A Study on the Impedance Matching Controller for Uniformly Varying Damped Mass-Spring Systems

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Abstract

This paper considers the analysis of the impedance matching controller for the *uniformly varying* damped mass-spring systems. The positive real property of the impedance matching controller is first investigated. The closed loop properties of the damped mass-spring systems controlled by the impedance matching controller is also discussed.

1 Introduction

Active vibration control of flexible structures has been greatly developed [1]. The wave control [2, 3, 4, 5] is one of the major concepts of the vibration control of flexible structures, and it is expected to be applicable to highly flexible or large scale structures [2]. However, most techniques of the wave control are proposed for simple uniform or repeated structures, and methodology of the wave control for non-uniform structures is still not clear from theoretical point of view. In [5], aiming to establish the wave control of non-uniform structures, we clarified a class of the damped mass-spring systems that can be analyzed by the wave-based analysis, and proposed the impedance matching controller for the system whose masses, spring constants and damping coefficients change with equal ratio with respect to the position (The system is called the *uniformly varying* damped mass-spring system). However, since the impedance matching controller is irrational function of the Laplace operator, properties of the impedance matching controller have not been investigated enough.

In this paper, we investigate the properties of the impedance matching controller and the resultant closed loop system by invoking the properties of the algebraic function [6]. We first investigate the analyticity and the positive real property of the impedance matching controller. The *ladder* structure of the feedback interconnection for the uniformly varying damped mass-spring system and the internal stability of the *non-uniform* damped mass-spring system controlled by the impedance matching controller are also discussed.

In Sec. 2, derivation of the impedance matching controller for the uniformly varying damped mass-spring systems is briefly summarized. In Sec. 3, the positive real property of the impedance matching controller is discussed. Section 4 gives the closed loop properties.

$\mathcal{R}_+$ and $\mathcal{C}_+$ denote the set of positive real number and the set of complex number with positive real part, respectively. $\mathcal{H}_\infty$ and $\mathcal{RH}_\infty$ denote the set of proper, stable transfer functions and the set of rational, proper, stable transfer functions, respectively.

2 Impedance Matching Controller for Uniformly Varying Damped Mass-Spring Systems

In this section, we briefly summarize the derivation of the impedance matching controller for the uniformly varying damped mass-spring systems. See Ref. [5] for more details.

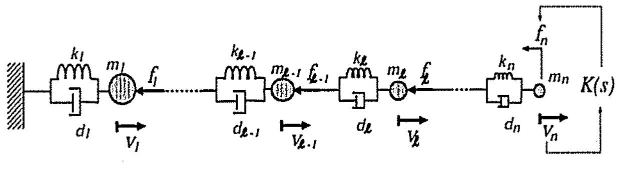

Figure 1: Uniformly Varying Damped Mass-Spring System

Figure 1 shows n cascade connected damped mass-spring systems fixed at the left end. ℓ $(= 1, \cdots, n)$ represents the position from the left end. $v_\ell(t)$[m/s] is the velocity of the mass m_ℓ[Kg] and $f_\ell(t)$[N] is the reaction force from the right side of the mass. We assume that the masses m_ℓ[Kg]> 0, viscous damping coefficients d_ℓ[Ns/m]> 0 and spring constants k_ℓ[N/m]> 0 change with equal ratio with respect to ℓ, namely they satisfy

$$\frac{m_{\ell+1}}{m_\ell} = \frac{d_{\ell+1}}{d_\ell} = \frac{k_{\ell+1}}{k_\ell} = a, \quad a \in \mathcal{R}_+. \tag{1}$$

The system satisfying (1) is called the uniformly varying damped mass-spring system.

Let $v_\ell(s)$ and $f_\ell(s)$ be the Laplace transform of $v_\ell(t)$ and $f_\ell(t)$ respectively, then $v_\ell(s)$ and $f_\ell(s)$ are represented by a recurrent formula

$$\begin{bmatrix} v_\ell(s) \\ f_\ell(s) \end{bmatrix} = A_\ell(s) \begin{bmatrix} v_{\ell-1}(s) \\ f_{\ell-1}(s) \end{bmatrix}, \qquad (2)$$

where

$$A_\ell(s) := \begin{bmatrix} 1 & -Z_\ell(s) \\ -Y_\ell(s) & 1+Z_\ell(s)Y_\ell(s) \end{bmatrix}, \qquad (3)$$

$$Z_\ell(s) := \frac{s}{d_\ell s + k_\ell} = \frac{1}{a} Z_{\ell-1}(s), \quad Y_\ell(s) := m_\ell s. \quad (4)$$

