Motions of ultrarelativistic particles accelerated in an oblique plasma wave

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The motion of ultrarelativistic particles in an oblique plasma wave is theoretically studied. Making use of the relation $v\,d\gamma/dt \gg \gamma dv/dt$, where γ is the Lorentz factor, the zeroth-order velocity and energy increase rate are obtained. This solution is applicable to any particle species. The particle velocity is nearly parallel to the external magnetic field when $v_{\mathrm{sh}} \sim c \cos\theta$, where v_{sh} is the wave propagation speed and θ is the angle between the wave normal and the external magnetic field. The perturbed motions of ions and of positrons are then separately discussed. Their perturbations are both nearly perpendicular to the zeroth-order velocity. The ion perturbation is one dimensional with the frequency $\omega \sim \Omega_{i0}\gamma^{-1/2}$, while that of the positrons is elliptic with $\omega \sim \Omega_{p0}\gamma^{-1}$, where Ω_{i0} and Ω_{p0} are the nonrelativistic gyrofrequencies of ions and positrons, respectively. © *2004 American Institute of Physics.* [DOI: 10.1063/1.1737741]

I. INTRODUCTION

Particle acceleration in both laboratory and astrophysical plasmas has been attracting a great deal of attention.[1–14] Recently, in particular, it has been recognized that nonlinear plasma waves can accelerate particles with nonstochastic mechanisms. Particle simulations have demonstrated that a magnetosonic shock wave (or pulse) can accelerate hydrogen[15–19] and heavy ions[20,21] to relativistic energies (with the Lorentz factor $\gamma \lesssim 10$) and electrons[22–24] to ultrarelativistic energies ($\gamma \gtrsim 100$) in very short time periods, of the order of (or shorter than) the ion gyroperiod. These explain the acceleration to the energy levels of solar energetic particles.[9]

One way of accelerating particles to much higher energies would be to make particles stay around the main pulse of a wave for long periods of time;[1,2,6,8,12] thereby, the interaction time between the particles and electric fields would become extremely long. In many cases, however, particles do not move with the same velocity as the plasma wave. After the encounter with the pulse, thus, they quickly go away from the pulse. Hence, it is important to find mechanisms that allow particles to move with the pulse for long periods of time. In plasma-based accelerators without external magnetic fields, for instance, long-time acceleration is made by use of plasma waves having propagation speeds close to the speed of light c.[6–8]

One such mechanism in an external magnetic field is the surfatron.[1] In this mechanism, it is assumed that the electric field is so strong that the condition $E_x/B_z > 1$ is satisfied, where E_x is the longitudinal electric field in the x direction and B_z is the magnetic field in the z direction. It is then argued that unlimited acceleration of particles in the y direction could occur. If $E_x/B_z < 1$, however, particles eventually move to the downstream region.[2–5]

The magnetic field **B** greatly suppresses the freedom of particle motion, particularly in the direction perpendicular to **B**. Under certain circumstances, this can make the interaction time between particles and wave fields significantly long. Indeed, since the particle speed is limited by the speed of light c, relativistic particles can move with a wave propagating obliquely to an external magnetic field $\mathbf{B}_0$ for long periods of time if

$$v_{\mathrm{sh}} \sim c \cos\theta \tag{1}$$

is satisfied, where v_{sh} is the wave propagation speed, and θ is the angle between the wave normal and the external magnetic field.[22–27]

It was shown in Refs. 25 and 26 that by virtue of this effect, nonthermal, fast ions with $\gamma \sim 1$ can be accelerated to ultrarelativistic energies with $\gamma > 100$ by an oblique shock wave satisfying Eq. (1). In this mechanism, that the particle speed is close to c has additional important effects; the gyration speed is comparable to or faster than the wave propagation speed, and the gyroradius is much larger than the width of the shock transition region. In association with the gyromotion, thus, the fast particles can move back and forth between the shock and upstream regions;[28,29] when they are in the shock region, they gain energy from the transverse electric field.

Also, a new acceleration mechanism for positrons in a shock wave with Eq. (1) has been found with particle simulations.[27] In this mechanism, the following property of γ plays a crucial role; if γ is large, a slight change in the speed can lead to a great change in γ, i.e.,

$$v\,d\gamma/dt \gg \gamma dv/dt. \tag{2}$$

Positrons that are prevented from passing through the pulse by the electric field along $\boldsymbol{B}$ move nearly parallel to $\boldsymbol{B}_0$ and are accelerated to ultrarelativistic energies. Unlike the above energetic ions, these positrons are in the shock transition region throughout the acceleration process.

