On Almost Invariant Subspaces of Structured Systems and Decentralized Control

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Abstract—**Graph-theoretic conditions are obtained for a structured system to have a property that the supremal $\mathcal{L}_p$-almost invariant (controllability) subspace is generically the entire state space and that the infimal $\mathcal{L}_p$-almost conditional invariant (complementary observability) subspace is generically the zero subspace. The conditions are used to determine stabilizability of structured interconnected systems via decentralized feedback control. Although the obtained graph-theoretic conditions are conservative, they are considered satisfactory to the extent that the benefits of easy testability involving only binary calculations outweigh the conservativeness in the results.**

I. Introduction

In formulating decentralized control strategies for linear time-invariant systems, there has been a considerable effort to obtain stabilizability conditions via the concept of structural controllability [1], [2], which are parameter independent. The notion of structurally fixed modes has been introduced [3] and a number of results have been obtained [4], [5]–[7], which provide conditions for existence of decentralized controllers [8] for stabilization of linear systems. Although the stabilizability conditions are parameter independent, the resulting stability of the closed-loop system is not. In the case of interconnected systems, the lack of robustness causes concern due to essential parameter uncertainty in the interconnections among the subsystems.

When a system is composed of interconnected subsystems, it can be stabilized by local state or output control provided certain conditions are satisfied involving the local feedback gains and structure of interconnections (e.g., [9]–[13]). Stability of the overall closed-loop system established in this way is invariant to arbitrary (bounded) values of the interconnection parameters. The approach, however, requires that each subsystem be transformed into a canonical form, which makes the parameter independence a conditional property of the resulting closed-loop system.

Recently, some new and promising results have been obtained for decentralized stabilization [14], [15], which utilize the concept of almost disturbance decoupling and small gain theorems in the context of almost invariant subspaces formulated in [16] and [17]. These results are free of canonical form restrictions and guarantee a type of connective stability of the closed-loop system, which tolerates parameter uncertainty in the interconnections among the subsystems. The stabilizability conditions, however, are parameter dependent because the relevant subspaces vary as functions of system parameters. One way to overcome this dependency is to select the cases in which the relevant subspaces are invariant under perturbations of system parameters. This leads to investigation of the subspaces in the framework of structured systems. Although the general problem is difficult, if not impossible, it can be solved effectively when the subspaces are equal to either the entire state space or the zero space.

In this paper, we introduce the conditions for a structured system to have the property that the supremal $\mathcal{L}_p$-almost invariant (controllability) subspace is generically the entire state space and that the infimal $\mathcal{L}_p$-almost conditional invariant (complementary observability) subspace is generically the zero space. The conditions are formulated in terms of directed graphs using the concept of generic rank of system matrices. In this way, we consider interconnection terms as disturbances of decoupled subsystems and use the conditions to determine stabilizability of structured interconnected systems by decentralized controllers. Although the obtained graph-theoretic conditions are conservative, they are considered satisfactory to the extent that the conservativeness is outweighed by easy testability of the conditions which require only binary combinatorial calculations.

II. Almost Invariant Subspaces and System Matrices

Consider a linear time-invariant system

$$S\colon \dot{x}=Ax+Bu,$$
$$y=Cx \tag{2.1}$$

where $x(t)\in\mathfrak{X}:=\boldsymbol{R}^n$, $u(t)\in\mathfrak{U}:=\boldsymbol{R}^m$, and $y(t)\in\mathfrak{Y}:=\boldsymbol{R}^l$ are the state, input, and output of S at time $t\in\boldsymbol{R}$, and $\mathfrak{X}$, $\mathfrak{U}$, and $\mathfrak{Y}$ are normed vector spaces. With S we associate the following almost invariant subspaces [16], [17]:

$\mathcal{V}^*_{b,\mathrm{Ker}\,C}$—the supremal $\mathcal{L}_p$-almost invariant subspace "contained" in Ker C; $\mathcal{R}^*_{b,\mathrm{Ker}\,C}$—the supremal $\mathcal{L}_p$-almost controllability subspace "contained" in Ker C; $\mathcal{S}^*_{b,\mathrm{Im}\,B}$—the infimal $\mathcal{L}_p$-almost conditional invariant subspace "containing" Im B; $\mathfrak{N}^*_{b,\mathrm{Im}\,B}$—the infimal $\mathcal{L}_p$-almost complementary observability subspace "containing" Im B.

These spaces can be computed by the well-known algorithms

$$\mathcal{V}_{k+1}=\mathrm{Ker}\,C\cap A^{-1}(\mathcal{V}_k+\mathrm{Im}\,B),\qquad \mathcal{V}_0=\mathfrak{X} \tag{2.2}$$

$$\mathcal{S}_{k+1}=\mathrm{Im}\,B+A\,(\mathrm{Ker}\,C\cap\mathcal{S}_k),\qquad \mathcal{S}_0=\{0\} \tag{2.3}$$

and

$$\mathcal{V}^*_{b,\mathrm{Ker}\,C}=\mathcal{V}_n+\mathcal{S}_n, \tag{2.4}$$

$$\mathcal{R}^*_{b,\mathrm{Ker}\,C}=\mathcal{S}_n, \tag{2.5}$$

$$\mathcal{S}^*_{b,\mathrm{Im}\,B}=\mathcal{V}_n\cap\mathcal{S}_n, \tag{2.6}$$

$$\mathfrak{N}^*_{b,\mathrm{Im}\,B}=\mathcal{V}_n. \tag{2.7}$$

We recall that the triple (A, B, C), or the system S, is called standard when the matrix B has full column rank and C has full row rank, and state the following result [18].

Lemma 2.8: Let a triple (A, B, C) be standard. Then, the state-space $\mathfrak{X}$ can be decomposed into four independent subspaces

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$\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3$, and $\mathfrak{X}_4$ (i.e., $\mathfrak{X} = \mathfrak{X}_1 \oplus \mathfrak{X}_2 \oplus \mathfrak{X}_3 \oplus \mathfrak{X}_4$) such that

$$\mathfrak{X}_1 \oplus \mathfrak{X}_2 = \mathcal{V}_n, \tag{2.9}$$

$$\mathfrak{X}_2 \oplus \mathfrak{X}_4 = \mathcal{S}_n \tag{2.10}$$

where $\mathcal{V}_n$ and $\mathcal{S}_n$ are defined in (2.2) and (2.3). Furthermore, there exist three nonsingular matrices T, H, G, and two matrices F, J such that

$$\begin{bmatrix} T^{-1} & T^{-1}J \\ 0 & H \end{bmatrix} \begin{bmatrix} sI_n - A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} T & 0 \\ FT & G \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cc} sI_{n_1}-A_1 & & & \bigcirc & 0 & 0 \\ & sI_{n_2}-A_2 & & & B_2 & 0 \\ & & sI_{n_3}-A_3 & & 0 & 0 \\ \bigcirc & & & sI_{n_4}-A_4 & 0 & B_4 \\ \hline 0 & 0 & C_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_4 & 0 & 0 \end{array}\right] \tag{2.11}$$

where the $n_i \times n_i$ matrix A_i corresponds to $T^{-1}A|_{\mathfrak{X}_i}T$, $i = 1, 2, 3, 4$, and B_2, B_4, C_3, and C_4 have dimensions $n_2 \times m_c$, $n_4 \times m_p$, $l_0 \times n_3$, and $l_p \times n_4$. In (2.11), the matrix A_1 is the rational form, the pairs (A_2, B_2) and (A_3^T, C_3^T) are in Brunovsky canonical form, and the triple (C_4, A_4, B_4) is in the prime canonical form.

Remark 2.12: We note that $m_p = l_p$ because (C_4, A_4, B_4) is prime.

From the above facts and, in particular (2.4)–(2.10), we can get the relations

$$\mathcal{V}^*_{b,\mathrm{Ker}\,C} = \mathfrak{X}_1 \oplus \mathfrak{X}_2 \oplus \mathfrak{X}_4, \tag{2.13}$$

$$\mathcal{R}^*_{b,\mathrm{Ker}\,C} = \mathfrak{X}_2 \oplus \mathfrak{X}_4, \tag{2.14}$$

$$\mathcal{S}^*_{b,\mathrm{Im}\,B} = \mathfrak{X}_2, \tag{2.15}$$

$$\mathfrak{N}^*_{b,\mathrm{Im}\,B} = \mathfrak{X}_1 \oplus \mathfrak{X}_2. \tag{2.16}$$

By $\rho(\cdot)$ we denote the rank of the indicated matrix and prove the following.