By a change of variables

$$\begin{bmatrix} v_\ell(s) \\ f_\ell(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{a}Z_\ell^-(s) & \frac{1}{a}Z_\ell^+(s) \\ -1 & 1 \end{bmatrix} \begin{bmatrix} f_\ell^-(s) \\ f_\ell^+(s) \end{bmatrix}, \qquad (5)$$

where

$$Z_\ell^-(s) := \frac{aZ_\ell(s)}{\lambda_1(s) - a}, \quad Z_\ell^+(s) := -\frac{aZ_\ell(s)}{\lambda_2(s) - a}, \qquad (6)$$

(3) is transformed to

$$\begin{bmatrix} f_\ell^-(s) \\ f_\ell^+(s) \end{bmatrix} = \begin{bmatrix} \lambda_1(s) & 0 \\ 0 & \lambda_2(s) \end{bmatrix} \begin{bmatrix} f_{\ell-1}^-(s) \\ f_{\ell-1}^+(s) \end{bmatrix}. \qquad (7)$$

$\lambda_1(s)$ and $\lambda_2(s)$ are the roots of a polynomial

$$p(s) := \lambda^2 - (a + 1 + Z_n(s)Y_n(s))\lambda + a. \qquad (8)$$

Note that $\lambda_1(s)$ and $\lambda_2(s)$ are the branches of the algebraic functions [6].

Let $\lambda_2(s)$ be the root satisfying $-\pi < \arg[\lambda_2(j\omega_0)] < 0$ at some $\omega_0 \in \mathcal{R}_+$, then $v_\ell^+(s) := \frac{1}{a}Z_\ell^+(s)f_\ell^+(s)$ is a traveling wave towards the positive direction of ℓ, and $v_\ell^-(s) := \frac{1}{a}Z_\ell^-(s)f_\ell^-(s)$ is a traveling wave towards the negative direction of ℓ. In addition, suppose the control force is given by $f_n(s) = K(s)v_n(s)$, then, from (5), the reflection coefficient of the velocity at n is given by

$$\rho_n^v(s) := \frac{v_n^-(s)}{v_n^+(s)} = \frac{Z_n^-(s)}{Z_n^+(s)} \times \frac{a - Z_n^+(s)K(s)}{a + Z_n^-(s)K(s)}. \qquad (9)$$

From (9), the impedance matching controller which renders $\rho_n^v(s) = 0$ is given by

$$f_n(s) = K_n^+(s)v_n(s), \qquad (10)$$

where

$$K_\ell^+(s) := \frac{a}{Z_\ell^+(s)} = \frac{1}{Z_{\ell+1}^+(s)}. \qquad (11)$$

$K_n^+(s)$ is a function of m_n, d_n, k_n and a.

3 Positive Real Property of the Impedance Matching Controller

In this section, we investigate the positive real property of the impedance matching controller.

Analyticity of $Z_\ell^-(s)$ and $Z_\ell^+(s)$ is first investigated.

Lemma 3. 1 *Let $\lambda_1(s)$ and $\lambda_2(s)$ be the roots of (8). $Z_\ell^-(s)$ and $Z_\ell^+(s)$ defined by (6) are analytic in $\mathcal{C}_+$.*

Proof:
Solving (6) for $\lambda_1(s)$ and $\lambda_2(s)$, and substituting them to (8), we get

$$Y_\ell(s)Z_\ell^+(s)^2 - (-a + 1 + Z_\ell(s)Y_\ell(s))Z_\ell^+(s) - aZ_\ell(s) = 0, \quad (12)$$

$$Y_\ell(s)(-Z_\ell^-(s))^2 - (-a + 1 + Z_\ell(s)Y_\ell(s))(-Z_\ell^-(s)) - aZ_\ell(s) = 0. \quad (13)$$

Observing the same structure of (12) and (13) with respect to $Z_\ell^+(s)$ and $-Z_\ell^-(s)$, we can confirm that $Z_\ell^+(s)$ and $-Z_\ell^-(s)$ are branches of an algebraic function $\mathbf{Z}(s)$ defined by

$$P(\mathbf{Z}, s) := (m_\ell d_\ell s^2 + m_\ell k_\ell s)\mathbf{Z}^2 - \{m_\ell s^2 + (-a+1)d_\ell s + (-a+1)k_\ell\}\mathbf{Z} - as = 0. \quad (14)$$

The algebraic functions have at most the algebraic singularities at the zeros of the first coefficient, the zeros of the discriminant and infinity [6]. Therefore, for the proof of the lemma, we only need to show that the zeros of the first coefficient and the discriminant are not in $\mathcal{C}_+$.