These results indicate that there exist acceleration mechanisms that act mainly on energetic particles. In this paper, we theoretically study the motions of highly relativistic particles in a plasma wave propagating in the x direction in an external magnetic field $\boldsymbol{B}_0$ in the (x,z) plane, with particular interest in the particles that move with and are

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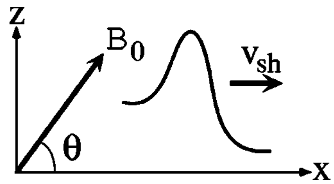

FIG. 1. Schematic diagram of pulse propagation. The wave is supposed to propagate in the x direction with a speed v_{sh} in an external magnetic field $\boldsymbol{B}_0$ in the (x,z) plane.

accelerated by the wave for long periods of time. Even though the motivation of the present work stems primarily from the above simulations on shock waves, the application of the theory will not be limited merely to magnetosonic waves. The only assumption for the wave is that it steadily propagates obliquely to $\boldsymbol{B}_0$.

In Sec. II, under the assumption (2), we derive the zeroth-order velocity and $d\gamma/dt$ of relativistic particles moving with the wave, $v_x \simeq v_{\mathrm{sh}}$. In the zeroth-order theory, we do not assume specific particle species. Unlike the surfatron acceleration model,[1] $E_x/B_z > 1$ is not assumed, and v_y can be small relative to the other velocity components. When $v_{\mathrm{sh}} \sim c \cos\theta$, this zeroth-order velocity is nearly parallel to the external magnetic field $\boldsymbol{B}_0$. We then study perturbations to this zeroth-order motion, presenting two different schemes. In Sec. III, we consider a perturbation scheme for energetic ions, while in Sec. IV, we investigate positron motions. It is shown that the perturbations of ions and of positrons are both nearly perpendicular to the zeroth-order velocity. The ion perturbation is one dimensional, and its frequency is much higher than the relativistic ion gyrofrequency; $\omega \sim \Omega_{i0}\gamma^{-1/2}$ where Ω_{i0} is the nonrelativistic ion gyrofrequency for B_0. The positron perturbation is elliptic, and its frequency is of the order of the relativistic gyrofrequency; $\omega \sim \Omega_{p0}\gamma^{-1}$. In these calculations, it is supposed that the particles are in the pulse (or in a small region in the wave). These ion and positron motions are therefore quite different from the energetic-ion motions discussed in Refs. 25 and 26; those energetic ions repeatedly go in and out of the shock region. In Sec. V, we compare the theory with the simulation result obtained in Ref. 27, where a positron motion that is quite similar to the one predicted by the present theory has been observed in particle simulations. Section VI gives a summary of our work.

II. MOTIONS OF ULTRARELATIVISTIC PARTICLES

We study particle motions in a stationary plasma wave propagating in the x direction in an external magnetic field $\boldsymbol{B}_0 = (B_{x0}, 0, B_{z0})$; for the definiteness, we take B_{x0} and B_{z0} to be positive (see Fig. 1). The angle between $\boldsymbol{B}_0$ and the x axis is denoted by θ; $\tan\theta = B_{z0}/B_{x0}$. We normalize the time, velocity, and length as $\hat{t} = \Omega_{j0} t$, $\hat{\mathbf{v}} = \boldsymbol{v}/c$, and $\hat{x} = x/(c/\Omega_{j0})$, where the nonrelativistic gyrofrequency Ω_{j0} is defined as

$$\Omega_{j0} = q_j B_0/(m_j c), \tag{3}$$

with m_j the rest mass of particle species j. The charge q_j is taken to be positive (extension to the case of negative charge would be straightforward). Also, the fields are normalized to B_0; $\hat{\boldsymbol{E}} = \boldsymbol{E}/B_0$ and $\hat{\boldsymbol{B}} = \boldsymbol{B}/B_0$. In the following, however, the hat is omitted.

We consider wave propagation and particle motion in the laboratory frame, where the plasma is at rest in an equilibrium state. For a stationary wave propagating with a speed v_{sh}, the field quantities may be written as $f(x,t) = f(\xi)$ with ξ defined as $\xi = x - v_{\mathrm{sh}} t$. From Faraday's law, then, we obtain the relations

$$E_y = v_{\mathrm{sh}}(B_z - B_{z0}), \tag{4}$$

$$E_z = -v_{\mathrm{sh}} B_y. \tag{5}$$

We also note that the x component of the magnetic field is constant, $B_x = B_{x0}$, because $\partial/\partial y = \partial/\partial z = 0$.