Theorem 2.17: Let the system S be standard. Then

i) $\mathcal{V}^*_{b,\mathrm{Ker}\,C} = \mathfrak{X}$ iff

$$\rho\left(\begin{bmatrix} sI-A & B \\ C & 0 \end{bmatrix}\right) = n+l \quad \text{for almost all } s \in \mathbf{C}; \tag{2.18}$$

ii) $\mathcal{R}^*_{b,\mathrm{Ker}\,C} = \mathfrak{X}$ iff

$$\rho\left(\begin{bmatrix} sI-A & B \\ C & 0 \end{bmatrix}\right) = n+l \quad \text{for all } s \in \mathbf{C}; \tag{2.19}$$

iii) $\mathcal{S}^*_{b,\mathrm{Im}\,B} = \{0\}$ iff

$$\rho\left(\begin{bmatrix} sI-A & B \\ C & 0 \end{bmatrix}\right) = n+m \quad \text{for almost all } s \in \mathbf{C}; \tag{2.20}$$

iv) $\mathfrak{N}^*_{b,\mathrm{Im}\,B} = \{0\}$ iff

$$\rho\left(\begin{bmatrix} sI-A & B \\ C & 0 \end{bmatrix}\right) = n+m \quad \text{for all } s \in \mathbf{C}. \tag{2.21}$$

Proofs of lemmas, theorems, and corollaries are given in the Appendix.

Remark 2.22: Even if S is not standard, parts i) and ii) of Theorem 2.17 hold when $C = 0$. In fact, if $C = 0$, i) is trivial because $\mathcal{V}^*_{b,\mathrm{Ker}\,0} = \mathfrak{X}$ and (2.18) holds for $l = 0$. Furthermore, ii) becomes the well-known rank condition for controllability because $\mathcal{R}^*_{b,\mathrm{Ker}\,0}$ is the controllable subspace. Similarly, iii) and iv) hold when $B = 0$, because iii) becomes a trivial statement and iv) becomes the rank condition for observability, since $\mathfrak{N}^*_{b,\mathrm{Im}\,0}$ is the unobservable subspace.

Theorem 2.17 establishes the relationships between the almost invariant subspaces and system matrices for an important special case which is of interest in this paper. In particular, the relationships are useful in formulating the generic properties of these subspaces in the graph-theoretic framework.

In the context of structural analysis, instead of a system S and a triple (A, B, C), we consider a structured system $\tilde{S}$ described by a triple of structured matrices $(\tilde{A}, \tilde{B}, \tilde{C})$ each having a number of fixed zero elements while the rest of the elements are independent free parameters. The parameter space $\mathbf{R}^\nu$ has the dimension ν equal to the total number of the indeterminate entries of $\tilde{A}, \tilde{B}, \tilde{C}$. When A, B, and C are numeric matrices obtained from $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ by fixing their indeterminates at some specific values in a set $\mathcal{P} \subset \mathbf{R}^\nu$, we write $(A, B, C) \in \mathcal{P}$.

Definition 2.23: Given a structured triple $(\tilde{A}, \tilde{B}, \tilde{C})$, we define the following.

i) The supremal $\mathcal{L}_p$-almost invariant (controllability) subspace contained in Ker C is said to be generically equal to the whole state-space $\mathfrak{X}$, which is denoted by $\mathcal{V}^*_{b,\mathrm{Ker}\,C} \stackrel{g}{=} \mathfrak{X}$ ($\mathcal{R}^*_{b,\mathrm{Ker}\,C} \stackrel{g}{=} \mathfrak{X}$), if there exists a proper algebraic variety $V \subset \mathbf{R}^\nu$ such that for any $(A, B, C) \in V^c$ we have $\mathcal{V}^*_{b,\mathrm{Ker}\,C} = \mathfrak{X}$ ($\mathcal{R}^*_{b,\mathrm{Ker}\,C} = \mathfrak{X}$).

ii) The infimal $\mathcal{L}_p$-almost conditionally invariant (complementary observability) subspace containing Im B is said to be generically equal to the zero space $\{0\}$, which is denoted by $\mathcal{S}^*_{b,\mathrm{Im}\,B} \stackrel{g}{=} \{0\}$ ($\mathfrak{N}^*_{b,\mathrm{Im}\,B} \stackrel{g}{=} \{0\}$), if there exists a proper algebraic variety $V \subset \mathbf{R}^\nu$ such that for any $(A, B, C) \in V^c$ we have $\mathcal{S}^*_{b,\mathrm{Im}\,B} = \{0\}$ ($\mathfrak{N}^*_{b,\mathrm{Im}\,B} = \{0\}$).

Let us recall the notion of generic rank $\tilde{\rho}(\cdot)$ (e.g., [2]). For a structured matrix $\tilde{M}$ with associated parameter space $\mathbf{R}^\nu$, $\tilde{\rho}(\tilde{M}) = r$ if there exists a proper algebraic variety $V \subset \mathbf{R}^\nu$ such that $\rho(M) = r$ for any $M \in V^c$. When we consider a structured triple $(\tilde{A}, \tilde{B}, \tilde{C})$ with a parameter space $\mathbf{R}^\nu$ and state that

$$\tilde{\rho}\left(\begin{bmatrix} sI-\tilde{A} & \tilde{B} \\ \tilde{C} & 0 \end{bmatrix}\right) = r \quad \text{for almost all } s \in \mathbf{C} \tag{2.24}$$

we mean that there exists a proper algebraic variety $V \subset \mathbf{R}^\nu$ such that for any $(A, B, C) \in V^c$ we have

$$\rho\left(\begin{bmatrix} sI-A & B \\ C & 0 \end{bmatrix}\right) = r \quad \text{for almost all } s \in \mathbf{C}. \tag{2.25}$$

We also say that a triple $(\tilde{A}, \tilde{B}, \tilde{C})$, or a system $\tilde{S}$, is a standard if $\tilde{B}$ has generic full column rank and $\tilde{C}$ has generic full row rank.

From Theorem 2.17 and Definition 2.23, the following result is automatic.

Theorem 2.26: Let a structured system $\tilde{S}$ be standard. Then,

i) $\mathcal{V}^*_{b,\mathrm{Ker}\,C} \stackrel{g}{=} \mathfrak{X}$ iff

$$\tilde{\rho}\left(\begin{bmatrix} sI-\tilde{A} & \tilde{B} \\ \tilde{C} & 0 \end{bmatrix}\right) = n+l \quad \text{for almost all } s \in \mathbf{C}; \tag{2.27}$$

ii) $\mathcal{R}^*_{b,\mathrm{Ker}\,C} \stackrel{g}{=} \mathfrak{X}$ iff

$$\tilde{\rho}\left(\begin{bmatrix} sI-\tilde{A} & \tilde{B} \\ \tilde{C} & 0 \end{bmatrix}\right) = n+l \quad \text{for all } s \in \mathbf{C}; \tag{2.28}$$

iii) $\mathcal{S}^*_{b,\mathrm{Im}\,B} \stackrel{g}{=} \{0\}$ iff

$$\tilde{\rho}\left(\begin{bmatrix} sI-\tilde{A} & \tilde{B} \\ \tilde{C} & 0 \end{bmatrix}\right) = n+m \quad \text{for almost all } s \in \mathbf{C}; \tag{2.29}$$

iv) $\mathfrak{N}^*_{b,\mathrm{Im}\,B} \stackrel{g}{=} \{0\}$ iff

$$\tilde{\rho}\left(\begin{bmatrix} sI-\tilde{A} & \tilde{B} \\ \tilde{C} & 0 \end{bmatrix}\right) = n+m \quad \text{for all } s \in \mathbf{C}. \tag{2.30}$$

The facts that the above generic rank conditions are difficult to test motivates a derivation of their graph-theoretic interpretation. This we consider next.