First, we immediately know the zeros of the first coefficient of (14) are $s = 0, -\frac{k_\ell}{d_\ell} \notin \mathcal{C}_+$. In the following, we examine the zeros of the discriminant of (14). Let $P_{\mathbf{Z}}(\mathbf{Z}, s)$ be the partial derivative of $P(\mathbf{Z}, s)$ with respect to $\mathbf{Z}$, namely,

$$P_{\mathbf{Z}}(\mathbf{Z}, s) := 2(m_\ell d_\ell s^2 + m_\ell k_\ell s)\mathbf{Z} - \{m_\ell s^2 + (-a+1)d_\ell s + (-a+1)k_\ell\}. \quad (15)$$

The discriminant $R(s)$ is the resultant of $P(\mathbf{Z}, s)$ and $P_{\mathbf{Z}}(\mathbf{Z}, s)$ [6]. From (14) and (15), $R(s)$ is given by

$$R(s) := m_\ell^2 s^4 + 2(a+1)m_\ell d_\ell s^3 + \{2(a+1)m_\ell k_\ell + (-a+1)^2 d_\ell^2\}s^2 + 2(-a+1)^2 d_\ell k_\ell s + (-a+1)^2 k_\ell^2 = 0. \quad (16)$$

From the Routh's stability criterion, we can show that (16) has no root in the closed right half-plane (See Appendix A). Therefore, $Z_\ell^-(s)$ and $Z_\ell^+(s)$ are analytic in $\mathcal{C}_+$. ■

In addition, $K_\ell^+(s)$ has the following property on the imaginary axis $s = j\omega$ [7].

Lemma 3. 2 *Suppose $\lambda_2(s)$ is the root of (8) satisfying $-\pi < \arg[\lambda_2(j\omega_0)] < 0$ at some $\omega_0 \in \mathcal{R}_+$. Then, $K_\ell^+(s)$ defined by (11) satisfies:*

(i) $K_\ell^+(j\omega)$ is continuous with respect to $\omega \in \mathcal{R}_+$.

(ii) $\mathrm{Re}[K_\ell^+(j\omega)] > 0$ for $^\forall\omega \in \mathcal{R}_+$.

(iii) $\lim_{\omega\to\infty} K_\ell^+(j\omega) = ad_\ell$

$$\lim_{\omega\to 0_+} K_\ell^+(j\omega) = \begin{cases} \lim_{\omega\to 0_+} \dfrac{a\cdot j\omega m_\ell}{1-a} = 0 & (a<1) \\ \sqrt{m_\ell k_\ell} & (a=1) \\ \lim_{\omega\to 0_+} \dfrac{k_\ell(a-1)}{j\omega} = -j\infty & (a>1). \end{cases}$$

Combining Lemma 3. 1 and Lemma 3. 2 , we can show the following Theorem.

Theorem 3. 1 *Let $\lambda_1(s)$ and $\lambda_2(s)$ be the roots of (8), and suppose $\lambda_2(s)$ is the root satisfying $-\pi < \arg[\lambda_2(j\omega_0)] < 0$ at some $\omega_0 \in \mathcal{R}_+$. Then, $K_\ell^+(s)$ defined by (11) has the following properties:*

(i) $K_\ell^+(s)$ is positive real.

(ii) $a \le 1$: $K_\ell^+(s) \in \mathcal{H}_\infty$.
$a > 1$: $K_\ell^+(s)$ is analytic in $\mathcal{C}_+$, and $|K_\ell^+(j\omega)| < \infty$, $^\forall\omega \in \mathcal{R}_+ \cup \{\infty\}$.

