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# Wave-based Analysis and Wave Control of Damped Mass-Spring Systems

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**Abstract**

This paper concerns with active vibration control of *non*-uniform damped mass-spring systems by the wave control. Especially, this study considers clarifying a class of a damped mass-spring system that can be analyzed by the wave-based analysis. Because the wave properties are determined by the propagation constants, three conditions for the propagation constants are considered. Necessary and sufficient condition for the physical parameters to hold the three conditions is obtained. Moreover, for this class of damped mass-spring systems, properties of the propagation constants and the characteristic impedances, which achieve the impedance matching, are studied. Numerical examples are shown to prove efficiency of the impedance matching controller.

## 1 Introduction

Recently active vibration control of flexible structures has been greatly developed [1]. Observing the concepts of vibration suppression of flexible structures, most techniques are based on the modal analysis [2, 3]. Control design based on the modal analysis is called the modal control. Modal control has shown efficiency for vibration control of flexible structures, however, if structures become more flexible, we need to control many modes, and must devote to hard control design. Moreover, in the modal control, control design usually aims to reduce vibration at the loop of the structure, however, especially for highly flexible structures, suppressing vibration at the loop sometimes results in poor performance at the other position.

The wave control that is similar concept to the impedance matching in the electric circuit theory is expected to be applicable to the above problems [4]. However, because the wave control is based on the wave-based analysis, most papers considered simple uniform structures such as uniform beams, wave guides, etc. [5, 6, 7, 8], and methodology of the wave control for general (non-uniform) structures is still not clear from theoretical point of view.

In this paper, aiming to establish the wave control of non-uniform structures, this study considers clarifying a class of a damped mass-spring system that can be analyzed by the wave-based analysis. Three conditions for the propagation constants are considered, and necessary and sufficient condition for the physical parameters of the system to hold the three conditions is obtained. For this class of damped mass-spring systems, properties of the characteristic impedances, which is an irrational function of the Laplace operator $s$, on the imaginary axis are also discussed to realize the impedance matching controller in finite dimension. Numerical examples are shown to prove efficiency of the obtained impedance matching controller.

## 2 Main Results

### 2.1 Damped Mass-Spring System to be Analyzed by the Wave-Based Analysis

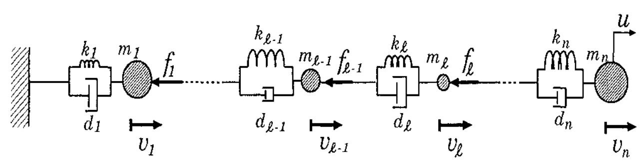


**Figure 1:** Non-uniform damped mass-spring system

Consider a non-uniform damped mass-spring system shown in Fig. 1. $\ell$ represents the position from the left end. $v_\ell(t)$[m/s] is the velocity of the corresponding mass $m_\ell$[Kg] and $f_\ell(t)$[N] is the reaction force from the right side of the mass. $d_\ell$[Ns/m] and $k_\ell$[N/m] are the viscous damping coefficient and the spring constant respectively. $u(t)$[N] is a control force acting on the right end. The Laplace transform of $v_\ell(t)$ and $f_\ell(t)$ are denoted by $v_\ell(s)$ and $f_\ell(s)$ respectively.

Defining

$$Z_\ell(s) := \frac{s}{d_\ell s + k_\ell}, \quad Y_\ell(s) := m_\ell s \tag{1}$$

then $v_\ell(s)$ and $f_\ell(s)$ are represented by a recurrent formula

$$X_\ell(s) = A_\ell(s) X_{\ell-1}(s), \tag{2}$$

where

$$X_\ell(s) := \begin{bmatrix} v_\ell(s) \\ f_\ell(s) \end{bmatrix},$$

$$A_\ell(s) := \begin{bmatrix} 1 & -Z_\ell(s) \\ -Y_\ell(s) & 1 + Z_\ell(s)Y_\ell(s) \end{bmatrix}.$$

Suppose there exists a transformation

$$X_\ell(s) = T_\ell(s)\widetilde{X}_\ell(s), \quad \widetilde{X}_\ell := \begin{bmatrix} f_\ell^-(s) \\ f_\ell^+(s) \end{bmatrix}, \qquad (3)$$

which transforms equation (2) into

$$\begin{aligned} \widetilde{X}_\ell(s) &= T_\ell(s)^{-1}A_\ell(s)T_{\ell-1}(s)\widetilde{X}_{\ell-1}(s) \\ &= \begin{bmatrix} \lambda_{1_\ell}(s) & 0 \\ 0 & \lambda_{2_\ell}(s) \end{bmatrix} \widetilde{X}_{\ell-1}(s). \end{aligned} \qquad (4)$$

If $m_\ell$, $d_\ell$, $k_\ell$ are independent of $\ell$ (uniform damped-mass spring), $\lambda_{1_\ell}(s)$ and $\lambda_{2_\ell}(s)$ are defined by the roots of

$$p(s) := \lambda^2 - (2 + Z(s)Y(s))\,\lambda + 1. \qquad (5)$$

From the relations between roots and coefficients for (5), following properties hold.