III. A Graph-Theoretic Characterization

We first recall some basic definitions concerning the system graph [19]. With a dynamic system $\tilde{S}$ we associate a directed graph (digraph) $D = (V, E)$ where $V = U \times X \times Y$ is a set of vertices and $U = \{u_1, u_2, \cdots, u_m\}$, $X = \{x_1, x_2, \cdots, x_n\}$, and $Y = \{y_1, y_2, \cdots, y_l\}$ are the input, state, and output vertices. $E = \{(u_j, x_i) | \tilde{b}_{ij} \neq 0\} \cup \{(x_j, x_i) | \tilde{a}_{ij} \neq 0\} \cup \{(x_j, y_i) | \tilde{c}_{ij} \neq 0\}$ is a set of edges where $\tilde{a}_{ij}$, $\tilde{b}_{ij}$, and $\tilde{c}_{ij}$ are the elements of the structured matrices $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$. A set of vertices V_s and edges E_s of a subgraph $D_s = (V_s, E_s)$ are denoted by $V(D_s)$ and $E(D_s)$, respectively. A path from the vertex v_1 to the vertex v_k is a subgraph $P_{v_1}^{v_k}$ with a set of vertices $V(P_{v_1}^{v_k}) = \{v_1, v_2, \cdots, v_k\}$ and a set of edges $E(P_{v_1}^{v_k}) = \{(v_1, v_2), (v_2, v_3), \cdots, (v_{k-1}, v_k)\}$. When the initial vertex v_1 or the terminal vertex v_k, or both, are not essential, we use the notation P^{v_k}, P_{v_1}, or P, depending on the context. The length of a path P is the number of elements in $E(P)$. A subgraph is called a cycle C if $V(C) = \{v_1, v_2, \cdots, v_k\}$ and $E(C) = \{(v_1, v_2), (v_2, v_3), \cdots, (v_{k-1}, v_k), (v_k, v_1)\}$. Subgraphs $D_1, D_2, \cdots, D_r$ are mutually disjoint if $V(D_i) \cap V(D_j) = \emptyset$ for $i \neq j$.

Of special interest in our consideration is the notion of a matching in a digraph $D = (V, E)$. A subgraph is called a matching from a set of vertices $V_{I_k} = \{v_{i_1}, v_{i_2}, \cdots, v_{i_k}\}$ to a set of vertices $V_{J_k} = \{v_{j_1}, v_{j_2}, \cdots, v_{j_k}\}$, which we denote by $M(V_{I_k}, V_{J_k})$, when it consists of k mutually disjoint paths from V_{I_k} to V_{J_k}. The length of a matching is the sum of the lengths of the corresponding paths, and we have the shortest matching $M^*(V_{I_k}, V_{J_k})$ when we choose the shortest path. Note that for given sets of V_{I_k} and V_{J_k}, the shortest matching and, thus, any matching need not be unique.

The following result provides a graph-theoretic characterization of the parts i) and iii) of Theorem 2.17.

Theorem 3.1: Let a structured system $\tilde{S}$ be standard. Then,

i) $\mathcal{V}^*_{b,\mathrm{Ker}\,C} \doteq \mathcal{X}$ iff there exists a matching $M(U_{I_l}, Y)$ in D;

ii) $\mathcal{S}^*_{b,\mathrm{Ker}\,C} \doteq \{0\}$ iff there exists a matching $M(U, Y_{J_m})$ in D.

Remark 3.2: The statement i) implies that the number of inputs should be greater than or equal to the number of outputs of S. For ii) to take place, the number of inputs should be less than or equal to the number of outputs.

To formulate graph-theoretic versions of parts ii) and iv) of Theorem 2.17, we need to use a subgraph $P_u = P_{u_{j_1}} \cup P_{u_{j_2}} \cup \cdots \cup P_{u_{j_p}}$ which is a union of mutually disjoint paths from vertices in U to arbitrary vertices of X. Similarly, $P_y = P^{y_{i_1}} \cup P^{y_{i_2}} \cup \cdots \cup P^{y_{i_z}}$ is defined with respect to X and Y. Finally, we need a subgraph $C_x = C_1 \cup C_2 \cup \cdots \cup C_r$ which is a union of mutually disjoint cycles of D.

Theorem 3.3: Let a structured system $\tilde{S}$ be standard. Then,

i) $\mathcal{R}^*_{b,\mathrm{Ker}\,C} \doteq \mathcal{X}$ iff the following two conditions hold.

i$_1$) There exists a subgraph $D_s = P_u \cup C_x \cup M(U_{I_l}, Y)$ in D such that $V_s \supset X$ and P_u, C_x, and $M(U_{I_l}, Y)$ are mutually disjoint.

i$_2$) For any vertex $x \in X - V(M^*)$, there exists a matching $M(U_{I_l} \cup \{u\}, Y \cup \{x\})$ in D for some $u \in U$, and $M^* = M^*(U_{I_l}, Y)$.

ii) $\mathcal{N}^*_{b,\mathrm{Im}\,B} \doteq \{0\}$ iff the following two conditions hold.

ii$_1$) There exists a subgraph $D_s = P_y \cup C_x \cup M(U, Y_{J_m})$ in D such that $V_s \supset X$, and P_y, C_x, and $M(U, Y_{J_m})$ are mutually disjoint.

ii$_2$) For any vertex $x \in X - V(M^*)$, there exists a matching $M(U \cup \{x\}, Y_{J_m} \cup \{y\})$ for some $y \in Y$, and $M^* = M^*(U, Y_{J_m})$.

Remark 3.4: Condition i) of the above theorem has two parts which correspond to the generic rank and input reachability properties of structural controllability [2]. Similarly, condition ii) resembles the structural observability conditions. It is not surprising, however, that Theorem 3.3 is more complex because it deals with both inputs and outputs simultaneously.

We illustrate Theorems 3.1 and 3.3 by several examples.

Example 3.5: Consider a structured system $\tilde{S}$ with the matrices

$$\tilde{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & * & 0 \\ 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & * & 0 \\ 0 & * & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} * & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & * \\ 0 & 0 \end{bmatrix},$$

$$\tilde{C} = \begin{bmatrix} 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * \end{bmatrix} \tag{3.6}$$

where as usual $*$ represents independent real numbers and 0 "hard zeros." The corresponding digraph D is shown in Fig. 1. It is easy to see that there exists no matching from $U = \{u_1, u_2\}$ to $Y = \{y_1, y_2\}$ and, therefore, we conclude from Theorem 3.1 that $\mathcal{V}^*_{b,\mathrm{Ker}\,C} \doteq \mathcal{X}$ does not hold. An implication of this fact is that the system S has no inverse no matter what are the values of free parameters.

Example 3.7: Let a system $\tilde{S}$ be specified by

$$\tilde{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & * \\ 0 & * & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} * & 0 & 0 \\ 0 & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & * & * \end{bmatrix},$$

$$\tilde{C} = \begin{bmatrix} 0 & * & 0 & 0 & * & 0 \\ * & 0 & 0 & 0 & * & 0 \end{bmatrix}. \tag{3.8}$$

From digraph D of Fig. 2, we see that there is a matching $M(\{u_1, u_2\}, \{y_1, y_2\}) = P_{u_1}^{y_1} \cup P_{u_2}^{y_2}$ with $P_{u_1}^{y_1} = (u_1, x_1, x_2, y_1)$ and $P_{u_2}^{y_2} = (u_2, x_3, x_4, x_5, y_2)$. By Theorem 3.1, $\mathcal{V}^*_{b,\mathrm{Ker}\,C} \doteq \mathcal{X}$. However, when we consider $\mathcal{R}^*_{b,\mathrm{Ker}\,C}$, we find that condition i) of Theorem 3.3 is not satisfied. Although i$_1$) is satisfied by $M(\{u_1, u_2\}, \{y_1, y_2\})$ and a self-loop at x_6, condition i$_2$) fails to hold. In fact, $M^*(\{u_1, u_2,\}, \{y_1, y_2\}) = P_{u_1}^{y_2} \cup P_{u_2}^{y_1}$ with $P_{u_1}^{y_2} = (u_1, x_1, y_2)$ and $P_{u_2}^{y_1} = (u_2, x_3, x_4, x_5, y_1)$, and $X - V(M^*) = \{x_2, x_6\}$, so that we have $M(\{u_1, u_2, u_3\}, \{y_1, y_2, x_6\})$ for x_6, but $M(\{u_1, u_2, u_3\}, \{y_1, y_2, x_2\})$ for x_2 does not exist.

If we modify the digraph D of Fig. 2 by moving the input u_3 from state x_6 to x_2, as shown in Fig. 3, we can establish condition $\mathcal{R}^*_{b,\mathrm{Ker}\,C} \doteq \mathcal{X}$. The new matrix $\tilde{B}$, which is given as

$$\tilde{B} = \begin{bmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & * & 0 \end{bmatrix} \tag{3.9}$$

leaves the set of vertices $X - V(M^*) = \{x_2, x_6\}$ unchanged, but the two matchings mentioned above are not present. The reason is that by moving u_3 to x_2, we create the path $P_{u_3}^{x_2} = (u_3, x_2)$ which provides for two required matchings in the modified graph D of Fig. 3.

Remark 3.10: Special properties of almost invariant spaces are established by existence of shortest matchings in a digraph D.