Proof:
(i): From the definition of the positive real function [8], we need to show:
(a) $K_\ell^+(s)$ is analytic in $s \in \mathcal{C}_+$;
(b) $K_\ell^+(s)$ is real for $s \in \mathcal{R}_+$;
(c) $K_\ell^+(s)^* + K_\ell^+(s) \ge 0$ for $s \in \mathcal{C}_+$,
where $K_\ell^+(s)^*$ represents the conjugate of $K_\ell^+(s)$

From Lemma 3. 1 , since $\dfrac{1}{s}$ is analytic in $\mathcal{C}_+$, $K_\ell^+(s) = \dfrac{a}{Z_\ell^+(s)}$ is analytic in $\mathcal{C}_+$, namely, (a) holds. For (b), from (11) and (6), we only need to show $\lambda_2(s)$, $s \in \mathcal{R}_+$ is real. This is true, since the discriminant of (8) for $s \in \mathcal{R}_+$ is

$$\left(a+1+\frac{m_\ell s^2}{d_\ell s + k_\ell}\right)^2 - 4a \ge (a-1)^2 \ge 0.$$

Finally, from Lemma 3. 2 , we can show (c) by using the maximum modulus theorem [6] (For $a > 1$, from Lemma 3. 2 (iii), note that the order of the pole at $s = 0$ is one and the residue is nonnegative.).
(ii): Obvious from (i) and Lemma 3. 2 . ■

4 Properties of the Closed Loop Systems

In this section, we investigate the properties of the damped mass-spring system controlled by the impedance matching controller.

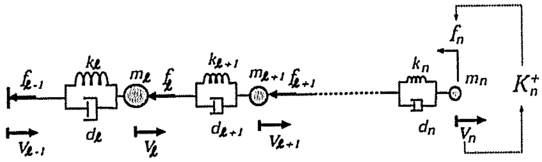

Figure 2: Damped Mass-Spring System Controlled by $K_n^+(s)$

4.1 The ladder structure

In this subsection, we consider the closed loop system for the uniformly varying damped mass-spring system satisfying (1). We especially concern ourselves with the closed loop system shown in Fig. 2 (Note that the boundary condition of the system shown in Fig. 1 is represented by $v_0(s) = 0.$).

In this case, the closed loop system and the impedance matching controller have the *ladder* structure.

Theorem 4. 1 *Consider $K_\ell^+(s)$ defined by (11). Then, the following equations hold.*

(i) $\mathcal{F}_\mathcal{L}(G_\ell(s), K_\ell^+(s)) = K_{\ell-1}^+(s)$,

(ii) $K_\ell^+(s) = \mathcal{F}_\mathcal{L}(G_\ell^\#(s), K_{\ell-1}^+(s))$,

(iii) $\mathcal{S}_\mathcal{P}(G_\ell(s), G_\ell^\#(s)) = J$,

where

$$G_\ell(s) := \begin{bmatrix} \dfrac{Y_\ell(s)}{1+Z_\ell(s)Y_\ell(s)} & \dfrac{1}{1+Z_\ell(s)Y_\ell(s)} \\ \dfrac{1}{1+Z_\ell(s)Y_\ell(s)} & -\dfrac{Z_\ell(s)}{1+Z_\ell(s)Y_\ell(s)} \end{bmatrix}, \quad (17)$$

$$G_\ell{}^\#(s) := \begin{bmatrix} -Y_\ell(s) & 1 \\ 1 & Z_\ell(s) \end{bmatrix}, \quad (18)$$

$$J := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad (19)$$

$\mathcal{F}_\mathcal{L}(\cdot,\cdot)$ and $\mathcal{S}_\mathcal{P}(\cdot,\cdot)$ represent the lower linear fractional transformation and the star product, respectively [9].

Proof:
In the proof, the variable s is omitted for notational simplicity.
(i): From (17) and (11), we get

$$\begin{aligned} \mathcal{F}_\mathcal{L}(G_\ell, K_\ell^+) &= \frac{Y_\ell}{1+Z_\ell Y_\ell} + \frac{1}{1+Z_\ell Y_\ell}\frac{1}{Z_{\ell+1}^+} \\ &\times \left\{1 - \left(-\frac{Z_\ell}{1+Z_\ell Y_\ell}\right)\frac{1}{Z_{\ell+1}^+}\right\}^{-1} \frac{1}{1+Z_\ell Y_\ell} \\ &= \frac{Y_\ell Z_{\ell+1}^+ + 1}{(1+Z_\ell Y_\ell)Z_{\ell+1}^+ + Z_\ell}. \end{aligned} \quad (20)$$

Moreover, from (12),

$$aY_\ell {Z_{\ell+1}^+}^2 - (-a+1+Z_\ell Y_\ell)Z_{\ell+1}^+ - Z_\ell = 0. \quad (21)$$