(a) $\lambda_{1_\ell}(s) \neq \lambda_{2_\ell}(s)$ ,

(b) $\lambda_{1_\ell}(s)\lambda_{2_\ell}(s) = a \in \mathcal{R}_+$,

(c) $\lambda_{1_\ell}(s)$ and $\lambda_{2_\ell}(s)$ are independent of $\ell$,

where $\mathcal{R}_+$ is the set of positive real number. Owing to these properties, $f_\ell^+(s)$ and $f_\ell^-(s)$ in (3) can be regarded as traveling waves. Physical meaning of these conditions are as follows. (a) requires that the wave properties of $f_\ell^+(s)$ and $f_\ell^-(s)$ are different. (b) requires that $f_\ell^+(s)$ and $f_\ell^-(s)$ travel in the opposite direction at the same speed. (c) requires that the wave properties of $f_\ell^+(s), f_\ell^-(s)$ are independent of position $\ell$, therefore they travel at the same speed at any position. In the electric circuit theory, $\lambda_{1_\ell}(s)$ and $\lambda_{2_\ell}(s)$ are called the propagation constants. See [8, 10] for more details of the wave-based analysis of the uniform damped-mass spring.

In the following, we characterize a class of the damped-mass spring system to be analyzed by the wave-based analysis from the properties of $\lambda_{1_\ell}(s)$ and $\lambda_{2_\ell}(s)$. Observing the analysis of the uniform case, it is probable that only the conditions (a) and (b) are dominant to be analyzed by the wave-based analysis, however, dependency on $\ell$ makes the problem much harder to solve. In this paper, as those in the uniform case, we also require independency of $\ell$ in addition. Considering condition (c), we use $\lambda_1(s)$, $\lambda_2(s)$ instead of $\lambda_{1_\ell}(s)$, $\lambda_{2_\ell}(s)$ hereafter. For the conditions (a), (b) and (c), the following theorem holds.

**Theorem 2. 1** *Consider damped mass-spring system shown in Fig. 1. There exists a transformation (3) which transforms (2) into (4) and the conditions (a), (b), (c) are satisfied iff physical parameters of the system satisfy*

$$\frac{m_\ell}{m_{\ell-1}} = \frac{d_\ell}{d_{\ell-1}} = \frac{k_\ell}{k_{\ell-1}} = a, \qquad \ell = 2, 3, \cdots, n. \qquad (6)$$

Proof:) See Appendix A.1.

### 2.2 Properties of the Propagation Constants and the Characteristic Impedances

In the proof of Theorem 2. 1 (Appendix A.1), important variables that characterize wave properties of $f_\ell^+(s)$ and $f_\ell^-(s)$ appeared: the propagation constants $\lambda_1(s)$, $\lambda_2(s)$ defined by (10) and the characteristic impedances $Z_\ell^-(s)$, $Z_\ell^+(s)$ defined by (9). In this subsection, we investigate properties of these variables on the imaginary axis $s = j\omega$ for the system which satisfies equation (6). Proofs are shown in Appendix A.2 and A.3.

**Property 2. 1** *Consider $\lambda_1(s)$ and $\lambda_2(s)$, the roots of (10). Suppose $\lambda_1(j\omega_0)$ locates in the upper half-plane of the complex plane, and $\lambda_2(j\omega_0)$ locates in the lower half-plane for some $\omega_0 \in \mathcal{R}_+$. Then, the following statements hold.*

*(i) $\lambda_1(j\omega), \lambda_2(j\omega)$ are continuous with respect to $\omega \in \mathcal{R}_+$. Moreover, for all $\omega \in R_+$, $\lambda_1(j\omega)$ locates in the upper half-plane, and $\lambda_2(j\omega)$ locates in the lower half-plane.*

*(ii) $0 < |\lambda_2(j\omega)| < \sqrt{a} < |\lambda_1(j\omega)| < \infty$ for $^\forall\omega \in R_+$.*

*(iii)* $\omega \to \infty$ : $\lim_{\omega\to\infty} \lambda_1(j\omega) = \left(a + 1 - \dfrac{m_n k_n}{{d_n}^2}\right) + j\infty$, $\lim_{\omega\to\infty} \lambda_2(j\omega) = 0$

$\omega \to 0_+$ :

$(a < 1)$ $\lim_{\omega\to 0_+} \lambda_1(j\omega) = 1$ , $\lim_{\omega\to 0_+} \lambda_2(j\omega) = a$

$(a = 1)$ $\lim_{\omega\to 0_+} \lambda_1(j\omega) = 1$ , $\lim_{\omega\to 0_+} \lambda_2(j\omega) = 1$

$(a > 1)$ $\lim_{\omega\to 0_+} \lambda_1(j\omega) = a$ , $\lim_{\omega\to 0_+} \lambda_2(j\omega) = 1$

Figure 2 shows typical loci of $\lambda_1(j\omega)$ (solid line) and $\lambda_2(j\omega)$ (dashed line). As indicated in Property 2. 1 , both starting from the real axis, $\lambda_1(j\omega)$ locates outside of the circle with radius $\sqrt{a}$ in the upper half-plane, while $\lambda_2(j\omega)$ locates inside of the circle in the lower half-plane. Using this property we can show the next property concerns about the characteristic impedances.