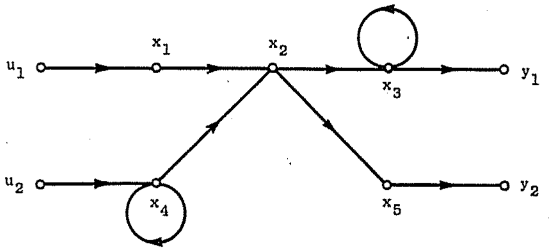

Fig. 1. Digraph D for Example 3.5.

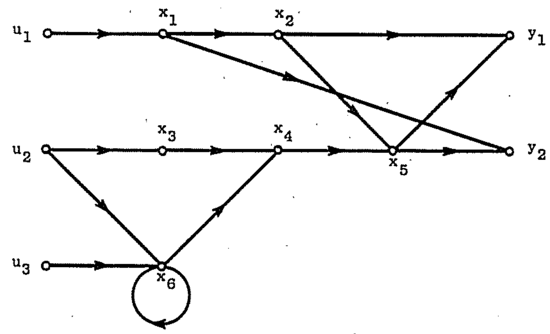

Fig. 2. Digraph D for Example 3.7.

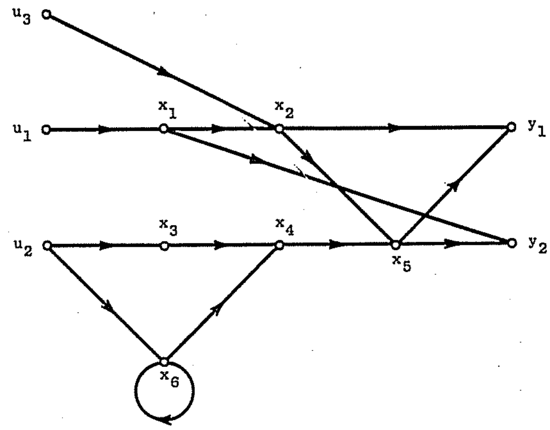

Fig. 3. Modified digraph D for Example 3.7.

This is computationally attractive because there are efficient algorithms for maximum flow in transportation networks which apply directly to this task (see, for example, [23]). In fact, first add the source vertex s and the sink vertex t to D, then add edges from s to every node in U and edges from every vertex in Y to t. Finally, assign capacity 1 to every edge of the newly formed digraph and apply any variant of the well-known Ford–Fulkerson's algorithm to get the desired result.

IV. Decentralized Control

We now apply the obtained graph-theoretic conditions to determine when structured interconnected systems can be stabilized by decentralized controllers. For simplicity in presentation, we consider an input and output decentralized system composed of only two subsystems. An extension of the obtained results to the general case is obvious, but may involve some combinatorial representations.

Let us consider an interconnected system

$$S: \begin{aligned} \dot{x}_1 &= A_1x_1 + B_1u_1 + A_{12}x_2 \\ \dot{x}_2 &= A_2x_2 + B_2u_2 + A_{21}x_1 \\ y_1 &= C_1x_1 \\ y_2 &= C_2x_2 \end{aligned} \tag{4.1}$$

where $x_i(t) \in \mathcal{X}_i := \boldsymbol{R}^{n_i}$, $u_i(t) \in \mathcal{U}_i := \boldsymbol{R}^{m_i}$, and $y_i(t) \in \mathcal{Y}_i := \boldsymbol{R}^{l_i}$ are the state, input, and output of the ith subsystem

$$S_i: \begin{aligned} \dot{x}_i &= A_ix_i + B_iu_i \\ y_i &= C_ix_i, \qquad i = 1, 2. \end{aligned} \tag{4.2}$$

By using local controllers with dynamic output feedback

$$C_i: \begin{aligned} \dot{z}_i &= F_iz_i + G_iy_i \\ u_i &= H_iz_i + K_iy_i, \qquad i = 1, 2 \end{aligned} \tag{4.3}$$

we obtain the overall closed-loop system

$$\hat{S}: \begin{bmatrix} \dot{x}_1 \\ \dot{z}_1 \\ \dot{x}_2 \\ \dot{z}_2 \end{bmatrix} = \left[\begin{array}{cc|cc} A_1 + B_1K_1C_1 & B_1H_1 & A_{12} & 0 \\ G_1C_1 & F_1 & 0 & 0 \\ \hline A_{21} & 0 & A_2 + B_2K_2C_2 & B_2H_2 \\ 0 & 0 & G_2C_2 & F_2 \end{array}\right] \begin{bmatrix} x_1 \\ z_1 \\ x_2 \\ z_2 \end{bmatrix} \tag{4.4}$$

which is an interconnection of two closed-loop subsystems

$$\hat{S}_i: \begin{bmatrix} \dot{x}_i \\ \dot{z}_i \end{bmatrix} = \begin{bmatrix} A_i + B_iK_iC_i & B_iH_i \\ G_iC_i & F_i \end{bmatrix} \begin{bmatrix} x_i \\ z_i \end{bmatrix}, \qquad i = 1, 2. \tag{4.5}$$

In the context of connective stability [19], we state the following.

Definition 4.6: An interconnected system S is said to be decentrally connectively stabilizable if there exist controllers C_1 and C_2 such that the closed-loop system $\hat{S}$ and subsystems $\hat{S}_1$ and $\hat{S}_2$ are *simultaneously* asymptotically stable.

The following lemma is a direct consequence of the result obtained by Willems and Ikeda [14]. In the lemma, we compute almost invariant subspaces relative to subsystems $S_i = (A_i, B_i, C_i)$. Then, by $\mathcal{R}^*_{b,\mathrm{Ker}\,H}$ and $\mathcal{N}^*_{b,\mathrm{Im}\,G}$ w.r.t. S_i we denote the supremal $\mathcal{L}_p$-almost controllability subspace contained in Ker H computed relative to (A_i, B_i) and the infimal $\mathcal{L}_p$-almost complementary observability subspace containing Im G computed relative to (A_i, C_i), respectively. We also recall that a system S is complete if it is controllable and observable.

Lemma 4.7: An interconnected system S is decentrally connectively stabilizable if one of the following conditions hold.

i) S_1 is complete and

$$\mathcal{R}^*_{b,\mathrm{Ker}\,A_{12}} = \mathcal{X}_2, \ \mathcal{N}^*_{b,\mathrm{Im}\,A_{21}} = \{0\} \qquad \text{w.r.t. } S_2. \tag{4.8}$$

ii) S_2 is complete and

$$\mathcal{R}^*_{b,\mathrm{Ker}\,A_{21}} = \mathcal{X}_1, \ \mathcal{N}^*_{b,\mathrm{Im}\,A_{12}} = \{0\} \qquad \text{w.r.t. } S_1. \tag{4.9}$$

With a system S of (4.1) we associate a structured system $\tilde{S}$ defined by the corresponding structured matrices. We assume that $\tilde{B}_1$ and $\tilde{B}_2$ have generic full column rank, and $\tilde{C}_1$ and $\tilde{C}_2$ have generic full row rank.

Definition 4.10: A structured interconnected system $\tilde{S}$ is said to be decentrally connectively stabilizable if there exists a proper variety $V \subset \boldsymbol{R}^{\nu}$ such that for any $(A_i, B_i, C_i, A_{ij}; i, j = 1, 2) \in V^c$, the system S is decentrally connectively stabilizable.

Before we state the main result of this section, we need to clarify some facts regarding the interconnection matrices A_{ij}. Note that in Lemma 4.7 the matrices A_{ij} do not necessarily have full rank. Thus, the structured matrices $\tilde{A}_{ij}$ need not have generic full rank either. In order to provide a structured version of Lemma 4.7 we have to modify the matrices $\tilde{A}_{ij}$ to obtain corresponding structured matrices having generic full rank. Let $\tilde{M}$ be a $p \times q$ matrix such that $\tilde{\rho}(\tilde{M}) = g$. Then, $\tilde{M}^r$ is a $g \times q$ submatrix of $\tilde{M}$ with $\tilde{\rho}(\tilde{M}^r) = g$ which can be obtained by removing $p - g$ rows of $\tilde{M}$. Similarly, we use $\tilde{M}^c$ of dimension $p \times g$ and $\tilde{\rho}(\tilde{M}^c) = g$ for columns of $\tilde{M}$. For example, if

$$\tilde{M} = \begin{bmatrix} 0 & * & 0 & 0 \\ 0 & * & 0 & * \\ 0 & 0 & 0 & * \end{bmatrix} \tag{4.11}$$

then any of the submatrices

$$\begin{bmatrix} 0 & * & 0 & 0 \\ 0 & * & 0 & * \end{bmatrix}, \begin{bmatrix} 0 & * & 0 & 0 \\ 0 & 0 & 0 & * \end{bmatrix}, \begin{bmatrix} 0 & * & 0 & * \\ 0 & 0 & 0 & * \end{bmatrix} \tag{4.12}$$

is a candidate for $\tilde{M}^r$. However, $\tilde{M}^c$ is the unique submatrix composed of the second and fourth column of $\tilde{M}$. We also use $D(\tilde{A}, \tilde{B}, \tilde{C})$ to identify the digraph D of a structured system $\tilde{S}$ having the triple $(\tilde{A}, \tilde{B}, \tilde{C})$.