A. Zeroth-order calculations

We investigate the motion of relativistic particles with $\gamma \gg 1$ being accelerated in the wave. An expression for the time derivative of γ,

$$\frac{d\gamma}{dt} = \frac{\gamma^3}{2}\frac{dv^2}{dt}, \tag{6}$$

shows that a slight change in the speed can give rise to a large change in energy if γ is much greater than unity. Indeed, if the particle speed v is very close to the speed of light, v cannot increase significantly, while γ can increase indefinitely. Hence, in the exact equation of motion,

$$\frac{d(\gamma\boldsymbol{v})}{dt} = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}, \tag{7}$$

we neglect the term $\gamma d\boldsymbol{v}/dt$ compared with $\boldsymbol{v}\, d\gamma/dt$:

$$\frac{d\gamma}{dt}\boldsymbol{v} = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}. \tag{8}$$

We use this equation for the lowest order calculations. [Higher order calculations based on Eq. (7) will be given later in Sec. III and Sec. IV.]

Suppose that a particle is moving with the wave,

$$v_x = v_{\mathrm{sh}}, \tag{9}$$

then, by virtue of Eqs. (4) and (5), the x, y, and z components of Eq. (8) can be written as

$$v_{\mathrm{sh}}\frac{d\gamma}{dt} = E_x + v_y B_z - v_z B_y, \tag{10}$$

$$v_y\frac{d\gamma}{dt} = v_z B_{x0} - v_{\mathrm{sh}} B_{z0}, \tag{11}$$

$$v_z\frac{d\gamma}{dt} = -v_y B_{x0}. \tag{12}$$

Eliminating $d\gamma/dt$, we obtain two equations:

$$-B_{x0} v_{\mathrm{sh}} v_y / v_z = E_x + v_y B_z - v_z B_y, \tag{13}$$

$$v_y^2 + v_z^2 = (B_{z0}/B_{x0}) v_{\mathrm{sh}} v_z. \tag{14}$$

We define the Lorentz factor γ_{sh} corresponding to the pulse propagation speed v_{sh} as $\gamma_{\mathrm{sh}} = (1 - v_{\mathrm{sh}}^2)^{-1/2}$. Then, substituting the relation

$$v_y^2+v_z^2=\gamma_{\rm sh}^{-2}-\gamma^{-2} \tag{15}$$

in Eq. (14), we find v_z as

$$v_z=\frac{B_{x0}}{B_{z0}}\frac{(\gamma_{\rm sh}^{-2}-\gamma^{-2})}{v_{\rm sh}}. \tag{16}$$

Combining Eqs. (13) and (16), we have

$$v_y=\frac{B_{x0}B_y(\gamma_{\rm sh}^{-2}-\gamma^{-2})^2-E_xB_{z0}v_{\rm sh}(\gamma_{\rm sh}^{-2}-\gamma^{-2})}{B_{z0}B_zv_{\rm sh}(\gamma_{\rm sh}^{-2}-\gamma^{-2})+B_{z0}^2v_{\rm sh}^3}. \tag{17}$$

Substituting Eqs. (16) and (17) in Eq. (12) yields

$$\frac{d\gamma}{dt}=\frac{E_xB_{z0}v_{\rm sh}-B_{x0}B_y(\gamma_{\rm sh}^{-2}-\gamma^{-2})}{B_z(\gamma_{\rm sh}^{-2}-\gamma^{-2})+B_{z0}v_{\rm sh}^2}. \tag{18}$$

Since we consider highly relativistic particles, we may use the approximation

$$\gamma_{\rm sh}^{-2}-\gamma^{-2}\simeq\gamma_{\rm sh}^{-2}. \tag{19}$$

Then, v_y, v_z, and $d\gamma/dt$ are given as

$$v_y=\frac{B_{x0}B_y\gamma_{\rm sh}^{-4}-E_xB_{z0}v_{\rm sh}\gamma_{\rm sh}^{-2}}{B_{z0}B_zv_{\rm sh}\gamma_{\rm sh}^{-2}+B_{z0}^2v_{\rm sh}^3}, \tag{20}$$

$$v_z=\frac{B_{x0}}{B_{z0}}\frac{(1-v_{\rm sh}^2)}{v_{\rm sh}}, \tag{21}$$

$$\frac{d\gamma}{dt}=\frac{E_xB_{z0}v_{\rm sh}-B_{x0}B_y\gamma_{\rm sh}^{-2}}{B_z\gamma_{\rm sh}^{-2}+B_{z0}v_{\rm sh}^2}. \tag{22}$$

Equations (9), (20), (21), and (22) suggest that a particle can move with a shock wave at a certain ξ position, $\xi=\xi_0$, with nearly a constant velocity, continuously absorbing energy from the wave.