**Property 2. 2** *Consider $Z_\ell^-(s)$ and $Z_\ell^+(s)$ defined by (9). Following statements hold.*

*(i) $Z_\ell^-(j\omega), Z_\ell^+(j\omega)$ are continuous with respect to $\omega \in \mathcal{R}_+$.*

*(ii)* $\mathrm{Re}[Z_\ell^-(j\omega)]$, $\mathrm{Re}[Z_\ell^+(j\omega)] > 0$ *for* $^\forall\omega \in \mathcal{R}^+$.

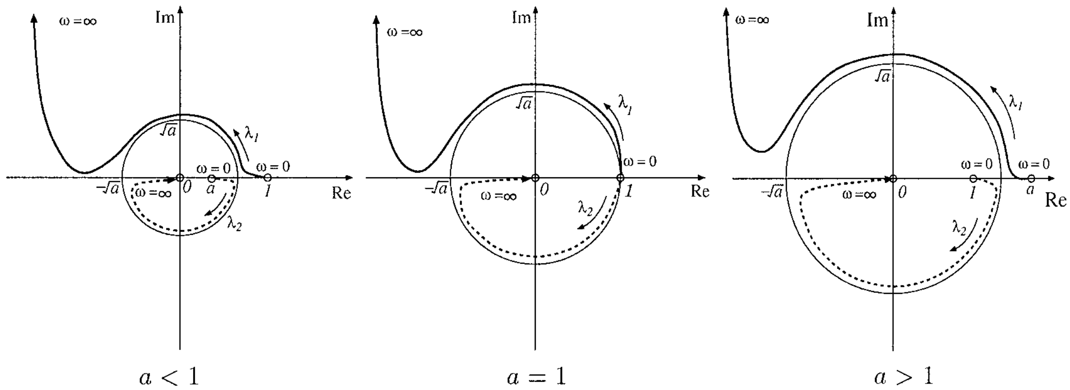


**Figure 2:** Loci of $\lambda_1(j\omega)$ (solid) and $\lambda_2(j\omega)$ (dash)

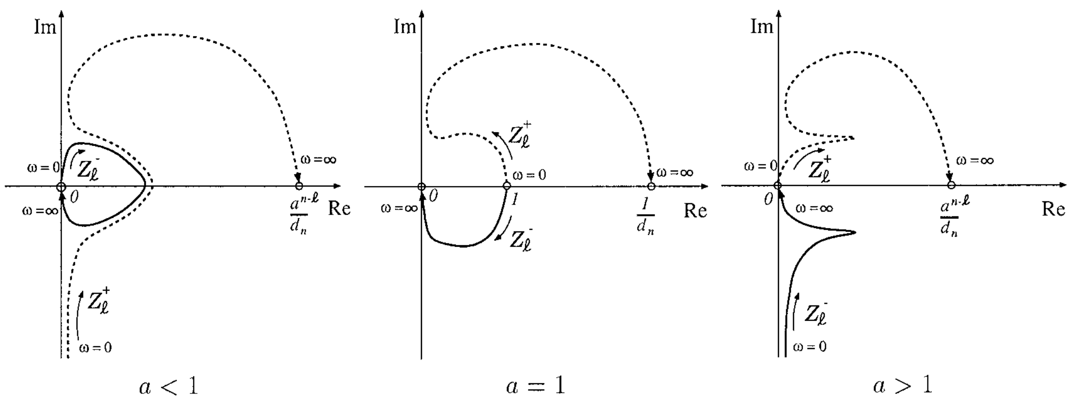


**Figure 3:** Loci of $Z_\ell^-(j\omega)$ (solid) and $Z_\ell^+(j\omega)$ (dash)

*(iii)* $\omega \to \infty : \lim_{\omega\to\infty} Z_\ell^+(j\omega) = \frac{a^{n-\ell}}{d_n}, \lim_{\omega\to\infty} Z_\ell^-(j\omega) = 0.$

$\omega \to 0_+$ :

$(a < 1) \quad \lim_{\omega\to 0_+} Z_\ell^+(j\omega) = -j\infty \ , \lim_{\omega\to 0_+} Z_\ell^-(j\omega) = 0,$

$(a = 1) \quad \lim_{\omega\to 0_+} Z_\ell^+(j\omega) = \lim_{\omega\to 0_+} Z_\ell^-(j\omega) = \sqrt{\frac{1}{m_n k_n}},$

$(a > 1) \quad \lim_{\omega\to 0_+} Z_\ell^+(j\omega) = 0 \ , \lim_{\omega\to 0_+} Z_\ell^-(j\omega) = -j\infty.$

Figure 3 shows typical loci of $Z_\ell^-(j\omega)$ (solid line) and $Z_\ell^+(j\omega)$ (dashed line). From the figure, we can confirm that $Z_\ell^-(s)$, $Z_\ell^+(s)$ has properties of the positive real function on the imaginary axis $s = j\omega$. Note, from Property 2. 2 (iii), that $Z_\ell^+(s)$ is a function with relative degree 0.