Theorem 4.13: A structured interconnected system $\tilde{S}$ is decentrally connectively stabilizable if one of the following conditions hold.

i) $\tilde{S}_1$ is structurally complete, and the digraphs $D(\tilde{A}_2, \tilde{B}_2, \tilde{A}^r_{12})$ and $D(\tilde{A}_2, \tilde{A}^c_{21}, \tilde{C}_2)$ satisfy conditions i) and ii) of Theorem 3.3, respectively.

ii) $\tilde{S}_2$ is structurally complete, and the digraphs $D(\tilde{A}_1, \tilde{B}_1, \tilde{A}^r_{12})$ and $D(\tilde{A}_1, \tilde{A}^c_{12}, \tilde{C}_1)$ satisfy conditions i) and ii) of Theorem 3.3, respectively.

Example 4.14: To illustrate the application of Theorem 4.13, let us consider a system $\tilde{S}$ described by the following triple $(\tilde{A}, \tilde{B}, \tilde{C})$:

$$\tilde{A} = \left[\begin{array}{c|c} \tilde{A}_1 & \tilde{A}_{12} \\ \hline \tilde{A}_{21} & \tilde{A}_2 \end{array}\right] = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & * \\ * & 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 \end{array}\right],$$

$$\tilde{B} = \left[\begin{array}{c|c} \tilde{B}_1 & 0 \\ \hline 0 & \tilde{B}_2 \end{array}\right] = \left[\begin{array}{cc|c} * & 0 & 0 \\ 0 & 0 & 0 \\ 0 & * & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & * \end{array}\right],$$

$$\tilde{C} = \left[\begin{array}{c|c} \tilde{C}_1 & 0 \\ \hline 0 & \tilde{C}_2 \end{array}\right] = \left[\begin{array}{ccc|ccc} 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & * \end{array}\right]. \tag{4.15}$$

The corresponding digraph $D(\tilde{A}, \tilde{B}, \tilde{C})$ is given in Fig. 4.

It is easy to see that

$$\tilde{A}^r_{21} = [*\ 0\ *],\ \tilde{A}^c_{12} = \begin{bmatrix} 0 \\ * \\ * \end{bmatrix}. \tag{4.16}$$

Therefore, we get the digraphs $D(\tilde{A}_1, \tilde{B}_1, \tilde{A}^r_{21})$ and $D(\tilde{A}_1, \tilde{A}^c_{12}, \tilde{C}_1)$ as shown in Fig. 5. Since $\tilde{S}_2$ is structurally complete, from Fig. 5 and condition ii) of Theorem 4.13, we conclude that the system $\tilde{S}$ specified by (4.15) is decentrally connectively stabilizable. We note that the digraph $D(\tilde{A}, \tilde{B}, \tilde{C})$ of Fig. 4 does not satisfy condition i) of Theorem 4.13.

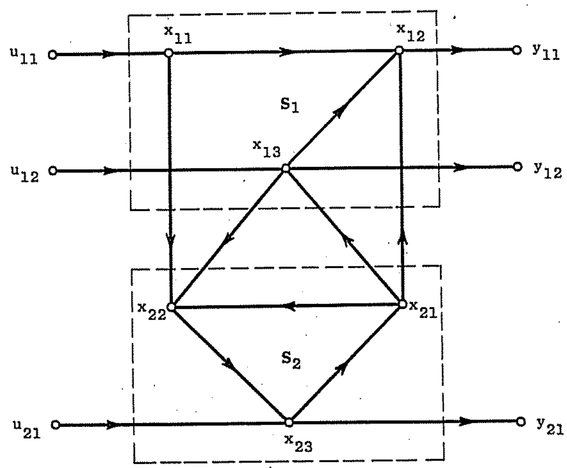

Fig. 4. Digraph $D(\tilde{A}, \tilde{B}, \tilde{C})$ for Example 4.14.

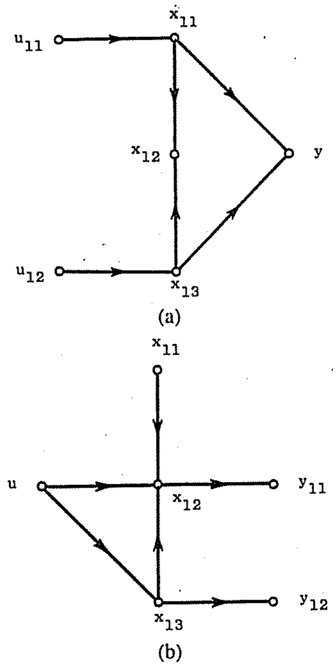

Fig. 5. Digraphs for Example 4.14. (a) $D(\tilde{A}_1, \tilde{B}_1, \tilde{A}^r_{21})$; (b) $D(\tilde{A}_1, \tilde{A}^c_{12}, \tilde{C}_1)$.

V. Conclusion

By using certain relations between properties of almost invariant subspaces and the rank of the system matrix, we have obtained a set of graph-theoretic conditions for the supremal $\mathcal{L}_p$-almost invariant (or controllability) subspace to be equal generically to the whole state space. Similar conditions have also been derived for the infimal $\mathcal{L}_p$-almost conditional invariant (or complementary observability) subspace to be equal generically to the zero subspace. By applying the conditions to interconnected systems, we can establish a parameter independent stabilizability involving decentralized feedback laws. The stabilizability test may appear conservative, but it is easy to apply because only binary computations are required.

Appendix

Proof of Theorem 2.17:

i) From (2.13), $\mathcal{V}^*_{b,\mathrm{Ker}\, C} = \mathcal{X}$ is equivalent to saying that $\mathcal{X}_3 =$

$\{0\}$ holds in Lemma 2.8, i.e., $n_3 = 0$ in (2.11). Notice that the submatrices $[sI_{n_2} - A_2 B_2]$ and

$$\begin{bmatrix} sI_{n_4}-A_4 & B_4 \\ C_4 & 0 \end{bmatrix}$$

in (2.11) have full row rank for all $s \in C$ because (A_2, B_2) is controllable and (C_4, A_4, B_4) is prime. Therefore, a necessary and sufficient condition for $\mathcal{V}^*_{b,\mathrm{Ker}\,C} = \mathcal{X}$ to hold is that a matrix on the right-hand side of (2.11) has full row rank for all $s \in C$ except at the eigenvalues of A_1. From the above fact and also from the nonsingularity of matrices

$$\begin{bmatrix} T^{-1} & T^{-1}J \\ 0 & H \end{bmatrix} \text{ and } \begin{bmatrix} T & 0 \\ FT & G \end{bmatrix},$$

the assertion i) is automatic.

ii) From (2.14), $\mathcal{R}^*_{b,\mathrm{Ker}\,C} = \mathcal{X}$ is equivalent to saying that $\mathcal{X}_1 = \mathcal{X}_3 = \{0\}$ holds in Lemma 2.8, i.e., $n_1 = n_3 = 0$ in (2.11). Therefore, the assertion ii) is easy to see by relying on the full row rank of $[sI_{n_2} - A_2 B_2]$ and

$$\begin{bmatrix} sI_{n_4}-A_4 & B_4 \\ C_4 & 0 \end{bmatrix}.$$

iii) From (2.15), $\mathcal{S}^*_{b,\mathrm{Im}\,B} = \{0\}$ is equivalent to saying that $\mathcal{X}_2 = \{0\}$ holds in Lemma 2.8, i.e., $n_2 = 0$ in (2.11). Therefore, the assertion iii) follows from the full column rank of

$$\begin{bmatrix} sI_{n_3}-A_3 \\ C_3 \end{bmatrix} \text{ and } \begin{bmatrix} sI_{n_4}-A_4 & B_4 \\ C_4 & 0 \end{bmatrix}$$

and also by using the same argument as used in i).

iv) From (2.16), $\mathcal{N}^*_{b,\mathrm{Im}\,B} = \{0\}$ is equivalent to saying that $\mathcal{X}_1 = \mathcal{X}_2 = \{0\}$ holds in Lemma 2.8, i.e., $n_1 = n_2 = 0$ in (2.11). Therefore, the assertion is trivial. Q.E.D.