Equation (12) indicates that, in order for the energy to increase, v_y must be negative (since v_z is positive). We therefore find from Eq. (20) a condition for the particle position ξ_0 as

$$E_x(\xi_0)-B_{x0}B_y(\xi_0)/(B_{z0}\gamma_{\rm sh}^2v_{\rm sh})>0. \tag{23}$$

Equation (22) directly gives this condition.

The zeroth-order theory leads to the condition that $v_{\rm sh}\gtrsim\cos\theta$. Substituting Eq. (16) in Eq. (15), we have another form of v_y:

$$v_y^2=\left(1-\frac{B_{x0}^2}{B_{z0}^2}\frac{(\gamma_{\rm sh}^{-2}-\gamma^{-2})}{v_{\rm sh}^2}\right)(\gamma_{\rm sh}^{-2}-\gamma^{-2}). \tag{24}$$

Since $v_y^2>0$ and $\gamma_{\rm sh}^{-2}>\gamma^{-2}$, we find that

$$v_{\rm sh}^2>(1-\gamma^{-2})\cos^2\theta, \tag{25}$$

which is equivalent to

$$v_{\rm sh}>v\cos\theta. \tag{26}$$

If v_y is small in magnitude, then, Eq. (14) gives

$$v_z/v_x\simeq B_{z0}/B_{x0}. \tag{27}$$

In this case, the particle moves nearly parallel to the external magnetic field. This type of particle motion has been observed in particle simulations of positron acceleration.[27] Equation (27) leads to the condition for the wave $v_{\rm sh}\sim\cos\theta$, which was assumed in Ref. 27. Conversely, if $v_{\rm sh}\sim\cos\theta$, then v_y is small. In fact, if $v_{\rm sh}\sim\cos\theta$, then $\gamma_{\rm sh}^{-2}\sim\sin^2\theta$. Equation (21) thus gives Eq. (27); hence, v_y is small.

These results would also be applicable to electrostatic waves if we put $B_y\rightarrow0$, $B_z\rightarrow B_{z0}$; hence, $E_y=E_z=0$.

B. Surfatron acceleration

We discuss the relation between the surfatron acceleration[1] and the present mechanism. In Ref. 1, an electrostatic wave propagating perpendicular to an external magnetic field was considered, under the assumption that $E_x/B_z>1$. It was then shown that arbitrarily large energy gain is theoretically possible, where v_y predominantly increases. In the present mechanism, however, ultrarelativistic-particle motions in oblique, electromagnetic waves are considered. The relation $E_x/B_z>1$ is not required. Also, v_y can be rather small. When $v_{\rm sh}\sim\cos\theta$, the particle velocity is nearly parallel to the external magnetic field $\boldsymbol{B}_0$.

Let us consider the limit of $\theta\rightarrow90°$ in the present model. For this case, $B_{x0}=0$, and Eq. (12) gives $v_z=0$. Neglecting the time derivative of v_y, we obtain from Eq. (11) that

$$v_y\gamma=-v_{\rm sh}B_{z0}t. \tag{28}$$

Also, since the particle speed is assumed to be close to c, we have

$$v_y\simeq-(1-v_{\rm sh}^2)^{1/2}. \tag{29}$$

These solutions agree with the particle velocity in the limit of $\gamma\gg1$ in the surfatron acceleration.[1] Hence, the present model with $\theta=90°$ includes the surfatron acceleration with $\gamma\gg1$.

If the propagation angle is $\theta=90°$, strong longitudinal electric fields are required even in the present model. Equations (10) and (11) give

$$\frac{v_{\rm sh}^2}{(B_z/B_{z0})(v_d/2)^2}+\frac{(v_y+v_d/2)^2}{(v_d/2)^2}=1, \tag{30}$$

where $v_d=E_x/B_z$. Since we are concerned with highly relativistic particles, the relation $v_{\rm sh}^2+v_y^2\simeq1$ must hold. Comparing this speed-of-light circle and ellipse (30) in the $(v_{\rm sh},v_y)$ plane, we see that we have solutions when $E_x/B_z\gtrsim1$ or $(B_z/B_{z0})[E_x/(2B_z)]^2\gtrsim1$ (see Fig. 2). We note that E_x/B_z is always smaller than unity in stationary nonlinear magnetosonic waves.[30]

III. PERTURBATIONS OF IONS

In the above calculations, changes in the particle position ξ and velocity $\boldsymbol{v}$ were neglected. We now investigate their perturbations around the zeroth-order solution. In Sec. III, we describe a perturbation scheme for ions; the nonrelativistic gyrofrequency is given by $\Omega_{i0}=q_iB_0/(m_ic)$ in Eq. (3).