## 3 Numerical Example

In this section, efficiency of the impedance matching controller for the system satisfying (6) is discussed. In numerical example, we consider 10-mass ($n = 10$) system shown in Fig. 4 with parameters $m_\ell = m_n * a^{\ell-10}$[Kg], $d_\ell = d_n * a^{\ell-10}$[Ns/m], $k_\ell = k_n * a^{\ell-10}$[N/m] and $m_n = 1$[Kg], $d_n = 1/300$[Ns/m], $k_n = 1$[N/m] and $a = 2/3$. In this case, $m_1$ is $2/3^{-9} \simeq 38.4$ times bigger

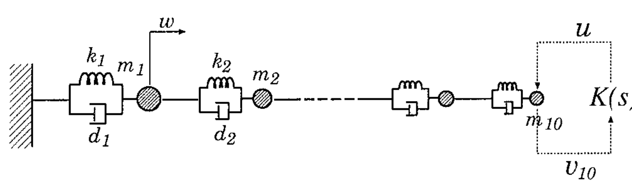


**Figure 4:** Damped mass-spring for numerical example

than $m_{10}$. $u$ is a control force acting on the right end and given by $u(s) = K(s)v_n(s)$. $w$ is a disturbance at position 1.

First, we derive the impedance matching controller for this system. Similar to the uniform case [8, 10], suppose $\lambda_1(s)$ is the root whose imaginary part is positive, then $f_\ell^+(s)$ and $f_\ell^-(s)$ in (3) are traveling waves towards the positive direction and negative direction respectively. Defining $v_\ell^+(s) := Z_\ell^+(s)f_\ell^+(s)$ and $v_\ell^-(s) := Z_\ell^-(s)f_\ell^-(s)$, $v_\ell^+(s)$ and $v_\ell^-(s)$ are also traveling waves. The velocity $v_\ell(s)$ is represented by the sum of these waves. The reflection coefficient of the velocity, the ratio of the reflected wave to the incident wave, is given by

$$\rho_n^v(s) := \frac{v_n^-(s)}{v_n^+(s)} = \frac{Z_n^-(s)}{Z_n^+(s)} \times \frac{a + Z_n^+(s)K(s)}{a - Z_n^-(s)K(s)}. \qquad (7)$$

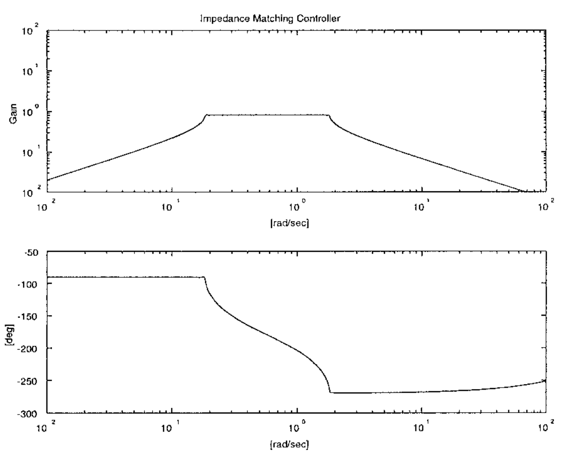


**Figure 5:** Bode plots of $\widehat{K}_n^+(s)$ (solid) and $K_n^+(s)$ (dash)

From (7), the impedance matching controller which renders $\rho_n^v(s) = 0$ is given by

$$K(s) = K_\ell^+(s) := -\frac{a}{Z_n^+(s)}. \tag{8}$$

Because $Z_n^+(s)$ defined by (9) includes $\lambda_2(s)$, $K_\ell^+(s)$ is an irrational function of $s$, therefore, we can not use the impedance matching controller directly in finite dimension. In fact,

$$\begin{aligned} K_n^+(s) &= \frac{d_n s + k_n}{s}\left(\frac{1-a}{2} + \frac{1}{2}\frac{m_n s^2}{d_n s + k_n}\right. \\ &\quad \left. - \sqrt{\left(\frac{1+a}{2} + \frac{1}{2}\frac{m_n s^2}{d_n s + k_n}\right)^2 - a}\right). \end{aligned}$$

Fortunately, from Property 2. 2 , $-K_n^+(s)$ is well approximated by a proper rational positive real function through complex-curve fitting [9] as those in [8]. Closed loop stability is guaranteed by the positive real property of the controller. Figure 5 shows the Bode plots of $K_n^+(s)$ (dashed line) and an approximation $\widehat{K}_n^+(s)$ (solid line) with degree 14. $\widehat{K}_n^+(s)$ and $K_n^+(s)$ coincide in this figure.