Proof of Theorem 2.26: Theorem 2.26 is obvious from Theorem 2.17, Definition 2.23, and the definition of generic rank. Q.E.D.

Proof of Theorem 3.1:

i) Sufficiency: Without loss of generality, we can assume that there exists a matching $M(\{u_1, u_2, \cdots, u_l\}, Y)$ in D. Let k be the total length of the shortest matching $M^*(\{u_1, u_2, \cdots, u_l\}, Y)$ and denote a set of all the shortest matching from $\{u_1, u_2, \cdots, u_l\}$ to Y by $\tilde{M}^*$. Then it is straightforward to verify [20] that

$$\det \begin{bmatrix} sI_n-\tilde{A} & \tilde{B}_l \\ \tilde{C} & 0 \end{bmatrix} = \alpha_0(p)s^{n-k}+\alpha_1(p)s^{n-k-1}+\cdots+\alpha_{n-k}(p), \tag{A.1}$$

where

$$\alpha_0(p) = \sum_{D_s \in \tilde{M}^*} \pm f(D_s) \tag{A.2}$$

$\tilde{B}_l$ is a submatrix of $\tilde{B}$ which is composed of the first l columns of $\tilde{B}$, p denotes the ν dimensional vector in the associated parameter space, and $f(\cdot)$ denotes the product of all nonfixed entries of $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ corresponding to all edges contained in the indicated subgraph. Notice that all $f(D_s)$'s in (A.2) are monomials of p and it holds that $f(D_{s_1}) \not\equiv (D_{s_2})$ for $D_{s_1}, D_{s_2} \in \tilde{M}^*$, $D_{s_1} \neq D_{s_2}$. This implies that $\alpha_0(p)$ in (A.1) is not identically zero as a polynomial of p because $\tilde{M}^*$ is not empty by the assumption. Therefore, a set $V = \{p \in R^\nu | \alpha_0(p) = 0\}$ is a proper variety in the parameter space, and it follows from (A.1) that for any

$$(A, B, C) \in V^c, \det \begin{bmatrix} sI_n-A & B_l \\ C & 0 \end{bmatrix}$$

is not identically zero as a polynomial of s, i.e., for any $(A, B, C) \in V^c$, the rank of

$$\begin{bmatrix} sI-A & B \\ C & 0 \end{bmatrix}$$

is equal to $n + l$ for almost all $s \in C$. From this and Theorem 2.17 i), we get $\mathcal{V}^*_{b,\mathrm{Ker}\,C} \stackrel{*}{=} \mathcal{X}$.

Necessity: From Theorem 2.17 i), $\mathcal{V}^*_{b,\mathrm{Ker}\,C} \stackrel{*}{=} \mathcal{X}$ implies that there exists at least one $(A, B, C) \in R^\nu$ such that

$$\begin{bmatrix} sI_n-A & B \\ C & 0 \end{bmatrix}$$

has full row rank for almost all $s \in C$. This fact implies that

$$\begin{bmatrix} sI_n-\tilde{A} & \tilde{B} \\ \tilde{C} & 0 \end{bmatrix}$$

has at least one minor with order $n + l$ that is not identically zero as a polynomial of s and p. Therefore, without loss of generality, assume that the minor that consists of the first n_1 columns of

$$\begin{bmatrix} sI_n-\tilde{A} \\ \tilde{C} \end{bmatrix}$$

and the first n_2 columns of $[{}^{\tilde{B}}_{0}]$ ($n_1 + n_2 = n + l$, $n_1 \le n$, $n_2 \ge l$) is not identically zero. In other words, $\det \tilde{\Gamma} \not\equiv 0$ where $\tilde{\Gamma}$ is an $(n + m) \times (n + m)$ matrix defined by

$$\tilde{\Gamma} = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{sI_n-\tilde{A}} & \multicolumn{2}{c}{\tilde{B}} \\ \hline \multicolumn{2}{c|}{\tilde{C}} & \multicolumn{2}{c}{0} \\ \hline 0 & \tilde{I}_{n-n_1} & 0 & 0 \\ 0 & 0 & 0 & \tilde{I}_{m-n_2} \end{array}\right]. \tag{A.3}$$

Observe [20] that $\det \tilde{\Gamma} \not\equiv 0$ if and only if the Coates graph $G_c(\tilde{\Gamma})$, which is associated with the matrix $\tilde{\Gamma}$, has a spanning subgraph $H = C_1 \cup \cdots \cup C_r$ where C_i's are mutually disjoint cycles. Notice that $G_c(\tilde{\Gamma})$ is identical to a digraph that is obtained from D as follows. Add $(m - l)$ nodes $y_{l+1}, \cdots, y_m$ and edges (x_i, x_i) for $i = 1, \cdots, n$, (x_{n_1+j}, y_{l+j}) for $j = l \cdots, m - l$, (u_{n_2+k}, y_{n_2+k}) for $k = 1, \cdots, m - n_2$ to D, and then identify u_i with y_i for $i = 1, 2, \cdots, m$. Therefore, it is easy to verify that the existence of the subgraph H in $G_c(\tilde{\Gamma})$ implies the existence of a subgraph $D_s = P^{y_1}_{u_{i_1}} \cup P^{y_2}_{u_{i_2}} \cdots \cup P^{y_l}_{u_{i_l}}$ in D where $P^{y_j}_{u_{i_j}}$'s are mutually disjoint. Notice that the subgraph D_s is a matching $M(U_{I_l}, Y)$.

ii) We omit the proof because it is similar to that of i). Q.E.D.

Proof of Theorem 3.3: Only part i) will be proved because part ii) can be derived by duality.

Before proving Theorem 3.3, we have to briefly review some results [21], [22] on structured controllability for a linear system in descriptor form

$$\Sigma\colon K\dot{\varphi}(t) = L\varphi(t) + M\omega(t)$$

where $\varphi(t) \in R^r$ and $\omega(t) \in R^p$ are the descriptor state and input of Σ, and K, L, and M are numerical matrices of appropriate dimensions.

Associated with the system Σ and the triple (K, L, M), consider a structured system $\tilde{\Sigma}$ described by a triple of structural matrices $(\tilde{K}, \tilde{L}, \tilde{M})$. Notice that the generic rank of $\tilde{K}$ is not necessarily equal to r, but it is assumed that $s\tilde{K} - \tilde{L}$ is a regular pencil generically. Let q be the degree of det $(s\tilde{K} - \tilde{L})$ with respect to s. Then it is easy to verify that there exist two permutation matrices Q_1 and Q_2 such that

$$\tilde{K}' := Q_1\tilde{K}Q_2 = \begin{bmatrix} \tilde{I}_q & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \tilde{K}'_{11} & \tilde{K}'_{12} \\ \tilde{K}'_{21} & \tilde{K}'_{22} \end{bmatrix}, \tag{A.4}$$

$$\tilde{L}' := Q_1\tilde{L}Q_2 = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{I}_{r-q} \end{bmatrix} + \begin{bmatrix} \tilde{L}'_{11} & \tilde{L}'_{12} \\ \tilde{L}'_{21} & \tilde{L}'_{22} \end{bmatrix} \tag{A.5}$$

$$\tilde{M}' := Q_1\tilde{M} = \begin{bmatrix} \tilde{M}'_1 \\ \tilde{M}'_2 \end{bmatrix}, \tag{A.6}$$

where $\tilde{I}_q$ is a $q \times q$ structured diagonal matrix. With the system $\tilde{\Sigma}$, we associate a digraph $D_\Sigma = (V_\Sigma, E_\Sigma)$ where the set of vertices V_Σ is defined as $V_\Sigma = \Phi \cup \Omega$ with $\Phi = \{\varphi_1, \varphi_2, \cdots, \varphi_r\}$ and $\Omega = \{\omega_1, \omega_2, \cdots, \omega_p\}$. The set of edges E_Σ is defined as $E_\Sigma = \{(\omega_j, \varphi_i) \mid \tilde{m}'_{ij} \neq 0\}$ and $\{(\varphi_j, \varphi_i) \mid \tilde{k}'_{ij} \neq 0 \text{ or } \tilde{l}'_{ij} \neq 0\}$ where $\tilde{k}'_{ij}$, $\tilde{l}'_{ij}$, and $\tilde{m}'_{ij}$ indicate the (i, j) entries of $\tilde{K}'$, $\tilde{L}'$, and $\tilde{M}'$, respectively. With these preliminaries the following result on structural controllability for linear system in descriptor form has been obtained in [21].