We expand the particle position and velocity as

$$\boldsymbol{x}=\boldsymbol{x}(0)+\boldsymbol{v}_0t+\epsilon\boldsymbol{x}_1(t)+\epsilon^2\boldsymbol{x}_2(t)+\cdots, \tag{31}$$

$$\boldsymbol{v}=\boldsymbol{v}_0+\epsilon'\boldsymbol{v}_1(t)+(\epsilon')^2\boldsymbol{v}_2(t)+\cdots, \tag{32}$$

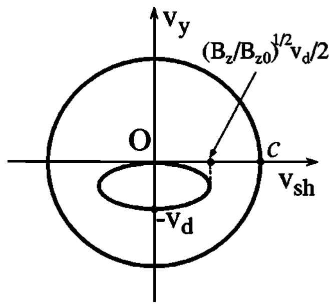

FIG. 2. Ellipse and speed-of-light circle in the (v_{sh}, v_y) plane. Here, unnormalized variables are used. The ellipse shows zeroth-order velocities for perpendicular waves.

where $\boldsymbol{x}(0)$ is the initial position, $\boldsymbol{v}_0$ is the zeroth-order solution (9), (16), and (17), and ϵ and ϵ' are smallness parameters. We introduce a parameter Γ showing large values,

$$\Gamma \sim \gamma \gg 1, \tag{33}$$

and assume that

$$dv_0/dt \sim \Gamma^{-3}. \tag{34}$$

Equation (6) then gives

$$d\gamma_0/dt \sim O(1), \tag{35}$$

with γ_0 the zeroth-order solution. For the perturbed velocity, we assume that

$$d\boldsymbol{v}_1/dt \sim \Gamma^{-1/2}\boldsymbol{v}_1, \tag{36}$$

i.e., the characteristic frequency ω is much higher than the relativistic gyrofrequency. Hence, from the relation $\boldsymbol{v} = d\boldsymbol{x}/dt$ and Eqs. (31) and (32), we have

$$\epsilon' \sim \Gamma^{-1/2}\epsilon. \tag{37}$$

Further, we assume that

$$\boldsymbol{v}_0 \cdot \boldsymbol{v}_1 \sim \Gamma^{-2} v_0 v_1, \tag{38}$$

which leads to

$$\epsilon' \boldsymbol{v}_0 \cdot \frac{d\boldsymbol{v}_1}{dt} \sim \epsilon' \Gamma^{-2} v_0 \left| \frac{d\boldsymbol{v}_1}{dt} \right| \sim \epsilon \Gamma^{-3}. \tag{39}$$

That is, the perturbed velocity $\boldsymbol{v}_1$ is almost perpendicular to the zeroth-order velocity $\boldsymbol{v}_0$; this will be examined after we obtain $\boldsymbol{v}_1$. Also, we expand the field quantities around ξ_0 as

$$E_x(\xi) = E_x(\xi_0) + \frac{dE_x}{d\xi_0}(\xi - \xi_0) + \cdots. \tag{40}$$

Here, $\xi_0 = x(0)$, and $dE_x/d\xi_0$ designates the value of $dE_x(\xi)/d\xi$ at $\xi = \xi_0$.

We apply this expansion scheme to the exact equation of motion, (7); for the details of calculations, see Appendix A. It is then found that the lowest order $[\sim O(1)]$ equations and solutions agree with those in Sec. II; i.e., we have Eqs. (9)–(12).

In the order of ϵ in the equation of motion, we obtain

$$v_{\mathrm{sh}} \gamma_0^3 \left(\boldsymbol{v}_0 \cdot \frac{d\boldsymbol{v}_1}{dt} \right) + \gamma_0 \frac{dv_{x1}}{dt} = \left(\frac{dE_x}{d\xi_0} + v_{y0} \frac{dB_z}{d\xi_0} - v_{z0} \frac{dB_y}{d\xi_0} \right) x_1. \tag{41}$$

The magnitude of the Lorentz force is $v_{y1} B_z(\xi_0) - v_{z1} B_y(\xi_0) \sim \epsilon \Gamma^{-1/2}$; thus, much smaller than ϵ. In the same way, we have the y and z component equations,

$$v_{y0} \gamma_0^3 \left(\boldsymbol{v}_0 \cdot \frac{d\boldsymbol{v}_1}{dt} \right) + \gamma_0 \frac{dv_{y1}}{dt} = 0, \tag{42}$$

$$v_{z0} \gamma_0^3 \left(\boldsymbol{v}_0 \cdot \frac{d\boldsymbol{v}_1}{dt} \right) + \gamma_0 \frac{dv_{z1}}{dt} = 0. \tag{43}$$

