Then there exists a set $D_r(\mathbb{R}) \subset \mathbb{R}$ satisfying $\{x \mid \|x\| \in D_r(\mathbb{R})\} \subset X_0$, and a set of differentiable operators $\lambda_i : D_r(\mathbb{R}) \to \mathbb{R}$'s and $x_i : D_r(\mathbb{R}) \to X_0$'s satisfying

$$f(x_i(s)) = \lambda_i(s)\, x_i(s), \quad \|x_i(s)\| = |s|,\ s \in D_r(\mathbb{R}). \tag{11}$$

Proof: Theorem 2 implies that there exists a differentiable operator $h : X_0 \to \mathbb{R}$ satisfying

$$\mathrm{d}h(x)(\mathrm{d}x) = \mathrm{Re}\langle f(x), \mathrm{d}x\rangle.$$

The operator h also satisfies

$$\mathrm{d}^2h(x)(\mathrm{d}x)(\mathrm{d}x) = \langle \mathrm{d}f(x)(\mathrm{d}x), \mathrm{d}x\rangle$$

Therefore we have

$$\begin{aligned} h(x) &= \langle f(0), x\rangle + \langle \mathrm{d}f(0)(x), x\rangle + o(\|x\|^2) \\ &= \langle \mathrm{d}f(0)(x), x\rangle + o(\|x\|^2) \\ &= \hat{h}(x) + o(\|x\|^2) \end{aligned}$$

with $\hat{h}(x) := \langle \mathrm{d}f(0)(x), x\rangle$ a quadratic approximation of $h(x)$. Now let $\hat{\lambda}_1 > \ldots > \hat{\lambda}_n$ denote the eigenvalues of $\mathrm{d}f(0) : X \to X$ and $\hat{x}_1, \ldots, \hat{x}_n$, $\|\hat{x}_i\| = 1$ denote the corresponding normalized eigenvectors. Since $\mathrm{d}f(0) : X \to X$ is self-adjoint, the upper and lower bounds of the ratio $\hat{h}/\|x\|^2$ on a sphere $S_r(X)$ coincide with $\hat{\lambda}_1$ and $\hat{\lambda}_n$, respectively.

$$\sup_{x\in S_r(X)} \frac{\hat{h}(x)}{\|x\|^2} = \hat{\lambda}_1, \qquad \inf_{x\in S_r(X)} \frac{\hat{h}(x)}{\|x\|^2} = \hat{\lambda}_n.$$

In fact, we have

$$\begin{aligned} \mathrm{d}\hat{h}(\pm r\hat{x}_i)(\mathrm{d}x) = 0, &\quad \forall \mathrm{d}x \text{ s.t. } (x, \mathrm{d}x) \in \mathrm{T}_x S_r(X) \\ \mathrm{d}^2\hat{h}(\pm r\hat{x}_1)(\mathrm{d}x)(\mathrm{d}x) < 0, &\quad \forall \mathrm{d}x \text{ s.t. } (x, \mathrm{d}x) \in \mathrm{T}_x S_r(X) \\ \mathrm{d}^2\hat{h}(\pm r\hat{x}_n)(\mathrm{d}x)(\mathrm{d}x) > 0, &\quad \forall \mathrm{d}x \text{ s.t. } (x, \mathrm{d}x) \in \mathrm{T}_x S_r(X). \end{aligned}$$

Namely, the operator $\hat{h}(x)/\|x\|^2$ restricted to the set $S_r(X)$ has its maximum value $\hat{\lambda}_1$ at $x = \pm r\hat{x}_1$ and its minimum $\hat{\lambda}_n$ at $x = \pm r\hat{x}_n$. Recall that the set $S_r(X)$ is compact, so the operator $h(x)/\|x\|^2$ also have its maximum and minimum on each set $S_r(X)$. Since $\hat{h}$ is a quadratic approximation of h at 0, there exists a neighborhood of 0 on which the operator $h(x)/\|x\|^2$ restricted to the set $S_r(X)$ has its (local) maximum at the points $x = x_1(r) \approx r\hat{x}_1$ and $x = x_1(-r) \approx -r\hat{x}_1$, and its minimum at the points $x = x_n(r) \approx r\hat{x}_n$ and $x = x_n(-r) \approx -r\hat{x}_n$. This proves the theorem in the case $n = 2$ indeed.

As h is continuously differentiable, $x_1(s)$ and $x_n(s)$ smoothly depends on s (in a neighborhood of 0). Since there exists an isometric diffeomorphism converting $x_1(s)$ and $x_n(s)$ into the coordinate axes [11], we can assume that $x_1(s)$ and $x_n(s)$ coincide with the coordinate axes without loss of generality.

The proof is completed by induction with respect to the dimension n.

(i) The case $n = 1$ is trivial, and the case $n = 2$ follows readily from the above discussion.