Figure 6 shows impulse response of the system disturbed by $w$. Solid lines represent the response with control and dashed lines represent the response without control. $v_1$, $v_5$, $v_{10}$ are the velocity of the corresponding masses and $u$ is the control force. From the response with control in Fig.6, reflection is suppressed at position 10 where controller exists, and vibration vanishes rapidly compare to the response without control.

## 4 Conclusion

In this paper, aiming to establish the wave control of non-uniform structures, this study considered clarifying a class of a damped mass-spring system that can

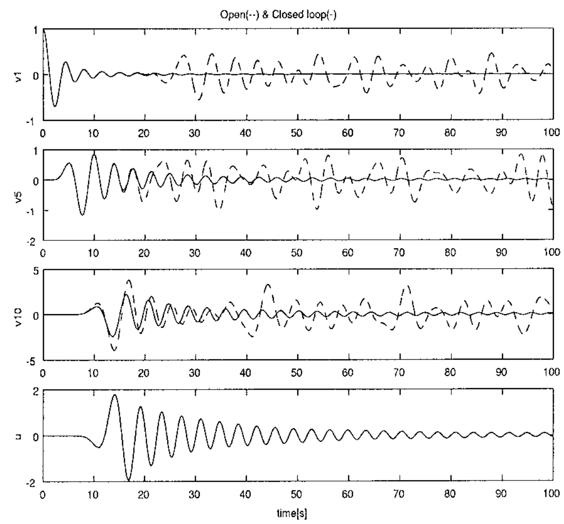


**Figure 6:** Impulse response with control (solid) and without control (dash)

be analyzed by the wave-based analysis. Three conditions for the propagation constants were considered, and necessary and sufficient condition for the physical parameters to hold the three conditions was obtained. Properties of the characteristic impedances were also discussed, and numerical examples were shown to prove efficiency of the impedance matching controller.

To loosen the conditions and formulation of the wave control as an impedance matching problem is our future work.

## A Appendix

### A.1: Proof of Theorem 2. 1

In the proof, $s$ is omitted for notational simplicity.
**Sufficiency)** Suppose (6) holds, then, from the definitions,

$$A_\ell = \begin{bmatrix} 1 & -\frac{1}{a^{\ell-n}}Z_n \\ -a^{\ell-n}Y_n & 1 + Z_n Y_n \end{bmatrix},\ Y_\ell = a^{\ell-n}Y_n,\ Z_\ell = \frac{1}{a^{\ell-n}}Z_n.$$

Define

$$T_{\ell-1} := \begin{bmatrix} Z_\ell^- & Z_\ell^+ \\ -1 & 1 \end{bmatrix},$$

$$\begin{aligned} Z_\ell^- &:= \frac{aZ_\ell}{\lambda_1 - a} = \frac{a \cdot a^{n-\ell}Z_n}{\lambda_1 - a}, \\ Z_\ell^+ &:= -\frac{aZ_\ell}{\lambda_2 - a} = -\frac{a \cdot a^{n-\ell}Z_n}{\lambda_2 - a}, \end{aligned} \tag{9}$$

where $\lambda_1$ and $\lambda_2$ are the roots of

$$p_\ell := \lambda^2 - (a + 1 + Z_n Y_n)\lambda + a, \tag{10}$$

which is independent of $\ell$. From (9),

$$\begin{aligned} T_{\ell-1}T_\ell^{-1} &= \begin{bmatrix} Z_\ell^- & Z_\ell^+ \\ -1 & 1 \end{bmatrix}\left\{\begin{bmatrix} 1/a & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} Z_\ell^- & Z_\ell^+ \\ -1 & 1 \end{bmatrix}\right\}^{-1} \\ &= Q_\ell, \quad Q_\ell := \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \end{aligned}$$

therefore, from (4),

$$\widetilde{X}_\ell = T_\ell{}^{-1} A_\ell T_{\ell-1} \widetilde{X}_{\ell-1} = T_{\ell-1}{}^{-1} Q_\ell A_\ell T_{\ell-1} \widetilde{X}_{\ell-1}. \quad (11)$$

Noting that (10) is the characteristic polynomial of

$$Q_\ell A_\ell = \begin{bmatrix} a & -\frac{a}{a^{\ell-n}} Z_n \\ -a^{\ell-n} Y_n & 1 + Z_n Y_n \end{bmatrix},$$

and $[Z_\ell^- \ -1]^T$, $[Z_\ell^+ \ 1]^T$ are the eigenvectors of $Q_\ell A_\ell$ corresponding to $\lambda_1$, $\lambda_2$ respectively, we can confirm that (11) is transformed into

$$\widetilde{X}_\ell = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \widetilde{X}_{\ell-1}. \quad (12)$$

Since $\lambda_1$ and $\lambda_2$ are the roots of (10), from the relations between roots and coefficients, (a), (b), (c) are satisfied. This proves the sufficiency.