Now, assuming that the perturbed quantities vary with time as $\exp(-i\omega t)$, we find from Eqs. (41)–(43) the frequency ω as

$$\omega^2 = -\frac{(1 - v_{\mathrm{sh}}^2)}{\gamma_0} \left(\frac{dE_x}{d\xi_0} + v_{y0} \frac{dB_z}{d\xi_0} - v_{z0} \frac{dB_y}{d\xi_0} \right). \tag{44}$$

Equation (44) is consistent with the ordering (36) and indicates that ω has real values when

$$\frac{dE_x}{d\xi_0} + v_{y0} \frac{dB_z}{d\xi_0} - v_{z0} \frac{dB_y}{d\xi_0} < 0. \tag{45}$$

If this condition is not met, then, the oscillation would grow with time; the particle would quickly escape from the wave.

The perturbed velocity $\boldsymbol{v}_1$ is given as

$$\boldsymbol{v}_1 = -i\omega x_1 (1, -\gamma_{\mathrm{sh}}^2 v_{\mathrm{sh}} v_{y0}, -\gamma_{\mathrm{sh}}^2 v_{\mathrm{sh}} v_{z0}). \tag{46}$$

We thus find that

$$\boldsymbol{v}_0 \cdot \boldsymbol{v}_1 = -i\omega x_1 v_{\mathrm{sh}} \gamma_{\mathrm{sh}}^2 / \gamma_0^2, \tag{47}$$

which is in agreement with the ordering (38), indicating that $\boldsymbol{v}_1$ is almost perpendicular to $\boldsymbol{v}_0$. Equation (46) also shows that the perturbation is one dimensional; $\boldsymbol{v}_1$ is parallel to the vector $(1, -\gamma_{\mathrm{sh}}^2 v_{\mathrm{sh}} v_{y0}, -\gamma_{\mathrm{sh}}^2 v_{\mathrm{sh}} v_{z0})$.

IV. PERTURBATIONS OF POSITRONS

We now discuss perturbations of positrons. Positrons have much higher gyrofrequency than ions if their Lorentz factors are the same. We thus need a slightly different perturbation scheme to treat positrons.

We expand the particle position and velocity as Eqs. (31)–(34). We normalize the time, velocity, and length using the positron gyrofrequency $\Omega_{p0} = eB_0/(m_p c)$ in Eq. (3); $\hat{t} = \Omega_{p0} t$, $\hat{\mathbf{v}} = \boldsymbol{v}/c$, and $\hat{x} = x/(c/\Omega_{p0})$. Again, the hat is omitted below. For the perturbed velocity, we assume that

$$d\boldsymbol{v}_1/dt \sim \Gamma^{-1}\boldsymbol{v}_1, \tag{48}$$

i.e., the characteristic frequency ω is of the order of the relativistic gyrofrequency. Hence, from the relation $\boldsymbol{v} = d\boldsymbol{x}/dt$ and Eqs. (31) and (32), it follows that

$$\epsilon' \sim \Gamma^{-1}\epsilon. \tag{49}$$

For the angle between $\boldsymbol{v}_0$ and $\boldsymbol{v}_1$, we assume Eq. (38); $\boldsymbol{v}_0 \cdot \boldsymbol{v}_1 \sim \Gamma^{-2} v_0 v_1$.

In the lowest order calculations, this expansion scheme also gives the zeroth-order equation of motion, (10)–(12), and velocities, (9) and (16)–(18).

In the order of ϵ' in the equation of motion, we have

$$\gamma_0^3\left(\boldsymbol{v}_0\cdot\frac{d\boldsymbol{v}_1}{dt}\right)\boldsymbol{v}_0+\gamma_0\frac{d\boldsymbol{v}_1}{dt}=\boldsymbol{v}_1\times\boldsymbol{B}. \tag{50}$$

Here, we have neglected the terms $\boldsymbol{v}_1\cdot d\boldsymbol{v}_0/dt$ and $\boldsymbol{v}_1 d\gamma_0/dt$ compared with $\boldsymbol{v}_0\cdot d\boldsymbol{v}_1/dt$ and $\gamma_0 d\boldsymbol{v}_1/dt$, respectively.