Lemma A.7: The following two conditions are equivalent.

i) $\tilde{\rho}([s\tilde{K} - \tilde{L}\tilde{M}]) = r$ for all $s \in C$.

ii) The following two conditions hold.

ii$_1$) $\tilde{\rho}([\tilde{L}\tilde{M}]) = r$.

ii$_2$) The nodes $\varphi_1, \cdots, \varphi_q$ among $\Phi = \{\varphi_1, \cdots, \varphi_q, \cdots, \varphi_r\}$ are reachable from Ω-nodes in D_Σ.

Next we show two more lemmas.

Lemma A.8:

$$\tilde{\rho}\left(\begin{bmatrix} sI_n - \tilde{A} & \tilde{B} \\ \tilde{C} & 0 \end{bmatrix}\right) = n + l \quad \text{for all } s \in C, \tag{A.9}$$

iff

$$\tilde{\rho}\left(\begin{bmatrix} s\tilde{I}_n - \tilde{A} & \tilde{B} \\ \tilde{C} & 0 \end{bmatrix}\right) = n + l \quad \text{for all } s \in C. \tag{A.10}$$

Proof:

Sufficiency: Trivial.

Necessity: Equation (A.10) implies that there exist a quadruple (E_0, A_0, B_0, C_0) and a positive number ϵ such that for any $(E, A, B, C) \in \mathbb{S} := \{(E, A, B, C) \in \boldsymbol{R}^{\nu+n} \mid \|E - E_0\| + \|A - A_0\| + \|B - B_0\| + \|C - C_0\| < \epsilon\}$,

$$\rho\left(\begin{bmatrix} sE - A & B \\ C & 0 \end{bmatrix}\right) = n + l \quad \text{for all } s \in C,$$

where $\boldsymbol{R}^\nu$ is the associated parameter space of $(\tilde{A}, \tilde{B}, \tilde{C})$ and E is diagonal. Therefore, we can always find a quadruple $(E_1, A_1, B_1, C_1) \in \mathbb{S}$ with E_1 being nonsingular.

This implies that

$$\rho\left(\begin{bmatrix} sI_n - E_1^{-1}A_1 & E_1^{-1}B_1 \\ C_1 & 0 \end{bmatrix}\right) = n + l \quad \text{for all } s \in C.$$

Notice that $(E_1^{-1}A_1, E_1^{-1}B_1, C_1) \in \boldsymbol{R}^\nu$ because E_1^{-1} is diagonal. Thus, (A.9) follows. Q.E.D.

Lemma A.11:

$$\tilde{\rho}\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & 0 \end{bmatrix} = n + l \tag{A.12}$$

iff condition i$_1$) of Theorem 3.3 holds.

Proof:

Necessity: By the same argument as we used in the proof of the necessity part of Theorem 3.1i), (A.12) implies that graph $G_c(\tilde{\Delta})$ has a spanning subgraph $H = C_1 \cup \cdots \cup C_r$ where C_i's are mutually disjoint cycles, and $\tilde{\Delta}$ is an $(n + m) \times (n + m)$ matrix defined by

$$\tilde{\Delta} = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\tilde{A}} & \multicolumn{2}{c}{B} \\ \multicolumn{2}{c|}{\tilde{C}} & \multicolumn{2}{c}{0} \\ \hline 0 & \tilde{I}_{n-n_1} & 0 & 0 \\ 0 & 0 & 0 & \tilde{I}_{m-n_2} \end{array}\right].$$

Notice that $G_c(\tilde{\Delta})$ is identical to a digraph that is obtained from D as follows: Add $(m - l)$ nodes $y_{l+1}, \cdots, y_m$ and edges (x_{n_1+j}, y_{l+j}) for $j = l, \cdots, m - l$, (u_{n_2+k}, y_{n+k}) for $k = l, \cdots, m - n_2$, and identify u_i with y_i for $i = 1, 2, \cdots, m$. Therefore, it is easy to see that the existence of the subgraph H in $G_c(\tilde{\Delta})$ implies condition i$_1$) of Theorem 3.3.

Sufficiency: Assume that condition i$_1$) of Theorem 3.3 is satisfied. Without loss of generality, we can assume that D has a subgraph $D_s = P_u \cup C_x \cup M(\{u_1, \cdots, u_l\}, Y)$ where $P_u = P_{u_{l+1}} \cdots P_{u_{l+p}}$, $C_x = C_1 \cup \cdots \cup C_r$, and $M(\{u_1, \cdots, u_l\}, Y) = P_{u_1}^{y_1} \cdots P_{u_l}^{y_l}$. Furthermore assume that the length of $P_{u_{l+j}}$ is l_{l+j} for $j = 1, \cdots, p$, the length of C_k is l_k^c for $k = 1, \cdots, r$, and the length of $P_{u_i}^{y_i}$ is l_i for $i = 1, \cdots, l$. Then, let A, B, and C be matrices obtained from $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ by substituting 1's for all the nonfixed entries corresponding to the edges contained in D_s and substituting 0's for the other nonfixed entries. It is easy to verify that there exists a permutation matrix Q such that

$$\begin{bmatrix} Q^T & 0 \\ 0 & I_l \end{bmatrix}\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}\begin{bmatrix} Q & 0 \\ 0 & I_m \end{bmatrix} = \left[\begin{array}{ccc|ccc} A_M & 0 & 0 & B_M & 0 & 0 \\ 0 & A_P & 0 & 0 & B_P & 0 \\ 0 & 0 & A_C & 0 & 0 & 0 \\ \hline C_M & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

where

$$A_M = \begin{bmatrix} N_{l_1} & & \bigcirc \\ & \ddots & \\ \bigcirc & & N_{l_l} \end{bmatrix} \quad B_M = \begin{bmatrix} e_{l_1}^c & & \bigcirc \\ & \ddots & \\ \bigcirc & & e_{l_l}^c \end{bmatrix},$$

$$C_M = \begin{bmatrix} e_{l_1}^r & & \bigcirc \\ & \ddots & \\ \bigcirc & & e_{l_l}^r \end{bmatrix},$$

$$A_P = \begin{bmatrix} N_{l_{l+1}} & & \bigcirc \\ & \ddots & \\ \bigcirc & & N_{l_{l+p}} \end{bmatrix} \quad B_P = \begin{bmatrix} e_{l_{l+1}}^c & & \bigcirc \\ & \ddots & \\ \bigcirc & & e_{l_{l+p}}^c \end{bmatrix},$$

$$A_C = \begin{bmatrix} \bar{N}_{l_1} & & \bigcirc \\ & \ddots & \\ \bigcirc & & \bar{N}_{l_r} \end{bmatrix}$$

and N_i, e_i^c, e_i^r, and $\bar{N}_i$ are $i \times i$, $i \times 1$, $1 \times i$, and $i \times i$ matrices defined as

$$N_i = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & \bigcirc & \\ & 1 & \ddots & & \\ & \bigcirc & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix}, \quad e_i^c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$e_i^r = [0 \ \cdots \ 0 \ 1]$$

$$\bar{N}_i = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & \bigcirc & \\ & 1 & \ddots & & \\ & \bigcirc & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix}.$$

This implies that we have found a triple $(A, B, C) \in \boldsymbol{R}^{\nu}$ such that $\rho([\begin{smallmatrix} A & B \\ C & 0 \end{smallmatrix}]) = n + l$. Thus, (A.12) is satisfied. Q.E.D.

Now we prove i) of Theorem 3.3.

Necessity: Assume that $\mathcal{R}^*_{b,\mathrm{Ker}\, C} \overset{*}{=} \mathcal{X}$. Then it follows from Theorem 2.24ii) and Lemma A.8 that a structured system $\tilde{\Sigma}$, i.e., a triple $(\tilde{K}, \tilde{L}, \tilde{M})$ satisfies condition i) of Lemma A.7 where

$$\tilde{K}=\begin{bmatrix} \tilde{I}_n & 0 \\ 0 & 0 \end{bmatrix}, \tilde{L}=\begin{bmatrix} \tilde{A} & \tilde{B}_I \\ \tilde{C} & 0 \end{bmatrix}, \tilde{M}=\begin{bmatrix} \tilde{B}_{II} \\ 0 \end{bmatrix} \quad \text{(A.13)}$$

where $\tilde{B} = [\tilde{B}_I \tilde{B}_{II}]$. Therefore, condition ii) of Lemma A.7 follows.

By Lemma A.11 it is easy to see that condition ii_1) of Lemma A.7 implies condition i_1) of Theorem 3.3.