**Necessity)** Suppose $T_\ell$ in (3) exists, then we can define

$$Q_\ell = \begin{bmatrix} \alpha_\ell & \beta_\ell \\ \gamma_\ell & \delta_\ell \end{bmatrix} := T_{\ell-1} T_\ell{}^{-1},$$

and equation (4) becomes

$$\widetilde{X}_\ell = T_{\ell-1}{}^{-1} Q_\ell A_\ell T_{\ell-1} \widetilde{X}_{\ell-1}.$$

Note that $\lambda_1$, $\lambda_2$ are eigenvalues of

$$Q_\ell A_\ell = \begin{bmatrix} \alpha_\ell - Y_\ell \beta_\ell & -Z_\ell \alpha_\ell + (1 + Z_\ell Y_\ell)\beta_\ell \\ \gamma_\ell - Y_\ell \delta_\ell & -Z_\ell \gamma_\ell + (1 + Z_\ell Y_\ell)\delta_\ell \end{bmatrix},$$

that is, the roots of

$$p_\ell = \lambda^2 - \Big\{ \alpha_\ell - Y_\ell \beta_\ell - Z_\ell \gamma_\ell + (1 + Z_\ell Y_\ell)\delta_\ell \Big\} \lambda + \alpha_\ell \delta_\ell - \beta_\ell \gamma_\ell. \quad (13)$$

In the following, we try to find a condition for $m_\ell$, $d_\ell$, $k_\ell$ such that the eigenvalues of $Q_\ell A_\ell$ satisfy (a), (b), (c).

From the condition (a), the transformation matrix $T_{\ell-1}$ which diagonalize $Q_\ell A_\ell$ is described by using different eigenvalues $\lambda_1$, $\lambda_2$ as

$$T_{\ell-1} := \begin{bmatrix} Z_\ell^- & Z_\ell^+ \\ -1 & 1 \end{bmatrix},$$

where

$$\begin{aligned} Z_\ell^- &:= -\frac{\lambda_1 - \{-Z_\ell \gamma_\ell + (1 + Z_\ell Y_\ell)\delta_\ell\}}{\gamma_\ell - Y_\ell \delta_\ell}, \\ Z_\ell^+ &:= \frac{\lambda_2 - \{-Z_\ell \gamma_\ell + (1 + Z_\ell Y_\ell)\delta_\ell\}}{\gamma_\ell - Y_\ell \delta_\ell}. \end{aligned} \quad (14)$$

Using this $T_{\ell-1}$, and from the definition $Q_\ell := T_{\ell-1} T_\ell{}^{-1}$, we obtain

$$\begin{bmatrix} \alpha_\ell & \beta_\ell \\ \gamma_\ell & \delta_\ell \end{bmatrix} = \begin{bmatrix} \dfrac{Z_\ell^- + Z_\ell^+}{Z_{\ell+1}^- + Z_{\ell+1}^+} & \dfrac{-Z_\ell^- Z_{\ell+1}^+ + Z_\ell^+ Z_{\ell+1}^-}{Z_{\ell+1}^- + Z_{\ell+1}^+} \\ 0 & 1 \end{bmatrix}. \quad (15)$$

Therefore,

$$\gamma_\ell = 0, \quad \delta_\ell = 1. \quad (16)$$

Using (16), (14) is simplified as

$$Z_\ell^- := \frac{\lambda_1 - (1 + Z_\ell Y_\ell)}{Y_\ell}, \quad Z_\ell^+ := -\frac{\lambda_2 - (1 + Z_\ell Y_\ell)}{Y_\ell}. \quad (17)$$

Noting that

$$\begin{aligned} Z_\ell^- + Z_\ell^+ &= \frac{\lambda_1 - \lambda_2}{Y_\ell}, \\ -Z_\ell^- Z_{\ell+1}^+ + Z_{\ell+1}^- Z_\ell^+ &= \frac{1}{Y_\ell Y_{\ell+1}} (\lambda_1 - \lambda_2)(Z_\ell Y_\ell - Z_{\ell+1} Y_{\ell+1}), \end{aligned}$$

(1, 1) and (1, 2) elements in (15) become

$$\alpha_\ell = \frac{Y_{\ell+1}}{Y_\ell}, \quad (18)$$

$$\beta_\ell = \frac{Z_\ell Y_\ell - Z_{\ell+1} Y_{\ell+1}}{Y_\ell} = Z_\ell - \alpha_\ell Z_{\ell+1}. \quad (19)$$