Taking the scalar product of Eq. (50) with $\boldsymbol{B}$, we obtain

$$\boldsymbol{U}\cdot\frac{d\boldsymbol{v}_1}{dt}=0, \tag{51}$$

where $\boldsymbol{U}$ is a vector defined by

$$\boldsymbol{U}=\gamma_0^2(\boldsymbol{v}_0\cdot\boldsymbol{B})\boldsymbol{v}_0+\boldsymbol{B}. \tag{52}$$

Since $\gamma_0\gg1$, $\boldsymbol{U}$ is nearly parallel to $\boldsymbol{v}_0$. Equation (51) indicates that the component of $\boldsymbol{v}_1$ parallel to the vector $\boldsymbol{U}$ is constant. If it is initially zero, the perturbed velocity $\boldsymbol{v}_1$ is always in the plane perpendicular to $\boldsymbol{U}$.

Equation (50) does not allow exactly circular motions. Forming the scalar product of Eq. (50) with $\boldsymbol{v}_1$, we find that

$$\frac{d}{dt}(\boldsymbol{v}_1\cdot\boldsymbol{v}_1)=-\gamma_0^2\frac{d}{dt}(\boldsymbol{v}_0\cdot\boldsymbol{v}_1)^2. \tag{53}$$

The value of $(\boldsymbol{v}_1\cdot\boldsymbol{v}_1)$ would vary with time as $(\boldsymbol{v}_0\cdot\boldsymbol{v}_1)$ changes. Obviously, a circular motion $[(\boldsymbol{v}_1\cdot\boldsymbol{v}_1)=\text{constant}]$ that is perpendicular to both $\boldsymbol{v}_0$ and $\boldsymbol{U}$ does not exist.

Assuming that the perturbed quantities vary with time as $\exp(-i\omega t)$, we obtain from Eq. (50) the frequency ω as

$$\omega^2=\gamma_0^{-4}[\gamma_0^2(\boldsymbol{B}\cdot\boldsymbol{v}_0)^2+B^2]. \tag{54}$$

(For the details of the calculations, see Appendix B.) The velocity components v_{y1} and v_{z1} are related to v_{x1} as

$$v_{y1}=\frac{(\omega^2\gamma_0^4 v_{y0}v_{z0}+B_yB_z)-i\omega(\gamma_0^3 v_{x0}\boldsymbol{B}\cdot\boldsymbol{v}_0+\gamma_0 B_x)}{(\omega^2\gamma_0^4 v_{x0}v_{z0}+B_xB_z)+i\omega(\gamma_0^3 v_{y0}\boldsymbol{B}\cdot\boldsymbol{v}_0+\gamma_0 B_y)}v_{x1}, \tag{55}$$

$$v_{z1}=\frac{(\omega^2\gamma_0^4 v_{y0}v_{z0}+B_yB_z)+i\omega(\gamma_0^3 v_{x0}\boldsymbol{B}\cdot\boldsymbol{v}_0+\gamma_0 B_x)}{(\omega^2\gamma_0^4 v_{x0}v_{y0}+B_xB_y)-i\omega(\gamma_0^3 v_{z0}\boldsymbol{B}\cdot\boldsymbol{v}_0+\gamma_0 B_z)}v_{x1}. \tag{56}$$

Using this solution, one can show that $\boldsymbol{U}\cdot\boldsymbol{v}_1=0$, which is consistent with Eq. (51).

This motion is elliptic. We can easily see this in a different coordinate system $(\tilde{x},\tilde{y},\tilde{z})$ where the $\tilde{z}$ axis is parallel to $\boldsymbol{U}$ (see Fig. 3). Noting that the three vectors $\boldsymbol{B}$, $\boldsymbol{v}_0$, and $\boldsymbol{U}$ are in one plane, we can take the $\tilde{x}$ axis such that both $\boldsymbol{B}$ and $\boldsymbol{v}_0$ are in the $(\tilde{x},\tilde{z})$ plane. The $\tilde{x}$ axis is taken to be parallel to $\boldsymbol{v}_0-(\boldsymbol{U}\cdot\boldsymbol{v}_0)\boldsymbol{U}/U^2$. We may thus write $\boldsymbol{B}$ and $\boldsymbol{v}_0$ as $\boldsymbol{B}=(B_{\tilde{x}},0,B_{\tilde{z}})$ and $\boldsymbol{v}_0=(v_{\tilde{x}0},0,v_{\tilde{z}0})$ with

$$v_{\tilde{x}0}=[v_0^2-(\boldsymbol{U}\cdot\boldsymbol{v}_0)^2/U^2]^{1/2}. \tag{57}$$