(ii) The case $n = k > 2$: Suppose that the theorem in the case $n = k-1$ holds and that the coordinate axes x_1 and x_n are chosen in the above way. Consider a plane $x_1 = t$. Then the case $n = k-1$ implies that there exists a neighborhood of 0 on which there exist $k-1$ independent solutions of (11), that is, there exist functions $\tilde{\lambda}_i(\tilde{s})$'s and $\tilde{x}_i(\tilde{s})$'s satisfying

$$\begin{aligned} f^j(\tilde{x}_i(\tilde{s})) &= \tilde{\lambda}_i(\tilde{s})\tilde{x}_i^j(\tilde{s}), \quad j = 2, \ldots, k \\ \|\tilde{x}_i(\tilde{s})\|^2 &= \tilde{s}^2 + t^2 \end{aligned}$$

with $x = (x^1, \ldots, x^k)$ and $f(x) = (f^1(x), \ldots, f^k(x))$ the expressions in local coordinates. Here what we need to prove is the existence of t and λ satisfying

$$\begin{aligned} f^1(t) &= \lambda\, t \\ \left(\sum_{j=2}^{k} f^j(\tilde{x}(\tilde{s}))^2\right)^{\frac{1}{2}} &= \lambda\, \tilde{s} \end{aligned}$$

This problem reduces to the eigenvalue problem in the case $n = 2$ by regarding $(\tilde{s}, t)$ as local coordinates. Hence there exists an eigenvector $(\tilde{s}, t) = (\tilde{s}(s), t(s))$ for each i whose corresponding eigenvalue has to coincide with $\tilde{\lambda}_i(\tilde{s}(s))$. This proves the theorem in the case $n = k$.

The steps (i) and (ii) imply the theorem. ■

Furthermore, we can prove a relationship between the eigenvalues of f and the singular values of g. Recall the linear case and let σ_i's and λ_i's denote the singular values of $A : X \to Y$ and the eigenvalues of $A^*A : X \to X$. Then clearly we have

$$\lambda_i = \sigma_i^2 \tag{12}$$

due to the definition (2). The nonlinear counterpart of this equation is given by the following theorem.

Theorem 5: Consider Hilbert spaces $X = \mathbb{R}^n$ and Y with a field $\mathbb{K}$, and a bounded nonlinear operator $g : X_0 \to Y$ with an open set X_0 satisfying $0 \in X_0 \subset X$. Suppose that the operator f defined by (7) satisfies the assumptions in Theorem 4. Then the singular values $\sigma_i(s)$'s of g defined by $\sigma_i(s) := \|g(x_i(s))\|/\|x_i(s)\|$ and the eigenvalues $\lambda_i(s)$'s of f satisfy

$$\lambda_i(s) = \sigma_i(s)^2 + s\, \sigma_i(s)\frac{\mathrm{d}\sigma_i(s)}{\mathrm{d}s}. \tag{13}$$

Further, the converse relation is given by

$$\sigma_i(s)^2 = \frac{2}{s^2}\int_0^s s\ \lambda_i(s)\mathrm{d}s. \tag{14}$$

Proof: First of all, the second equation in (11) is

$$s^2 = \|x_i(s)\|^2$$

which implies

$$2\ s\ \mathrm{d}s = \mathrm{d}(\|x_i(s)\|^2)(\mathrm{d}s) = 2\langle x_i, \mathrm{d}x_i(s)(\mathrm{d}s)\rangle. \tag{15}$$

We have another relation

$$\sigma_i(s)^2 = \frac{\|g(x_i(s))\|^2}{\|x_i(s)\|^2}.$$

This reduces to

$$\begin{aligned}
&\mathrm{d}(\sigma_i(s)^2)(\mathrm{d}s)\\
&= \mathrm{d}\left(\frac{\|g(x_i(s))\|^2}{\|x_i(s)\|^2}\right)(\mathrm{d}s)\\
&= \frac{2\|x_i\|^2\langle g(x_i), \mathrm{d}g(x_i)(\mathrm{d}x)\rangle - 2\|g(x_i)\|^2\langle x_i, \mathrm{d}x\rangle}{\|x_i\|^4}\\
&= \frac{2\langle(\mathrm{d}g(x_i))^* g(x_i) - \sigma_i^2 x_i, \mathrm{d}x\rangle}{\|x_i\|^2}\\
&= \frac{2\langle f(x_i) - \sigma_i^2 x_i, \mathrm{d}x\rangle}{s^2}\\
&= \frac{2(\lambda_i - \sigma_i^2)\langle x_i, \mathrm{d}x_i(s)(\mathrm{d}s)\rangle}{s^2}\\
&= \frac{2(\lambda_i - \sigma_i^2)}{s}\mathrm{d}s.
\end{aligned}$$

Here we use the relation (15). Therefore,

$$\lambda_i(s) = \sigma_i(s)^2 + \frac{s}{2}\frac{\mathrm{d}\sigma_i(s)^2}{\mathrm{d}s} = \sigma_i(s)^2 + s\ \sigma_i(s)\frac{\mathrm{d}\sigma_i(s)}{\mathrm{d}s} \tag{16}$$

which is equivalent to (13). Moreover, multiplying the above (left) equation by s, we obtain a form

$$s\ \lambda_i(s) = \frac{1}{2}\frac{\mathrm{d}(s^2\sigma_i(s)^2)}{\mathrm{d}s}.$$

Hence, integrating this equation, we obtain

$$\sigma_i(s)^2 = \frac{2}{s^2}\int_0^s s\ \lambda_i(s)\mathrm{d}s + c$$

with a constant c. Further, Equation (16) implies that $c = 0$ which reduces the above equation to Equation (14). This completes the proof. ■

Theorem 5 shows the fact that there is a one-to-one relationship between $\lambda_i(s)$ and $\sigma_i(s)$. In the linear case, both $\sigma_i(s)$'s and $\lambda_i(s)$'s are constant, so Equations (13) and (14) recover the straightforward relationship (12).

V. CONCLUSION

This paper proposed a novel framework for singular value analysis of nonlinear operators. First of all, a natural definition of nonlinear singular values is proposed. Second, it is shown that the singular values thus defined can be obtained by solving a special class of nonlinear eigenvalue problems with respect to differentially self-adjoint operators. Third, some properties of singular values are clarified. Finally, a sufficient condition for the existence of singular values is proved. It is expected that this framework will provide a useful analysis tools for nonlinear control systems theory as in the linear case.

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