Moreover, from (16), since the characteristic polynomial (13) is simplified as

$$p_\ell = \lambda^2 - (\alpha_\ell - Y_\ell \beta_\ell + 1 + Z_\ell Y_\ell)\lambda + \alpha_\ell, \quad (20)$$

therefore, from the relations between roots and coefficients of (20) and the condition (b), we get

$$\alpha_\ell = a. \quad (21)$$

Combining (18), (21) and $Y_\ell := m_\ell s$, we get

$$\frac{m_\ell}{m_{\ell-1}} = a. \quad (22)$$

Finally, since the condition (c) requires $p_{\ell+1} = p_\ell$, from (20) with $\alpha_\ell = a$ and $\beta_\ell = Z_\ell - a Z_{\ell+1}$,

$$\lambda^2 - (a + 1 + a Z_{\ell+1} Y_\ell)\lambda + a = \lambda^2 - (a + 1 + a Z_{\ell+2} Y_{\ell+1})\lambda + a,$$

namely,

$$\frac{Y_{\ell+1}}{Y_\ell} = \frac{Z_{\ell+1}}{Z_{\ell+2}} = a.$$

Because $Z_\ell := \dfrac{s}{d_\ell s + k_\ell}$, the above equation means,

$$\frac{d_\ell}{d_{\ell-1}} = \frac{k_\ell}{k_{\ell-1}} = a. \quad (23)$$

(22) and (23) prove the necessity. ■

## A.2: Proof of Property 2. 1

(i) From the definition (1), $Z_n(j\omega)$ and $Y_n(j\omega)$ are continuous with respect to $\omega \in \mathcal{R}_+$. Therefore, according to continuity of the roots with respect to the coefficients, $\lambda_1(j\omega)$, $\lambda_2(j\omega)$ are continuous with respect to $\omega \in \mathcal{R}_+$. Moreover, from the relations between roots and coefficients for (10), we get

$$\begin{aligned} \lambda_1(j\omega)\lambda_2(j\omega) &= a, \quad (24) \\ \lambda_1(j\omega) + \lambda_2(j\omega) &= a + 1 + Z_n(j\omega) Y_n(j\omega) \\ &= a + 1 - \frac{\omega^2 m_n k_n}{\omega^2 d_n^2 + k_n^2} + j \frac{\omega^3 m_n d_n}{\omega^2 d_n^2 + k_n^2}. \quad (25) \end{aligned}$$

From (24), if $\lambda_1(j\omega)$ moves from the upper half-plane to the lower half-plane, there must exist $\omega_c \in \mathcal{R}_+$ such that $\lambda_1(j\omega_c), \lambda_2(j\omega_c) \in \mathcal{R}_+$. However, from (25), since

$$\mathrm{Im}\{\lambda_1(j\omega) + \lambda_2(j\omega)\} = \frac{\omega^3 m_n d_n}{\omega^2 d_n^2 + k_n^2} > 0, \quad {}^\forall \omega \in \mathcal{R}_+, \quad (26)$$

such $\omega_c$ does not exist. This proves (i).

**(ii)** From (24), $\arg[\lambda_1(j\omega)] = -\arg[\lambda_2(j\omega)]$. Combining this fact with (26), modulus of $\lambda_1(j\omega)$ should be greater than $\lambda_2(j\omega)$. Moreover, since $\lambda_1(j\omega)\lambda_2(j\omega) = a$,

$$|\lambda_2(j\omega)| < \sqrt{a} < |\lambda_1(j\omega)|. \tag{27}$$

$0 < |\lambda_2(j\omega)| < \infty$, $0 < |\lambda_1(j\omega)| < \infty$ are obvious from (24) and the continuity.

**(iii)** $\boxed{\omega \to \infty}$ : From (25),

$$\lim_{\omega\to\infty}\{\lambda_1(j\omega)+\lambda_2(j\omega)\} = \left(a+1-\frac{m_n k_n}{d_n^2}\right) + j\infty.$$

Since $\lambda_1(j\omega)$ locates in the upper half plane and $\lambda_2(j\omega)$ locates in the lower half plane satisfying (24),

$$\lim_{\omega\to\infty}\lambda_1(j\omega) = \left(a+1-\frac{m_n k_n}{{d_n}^2}\right)+j\infty, \quad \lim_{\omega\to\infty}\lambda_2(j\omega) = 0. \tag{28}$$

$\boxed{\omega \to 0_+}$ : From (25), $\lim_{\omega\to 0_+}\{\lambda_1(j\omega)+\lambda_2(j\omega)\} = a+1$. Since $\lambda_1(j\omega)\lambda_2(j\omega) = a$, the roots of (10) are $a$ and 1. Because $|\lambda_2(j\omega)| < |\lambda_1(j\omega)|$, we get the property. ■

## A.3: Proof of Property 2. 2

**(i)** From (9), continuity of $Z_n(j\omega)$ and $\lambda_1(j\omega)$, $\lambda_2(j\omega) \notin \mathcal{R}_+$, continuity of $Z_\ell^-(j\omega)$, $Z_\ell^+(j\omega)$ are obvious.