Next we will prove that condition ii_2) of Lemma A.7 implies condition i_2) of Theorem 3.3 under condition ii_1) of Lemma A.7. To do this we need to investigate a relationship between two digraphs $\boldsymbol{D}_\Sigma$ and $\boldsymbol{D}$. Recall that condition ii_1) of Lemma A.7 implies condition i_1) of Theorem 3.3. Therefore, without loss of generality, we can assume that there exists a matching $\boldsymbol{M}(\{u_1, u_2, \cdots, u_l\}, Y)$ in $\boldsymbol{D}$ and a shortest matching is $\boldsymbol{M}^* = \boldsymbol{P}^{y_1}_{u_1} \cup \cdots \cup \boldsymbol{P}^{y_l}_{u_l}$ where the length of $\boldsymbol{P}^{y_i}_{u_i}$ is l_i for $i = 1, \cdots, l$. Let $q = n - \sum_{i=1}^{l} l_i$. Notice that q is the degree of det $[s\tilde{K} - \tilde{L}]$ with respect to s. Furthermore, without loss of generality, it can be assumed that $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ have the following structures:

$$\tilde{A}=\begin{bmatrix} \tilde{A}_{00} & \tilde{A}_{01} & \cdots & \tilde{A}_{0l} \\ \tilde{A}_{10} & \tilde{A}_{11} & \cdots & \tilde{A}_{1l} \\ \vdots & \vdots & & \vdots \\ \tilde{A}_{l0} & \tilde{A}_{l1} & \cdots & \tilde{A}_{ll} \end{bmatrix}, \quad B=\left[\begin{array}{ccc|c} & B_0 & & \\ \hline \tilde{b}_{11} & \cdots & \tilde{b}_{1l} & \tilde{B}_{II} \\ \vdots & & \vdots & \\ \tilde{b}_{l1} & \cdots & \tilde{b}_{ll} & \end{array}\right],$$

$$\tilde{C}=\left[\begin{array}{c|ccc} & \tilde{c}_{11} & \cdots & \tilde{c}_{1l} \\ \tilde{C}_0 & \vdots & & \vdots \\ & \tilde{c}_{l1} & \cdots & \tilde{c}_{ll} \end{array}\right] \quad \text{(A.14)}$$

where $\tilde{A}_{ii}$, $\tilde{b}_{ii}$, and $\tilde{c}_{ii}$ are $l_i \times l_i$, $l_i \times 1$, $1 \times l_i$ structured matrices for $i = 1, \cdots, l$ defined as

$$\tilde{A}_{ii}=\begin{bmatrix} \times & & & \\ & \times & & \times \\ \oplus & & \ddots & \\ & \oplus & & \ddots \\ \bigcirc & & \ddots & \\ & & \oplus & \times \end{bmatrix}, \quad \tilde{b}_{ii}=\begin{bmatrix} \oplus \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$\tilde{c}_{ii}=[0 \ \cdots \ 0 \ \oplus]$$

where $\oplus$'s represent the nonfixed entries, 0's represent the fixed (zero) entries, and $\times$'s represent either the nonfixed or the fixed entries. Notice that 0's in $\tilde{A}_{ii}$, $\tilde{b}_{ii}$, and $\tilde{c}_{ii}$ are due to the fact that $\boldsymbol{M}^*$ is the shortest matching, and $\tilde{A}_{00}$ is a $q \times q$ structured matrix. From (A.13) and (A.14) we can choose Q_1 and Q_2 in (A.4)–(A.6) as follows:

$$Q_1=\left[\begin{array}{c|c|c|c|c} I_q & \bigcirc & \cdots & \bigcirc & \bigcirc \\ \hline \bigcirc & I_{l_1} & \cdots & \bigcirc & \bigcirc \\ \hline 0\cdots 0 & 0 \cdots 0 & \cdots & 0 \cdots 0 & 1\, 0 \cdots 0 \\ \hline \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline \bigcirc & \bigcirc & \cdots & I_{l_l} & \bigcirc \\ \hline 0 \cdots 0 & 0 \cdots 0 & \cdots & 0 \cdots 0 & 0 \cdots 0\, 1 \end{array}\right],$$

$$Q_2=\left[\begin{array}{c|c|c|c|c|c} I_q & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \bigcirc & \cdots & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \bigcirc \\ \hline \bigcirc & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & I_{l_1} & \cdots & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \bigcirc \\ \hline \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline \bigcirc & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \bigcirc & \cdots & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & I_{l_l} \\ \hline \bigcirc & \begin{matrix}1\\ 0\\ \vdots\\ 0\end{matrix} & \bigcirc & \cdots & \begin{matrix}0\\ 0\\ \vdots\\ 1\end{matrix} & \bigcirc \end{array}\right].$$

Therefore, by introducing the following notation about X-nodes of $\boldsymbol{D}$ and X-nodes of $\boldsymbol{D}_\Sigma$:

$$x_j^{(i)} := \begin{cases} x_j & \text{for } i=0,\ 1 \le j \le q \\ x_{q+l_1+\cdots+l_{i-1}+j} & \text{for } 1 \le i \le l,\ 1 \le j \le l_i \end{cases}$$

$$\varphi_k^{(i)} := \begin{cases} \varphi_k & \text{for } i=0,\ 1 \le k \le q \\ \varphi_{q+l_1+\cdots+l_{i-1}+(i-1)+k} & \text{for } 1 \le i \le l,\ 1 \le k \le l_i+1 \end{cases}$$

we get one-to-one correspondence between $\tilde{E} := E \cup \{(x_i, x_i) \mid i = 1, \cdots, n\}$ and E_Σ as follows:

$$E \to E_\Sigma$$

$$(x_j^{(i)}, x_k^{(h)}) \mapsto \begin{cases} (\varphi_j^{(0)}, \varphi_k^{(h)}) & \text{for } i=0 \\ (\varphi_{j+1}^{(i)}, \varphi_k^{(h)}) & \text{for } 1 \le i \le l \end{cases}$$

$$(u_i, x_k^{(h)}) \mapsto \begin{cases} (\varphi_1^{(0)}, \varphi_k^{(h)}) & \text{for } 1 \le i \le l \\ (\omega_{i-l}, \varphi_k^{(h)}) & \text{for } l+1 \le i \le m \end{cases}$$

$$(x_j^{(i)}, y_k) \mapsto \begin{cases} (\varphi_j^{(0)}, \varphi_{l_k+1}^{(k)}) & \text{for } i=0 \\ (\varphi_{j+1}^{(i)}, \varphi_{l_k+1}^{(k)}) & \text{for } 1 \le i \le l. \end{cases}$$

From the above relation between $\boldsymbol{D}$ and $\boldsymbol{D}_\Sigma$, it is easy to verify that condition ii_2) of Lemma A.7 implies condition i_2) of Theorem 3.3.

Sufficiency: Omitted because we can use the proof of necessity part in the reverse order. Q.E.D.

Proof of Lemma 4.7:

Recall [14] that an interconnected system $\boldsymbol{S}$ is decentrally connectively stabilizable if $\boldsymbol{S}_1$ is stabilizable and detectable, $\boldsymbol{S}_2$ is stabilizable and

$$\mathrm{Im}\, A_{21} \subset \mathcal{V}^+_{b,\mathrm{Ker}\, A_{12}},\ \mathcal{S}^+_{b,\mathrm{Im}\, A_{21}} = \{0\} \ \text{ w.r.t. } \boldsymbol{S}_2$$

where $\mathcal{V}^+_{b,\mathrm{Ker}\, A_{12}}$ is the supremal asymptotically stable invariant subspace contained in Ker A_{12}. Therefore, it is trivial that the system $\boldsymbol{S}$ is decentrally connectively stabilizable if condition i) of Lemma 4.7 holds. Condition ii) is also derived from Willems and Ikeda [14]. Q.E.D.

Proof of Theorem 4.13: From Lemma 4.7 and Definition 4.10, it is easy to see that the structured interconnected system $\boldsymbol{S}$ is decentrally connectively stabilizable if one of the following two conditions holds.

i) S_1 is structurally complete and

$$\mathcal{R}^*_{b,\mathrm{Ker}\,A^r_{12}} \overset{*}{=} \mathcal{X}_2,\ \mathcal{N}^*_{b,\mathrm{Im}\,A^c_{21}} \overset{*}{=} \{0\} \text{ w.r.t. } S_2. \tag{A.15}$$

ii) S_2 is structurally complete and

$$\mathcal{R}^*_{b,\mathrm{Ker}\,A^r_{21}} \overset{*}{=} \mathcal{X}_1,\ \mathcal{N}^*_{b,\mathrm{Im}\,A^c_{12}} \overset{*}{=} \{0\} \text{ w.r.t. } S_1. \tag{A.16}$$

Notice that (A.15) and (A.16) are characterized in terms of associated system graphs in Theorem 3.3. Therefore, the assertion is trivial. Q.E.D.

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