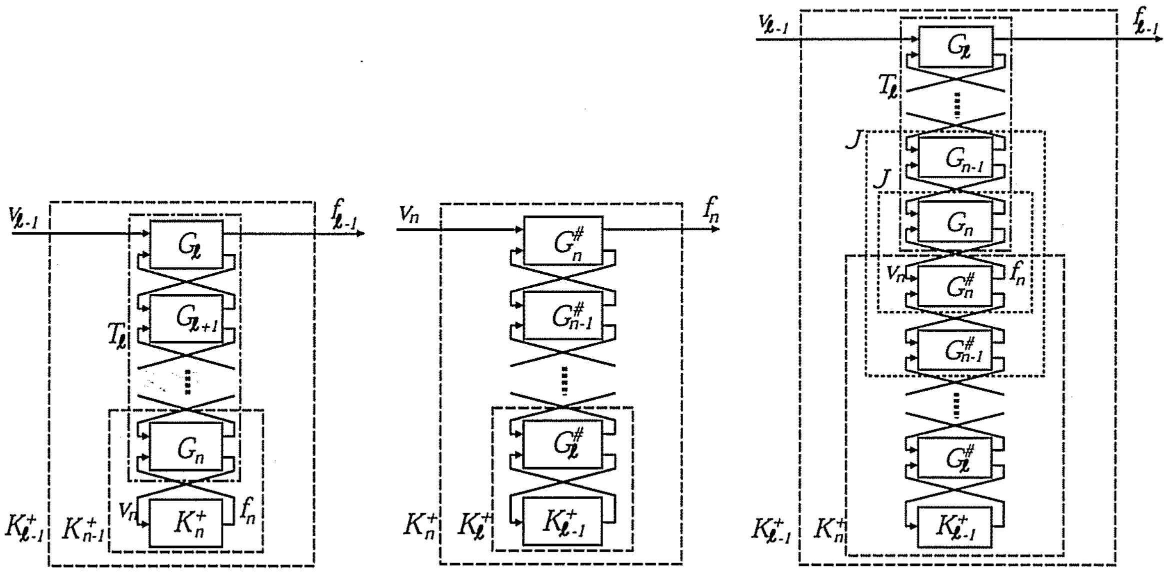

Figure 3: Ladder Structure of the Impedance Matching Controller

Therefore, from (20), (21) and (11), we get

$$\mathcal{F}_{\mathcal{L}}(G_\ell, K_\ell^+) = \frac{(Y_\ell Z_{\ell+1}^+ + 1)}{a(Y_\ell Z_{\ell+1}^{+\,2} + Z_{\ell+1}^+)} = \frac{1}{aZ_{\ell+1}^+} = K_{\ell-1}^+.$$

(ii): Similar to (i), from (18), (11) and (12), we get

$$\begin{aligned}\mathcal{F}_{\mathcal{L}}(G_\ell^\#, K_{\ell-1}^+) &= -Y_\ell + \frac{1}{Z_\ell^+}\left(1 - Z_\ell \frac{1}{Z_\ell^+}\right)^{-1} \\ &= \frac{1}{Z_\ell^+}\left(\frac{-Y_\ell Z_\ell^{+2} + (Z_\ell Y_\ell + 1)Z_\ell^+}{Z_\ell^+ - Z_\ell}\right) \\ &= \frac{1}{Z_\ell^+}\left(\frac{aZ_\ell^+ - aZ_\ell}{Z_\ell^+ - Z_\ell}\right) = K_\ell^+.\end{aligned}$$

(iii): From (17) and (18), we get (iii) from direct calculations. ■

Remark 1:
From (2), since

$$\begin{bmatrix} f_{\ell-1}(s) \\ v_\ell(s) \end{bmatrix} = G_\ell(s) \begin{bmatrix} v_{\ell-1}(s) \\ f_\ell(s) \end{bmatrix}, \qquad (22)$$

$G_\ell(s)$ can be interpreted as the transfer function of the uniformly varying damped mass-spring system of *ℓ-th stage* from $[v_{\ell-1}\ f_\ell]^T$ to $[f_{\ell-1}\ v_\ell]^T$. In addition, let $T_\ell(s)$ be the transfer function of the right hand side of the uniformly varying damped mass spring system at ℓ from $[v_{\ell-1}\ f_n]^T$ to $[f_{\ell-1}\ v_n]^T$ (See Fig. 2). $T_\ell(s)$ is represented by the star product of $G_\ell(s)$:

$$T_\ell(s) := \mathcal{S}_{\mathcal{P}}(G_\ell(s), T_{\ell+1}(s)), \quad T_n(s) := G_n(s). \quad (23)$$