**(ii)** Since $\lambda_1(j\omega)$ and $\lambda_2(j\omega)$ are the roots of (10), $\lambda_i(j\omega)^2 - \left\{a+1+Z_n(j\omega)Y_n(j\omega)\right\}\lambda_i(j\omega)+a = 0$ $(i = 1,\ 2)$, therefore,

$$Z_n(j\omega) = \frac{(\lambda_i(j\omega)-1)(\lambda_i(j\omega)-a)}{Y_n(j\omega)\lambda_i(j\omega)}, \quad i = 1,\ 2. \tag{29}$$

From (9), by using (24) and (29) with $i = 2$, we get

$$\begin{aligned} Z_\ell^+(j\omega) &= -\frac{\lambda_1(j\omega)\lambda_2(j\omega)\cdot a^{n-\ell}}{\lambda_2(j\omega)-a}\frac{(\lambda_2(j\omega)-1)(\lambda_2(j\omega)-a)}{Y_n(j\omega)\lambda_2(j\omega)} \\ &= \frac{a^{n-\ell}}{Y_n(j\omega)}(\lambda_1(j\omega)-a). \end{aligned} \tag{30}$$

Because $0 < \arg[\lambda_1(j\omega)-a] < \pi$ (Property 2. 1 (i)) and $\arg[1/Y_n(j\omega)] = \arg[1/j\omega m_n] = -\pi/2$, we get

$$-\frac{\pi}{2} < \arg\left[\frac{a^{n-l}}{Y_n(j\omega)}(\lambda_1(j\omega)-a)\right] < \frac{\pi}{2},$$

namely, $\mathrm{Re}[Z_\ell^+(j\omega)] > 0$. We can also prove $\mathrm{Re}[Z_\ell^-(j\omega)] > 0$ in similar way.

**(iii)** $\boxed{\omega \to \infty}$ : From (9) and Property 2. 1 (iii),

$$\lim_{\omega\to\infty} Z_\ell^+(j\omega) = \lim_{\omega\to\infty}\frac{-a\cdot a^{n-\ell}}{\lambda_2(j\omega)-a}\frac{j\omega}{j\omega d_n+k_n} = \frac{a^{n-\ell}}{d_n}. \tag{31}$$

From (9), by using (24) and (29) with $i = 1$,

$$\begin{aligned} \lim_{\omega\to\infty} Z_\ell^-(j\omega) &= \lim_{\omega\to\infty}\frac{\lambda_1(j\omega)\lambda_2(j\omega)\cdot a^{n-\ell}}{\lambda_1(j\omega)-a}\frac{(\lambda_1(j\omega)-1)(\lambda_1(j\omega)-a)}{Y_n(j\omega)\lambda_1(j\omega)} \\ &= \lim_{\omega\to\infty}\frac{a^{n-\ell}}{j\omega m_n}(a-\lambda_2(j\omega)) = 0. \end{aligned} \tag{32}$$

$\boxed{\omega \to 0_+}$ : From (9), (24) and (29),

$$Z_\ell^+(j\omega) = \frac{-a\cdot a^{n-\ell}}{\lambda_2(j\omega)-a}\frac{j\omega}{j\omega d_n+k_n} = \frac{a^{n-\ell}}{j\omega m_n}(\lambda_1(j\omega)-a) \tag{33}$$

$$Z_\ell^-(j\omega) = \frac{a\cdot a^{n-l}}{\lambda_1(j\omega)-a}\frac{j\omega}{j\omega d_n+k_n} = \frac{a^{n-\ell}}{j\omega m_n}(a-\lambda_2(j\omega)). \tag{34}$$

In the case of $a < 1$ and $a > 1$, we can confirm the property by using the above equations with $\omega \to 0_+$.

In the case of $a = 1$, noting from (33) that $\left\{Z_\ell^+(j\omega)\right\}^2 = \frac{1}{\lambda_2(j\omega)}\frac{1}{j\omega m_n}\frac{j\omega}{j\omega d_n+k_n}$, we get $\lim_{\omega\to 0_+} Z_\ell^+(j\omega) = \sqrt{\frac{1}{m_n k_n}}$ owing to $\mathrm{Re}[Z_\ell^+(j\omega)] > 0$. Moreover, from (9) and (24),

$$\frac{Z_\ell^-(j\omega)}{Z_\ell^+(j\omega)} = -\frac{\lambda_2(j\omega)-1}{\lambda_1(j\omega)-1} = \lambda_2(j\omega)\frac{\lambda_1(j\omega)-1}{\lambda_1(j\omega)-1} = \lambda_2(j\omega).$$

Since $\lim_{\omega\to 0_+}\lambda_2(j\omega) = a$, we get $\lim_{\omega\to 0_+} Z_\ell^-(j\omega) = \sqrt{\frac{1}{m_n k_n}}$. ■

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