From Remark 1, Theorem 4. 1 can be well interpreted by the feedback interconnection in Fig. 3. We can see the *ladder* structure of the impedance matching controller: The left figure represents that the transfer function of the uniformly varying damped mass-spring system controlled by $K_n^+(s)$ equals $K_{\ell-1}^+(s)$ (Theorem 4. 1 (i)). The middle figure represents that $K_n^+(s)$ is composed of the feedback interconnection of $G_\ell{}^\#(s), \cdots, G_n{}^\#(s)$ and $K_{\ell-1}^+(s)$ (Theorem 4. 1 (ii)). Combining these results, as shown in the right figure, we can see that $K_n^+(s)$ eliminates $G_n(s), \cdots, G_\ell(s)$ by using the inverse system $G_n{}^\#(s), \cdots, G_\ell{}^\#(s)$ (Theorem 4. 1 (iii)). In other words, the impedance matching controller makes the closed loop system be the impedance matching controller.

4.2 Internal stability

In this subsection, we consider the internal stability of the closed loop system. In this case, we do *not* assume the condition (1) (The system is called the *non*-uniform damped mass-spring system). The same symbols used for the uniformly varying damped mass-spring systems are also used for the non-uniform varying damped mass-spring systems by attaching '$\sim$'.

Consider the feedback interconnection shown in Fig. 4. Similar to the previous subsection, $\widetilde{T}_\ell(s)$ represents

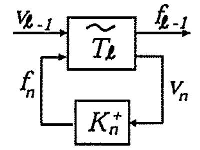

Figure 4: Feedback Interconnection of $\widetilde{T}_\ell(s)$ and $K_n^+(s)$

the transfer function of the right hand side of the *non*-uniform damped mass spring system from $[v_{\ell-1}\ f_n]^T$

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to $[f_{\ell-1}\ v_n]^T$. $\widetilde{T}_\ell(s)$ is represented by

$$\widetilde{T}_\ell(s) = \begin{bmatrix} \widetilde{T}_\ell^{11}(s) & \widetilde{T}_\ell^{12}(s) \\ \widetilde{T}_\ell^{21}(s) & \widetilde{T}_\ell^{22}(s) \end{bmatrix} := \mathcal{S}_\mathcal{P}(\widetilde{G}_\ell(s), \widetilde{T}_{\ell+1}(s)),\ \widetilde{T}_n(s) := \widetilde{G}_n(s), \quad (24)$$

where

$$\widetilde{G}_\ell(s) := \begin{bmatrix} \dfrac{\widetilde{Y}_\ell(s)}{1+\widetilde{Z}_\ell(s)\widetilde{Y}_\ell(s)} & \dfrac{1}{1+\widetilde{Z}_\ell(s)\widetilde{Y}_\ell(s)} \\ \dfrac{1}{1+\widetilde{Z}_\ell(s)\widetilde{Y}_\ell(s)} & -\dfrac{\widetilde{Z}_\ell(s)}{1+\widetilde{Z}_\ell(s)\widetilde{Y}_\ell(s)} \end{bmatrix}. \quad (25)$$

First, we investigate the property of $\widetilde{T}_\ell(s)$.

Lemma 4.1 *Consider $\widetilde{T}_\ell(s)$ defined by (24). $\widetilde{T}_\ell(s)$ has the following properties.*

(i) $\lim_{\omega\to 0+} \widetilde{T}_\ell(j\omega) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\lim_{\omega\to\infty} \widetilde{T}_\ell(j\omega) = \begin{bmatrix} d_\ell & 0 \\ 0 & 0 \end{bmatrix}$.

(ii) $\widetilde{T}_\ell(s) \in \mathcal{RH}_\infty$.

(iii) $-\widetilde{T}_\ell^{22}(s)$ *is positive real.*

Proof:
See Appendix B. ■

From Theorem 3.1 and Lemma 4.1, we can show the internal stability of the non-uniform damped mass-spring system controlled by the impedance matching controller.

Theorem 4.2 *Consider $\widetilde{T}_\ell(s)$ and $K_n^+(s)$ defined by (24) and (11), and suppose $a \le 1$. Then, the feedback interconnection shown in Fig. 4 is internally stable.*

Proof:
Since $\widetilde{T}_\ell(s) \in \mathcal{RH}_\infty$, we need to show the feedback interconnection of $\widetilde{T}_\ell^{22}(s)$ and $K_n^+(s)$ is the internally stable. From Lemma 3.2 and 4.1 and Theorem 3.1, this is true from the Nyquist criterion by using the following conditions:

$$-\widetilde{T}_\ell^{22}(s) \in \mathcal{RH}_\infty,\ K_n^+(s) \in \mathcal{H}_\infty,$$

$$\left|\arg\left[-\widetilde{T}_\ell^{22}(j\omega)K_n^+(j\omega)\right]\right| < \pi,\ \forall\omega \in \mathcal{R}_+,$$

$$\lim_{\omega\to 0+} -\widetilde{T}_\ell^{22}(j\omega)K_n^+(j\omega) = \lim_{\omega\to\infty} -\widetilde{T}_\ell^{22}(j\omega)K_n^+(j\omega) = 0.$$

■

Remark 2:
From Theorem 4.2, we can choose the parameters of the impedance matching controller from m_n, d_n, $k_n > 0$ and $1 \ge a > 0$ for vibration control of the non-uniform damped mass-spring systems. Since the impedance matching controller well perform for vibration control of the uniform damped mass-spring systems [5], and $\mathcal{F}_\mathcal{L}(T_\ell(s), K_n^+(s)) = K_{\ell-1}^+(s)$ (Theorem 4.1), we can determine the parameters by minimizing the *distance* between $K_{\ell-1}^+(s)$ and $\mathcal{F}_\mathcal{L}(\widetilde{T}_\ell(s), K_n^+(s))$.

Remark 3:
In the case of $a > 1$, $\widetilde{T}_\ell^{22}(s)$ has a zero at $s = 0$, while $K_n^+(s)$ has a pole at $s = 0$. The internal stability is not guaranteed only because of this unstable pole-zero cancellation at $s = 0$.

5 Conclusion

In this paper, we studied the properties of the impedance matching controller for the uniformly varying damped mass-spring systems and the resultant closed loop systems. The positive real property of the impedance matching controller was first investigated. The closed loop properties of the damped mass-spring systems controlled by the impedance matching controller were also discussed.

To propose a general framework of the wave control for the non-uniform damped mass-spring systems is our future work.

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Appendix

A: Routh's stability criterion

The zeros of (16) can be examined by using the Routh's stability criterion [10]. From (16), $R(s)$ is written by

$$R(s) := a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0, \qquad (26)$$

where

$$\begin{aligned} a_0 &:= m_\ell^2 > 0, \\ a_1 &:= 2(a+1)m_\ell d_\ell > 0, \\ a_2 &:= 2(a+1)m_\ell k_\ell + (-a+1)^2 d_\ell^2 > 0, \\ a_3 &:= 2(-a+1)^2 d_\ell k_\ell > 0, \\ a_4 &:= (-a+1)^2 k_\ell^2 > 0. \end{aligned}$$

From the Routh's stability criterion, since all the coefficients of (26) are positive, we need to show all the elements in the first column of the following table are positive:

$$\begin{array}{ccc} a_0 & a_2 & a_4 \\ a_1 & a_3 & 0 \\ b_1 & b_2 & \\ c_1 & 0 & \\ d_1. & & \end{array}$$

This is true as in the following.

$$\begin{aligned} b_1 &:= \frac{a_1 a_2 - a_0 a_3}{a_1} = \frac{1}{a_1}[2(a^2+6a+1)m^2 dk \\ &\qquad +2(a+1)(-a+1)^2 m d^3] > 0, \\ b_2 &:= \frac{a_1 a_4 - a_0 \cdot 0}{a_1} = a_4 > 0, \\ c_1 &:= \frac{b_1 a_3 - a_1 b_2}{b_1} = \frac{1}{a_1 b_1}\{16a(-a+1)^2 m_\ell^2 d_\ell^2 k_\ell^2 \\ &\qquad +4(a+1)(-a+1)^4 m_\ell d_\ell^4 k_\ell\} > 0, \\ d_1 &:= \frac{c_1 b_2 - b_1 \cdot 0}{c_1} = b_2 = a_4 > 0. \end{aligned}$$

B: Proof of Lemma 4. 1

(i): From (25), the results follows from the facts

$$\lim_{\omega\to 0} \widetilde{G}_\ell(j\omega) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \lim_{\omega\to\infty} \widetilde{G}_\ell(j\omega) = \begin{bmatrix} \widetilde{d}_\ell & 0 \\ 0 & 0 \end{bmatrix}.$$

(ii): From the state space representation, we get another representation of $\widetilde{T}_\ell(s)$:

$$\widetilde{T}_\ell(s) = \left(\overline{T}(s) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) \begin{bmatrix} \widetilde{d}_\ell + \dfrac{\widetilde{k}_\ell}{s} & 0 \\ 0 & 1 \end{bmatrix}, \qquad (27)$$

where

$$\overline{T}(s) := \left[\begin{array}{c|c} -\widetilde{k}_\ell\ 0 \cdots 0 & -\widetilde{d}_\ell\ 0 \cdots 0 \\ \hline 0 \cdots\cdots 0 & 0 \cdots 0\ 1 \end{array}\right]$$
$$\times \left(sI - \begin{bmatrix} 0_{n_\ell \times n_\ell} & I_{n_\ell} \\ -\widetilde{M}_\ell^{-1}\widetilde{K}_\ell & -\widetilde{M}_\ell^{-1}\widetilde{D}_\ell \end{bmatrix}\right)^{-1} \left[\begin{array}{c} 0_{n_\ell \times 2} \\ \hline \widetilde{M}_\ell^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \vdots \\ \vdots & 0 \\ 0 & -1 \end{bmatrix} \end{array}\right],$$

$$\widetilde{M}_\ell := \mathrm{diag}(\widetilde{m}_\ell, \widetilde{m}_{\ell+1}, \cdots, \widetilde{m}_n),$$

$$\widetilde{D}_\ell := \begin{bmatrix} \widetilde{d}_\ell + \widetilde{d}_{\ell+1} & -\widetilde{d}_{\ell+1} & 0 & \cdots & 0 \\ -\widetilde{d}_{\ell+1} & \widetilde{d}_{\ell+1} + \widetilde{d}_{\ell+2} & -\widetilde{d}_{\ell+2} & \ddots & \vdots \\ 0 & -\widetilde{d}_{\ell+2} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \widetilde{d}_{n-1} + \widetilde{d}_n & -\widetilde{d}_n \\ 0 & \cdots & 0 & -\widetilde{d}_n & \widetilde{d}_n \end{bmatrix}$$

$$\widetilde{K}_\ell := \begin{bmatrix} \widetilde{k}_\ell + \widetilde{k}_{\ell+1} & -\widetilde{k}_{\ell+1} & 0 & \cdots & 0 \\ -\widetilde{k}_{\ell+1} & \widetilde{k}_{\ell+1} + \widetilde{k}_{\ell+2} & -\widetilde{k}_{\ell+2} & \ddots & \vdots \\ 0 & -\widetilde{k}_{\ell+2} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \widetilde{k}_{n-1} + \widetilde{k}_n & -\widetilde{k}_n \\ 0 & \cdots & 0 & -\widetilde{k}_n & \widetilde{k}_n \end{bmatrix}$$

($n_\ell := n - \ell + 1$). Note that $\overline{T}(s) \in \mathcal{RH}_\infty$, since $\widetilde{M}_\ell, \widetilde{D}_\ell, \widetilde{K}_\ell > 0$ [11]. From (27), since $\overline{T}(s) \in \mathcal{RH}_\infty$, $\widetilde{T}_\ell(s)$ only could have an unstable pole at $s = 0$. However, this is not the case from (i).
(iii) From (27), $-\widetilde{T}_\ell^{22}(s) = \widetilde{C}_\ell^{22}(sI - \widetilde{A}_\ell^{22})^{-1}\widetilde{B}_\ell^{22}$ has an realization:

$$\begin{aligned} \widetilde{A}_\ell^{22} &:= \left[\begin{array}{c|c} 0_{n_\ell \times n_\ell} & I_{n_\ell} \\ \hline -\widetilde{M}_\ell^{-1}\widetilde{K}_\ell & -\widetilde{M}_\ell^{-1}\widetilde{D}_\ell \end{array}\right], \\ \widetilde{B}_\ell^{22} &:= \left[\begin{array}{c} 0_{n_\ell \times 1} \\ \hline \widetilde{M}_\ell^{-1} \begin{bmatrix} 0_{(n_\ell - 1)\times 1} \\ 1 \end{bmatrix} \end{array}\right], \\ \widetilde{C}_\ell^{22} &:= [\, 0_{1\times n_\ell} \mid 0_{1\times(n_\ell - 1)}\ 1 \,]. \end{aligned}$$

Therefore, $-\widetilde{T}_\ell^{22}(s)$ is a positive real function from the positive real lemma [7, 8]. ■