Since $\boldsymbol{U}$ is nearly parallel to $\boldsymbol{v}_0$, the magnitude of $v_{\tilde{x}0}$ is quite small; with the aid of Eq. (52), we see that $v_{\tilde{x}0}=[v_0^2B^2-(\boldsymbol{v}_0\cdot\boldsymbol{B})^2]^{1/2}/U\sim\gamma_0^{-2}$. Taking the scalar product of Eq. (52) with $\boldsymbol{B}$ and with $\boldsymbol{v}_0$, we have

$$UB_{\tilde{z}}=\gamma_0^2(\boldsymbol{v}_0\cdot\boldsymbol{B})^2+B^2, \tag{58}$$

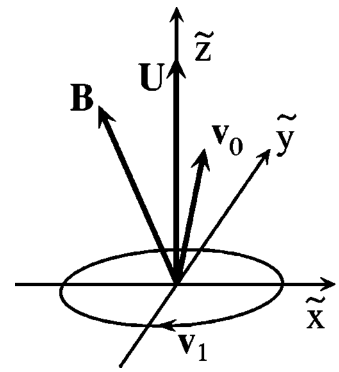

FIG. 3. Schematic diagram of the coordinate system $(\tilde{x},\tilde{y},\tilde{z})$. The $\tilde{z}$ axis is taken to be parallel to $\boldsymbol{U}$. The vectors $\boldsymbol{U}$, $\boldsymbol{B}$, and $\boldsymbol{v}_0$ are in the $(\tilde{x},\tilde{z})$ plane, while $\boldsymbol{v}_1$ is in the $(\tilde{x},\tilde{y})$ plane. The vectors in the figure do not show exact magnitudes.

$$Uv_{\tilde{z}0}=\gamma_0^2(\boldsymbol{v}_0\cdot\boldsymbol{B}), \tag{59}$$

where use has been made of the relation $\gamma_0^2=\gamma_0^2v_0^2+1$. Also, the $\tilde{x}$ component of Eq. (52) leads to

$$B_{\tilde{x}}=-\frac{\gamma_0^2v_{\tilde{x}0}v_{\tilde{z}0}}{\gamma_0^2v_{\tilde{x}0}^2+1}B_{\tilde{z}}. \tag{60}$$

Because of the relation (51), we assume that $v_{\tilde{z}1}=0$. From Eq. (50), then, we obtain equations for $(v_{\tilde{x}1},v_{\tilde{y}1})$:

$$-i\omega\gamma_0(\gamma_0^2v_{\tilde{x}0}^2+1)v_{\tilde{x}1}-B_{\tilde{z}}v_{\tilde{y}1}=0, \tag{61}$$

$$B_{\tilde{z}}v_{\tilde{x}1}-i\omega\gamma_0v_{\tilde{y}1}=0, \tag{62}$$

$$-i\omega\gamma_0^3v_{\tilde{x}0}v_{\tilde{z}0}v_{\tilde{x}1}+B_{\tilde{x}}v_{\tilde{y}1}=0. \tag{63}$$

Substituting Eq. (60) in Eq. (63), we see that Eq. (63) is the same as Eq. (61). The condition for a nontrivial solution of Eqs. (61)–(63) therefore reads as

$$\omega^2=\frac{B_{\tilde{z}}^2}{\gamma_0^2(\gamma_0^2v_{\tilde{x}0}^2+1)}. \tag{64}$$

On account of Eqs. (52), (57), and (58), one can show that Eq. (64) is identical to Eq. (54). The velocity component $v_{\tilde{y}1}$ is related to $v_{\tilde{x}1}$ as

$$v_{\tilde{y}1}=-i(\gamma_0^2v_{\tilde{x}0}^2+1)^{1/2}v_{\tilde{x}1}. \tag{65}$$

This motion is elliptic in the plane perpendicular to $\boldsymbol{U}$.

V. COMPARISON WITH PARTICLE SIMULATIONS

This theory explains the positron motion that has recently been observed in particle simulations for oblique shock waves in an electron-positron-ion plasma.[27] We compare the present theory with Fig. 4 in Ref. 27, which displays time variations of position (x,y,z) and γ for a positron accelerated to $\gamma\simeq600$ in a shock wave with $v_{\rm sh}\simeq c\cos\theta$; $v_{\rm sh}=2.4v_{\rm A}$ with $v_{\rm A}/c=0.3$ and $\theta=42°$. (To compare with the simulation, we here use unnormalized variables.)

That positron encountered the shock wave at $\omega_{pe}t\simeq220$ and then began to move with it. Its average velocity during the acceleration, i.e., from $\omega_{pe}t=230$ to $\omega_{pe}t=1000$, was $\boldsymbol{v}_0/c=(0.74,-0.15,0.65)$. That is, it was nearly parallel to the external magnetic field $\boldsymbol{B}_0$; v_z/v_x

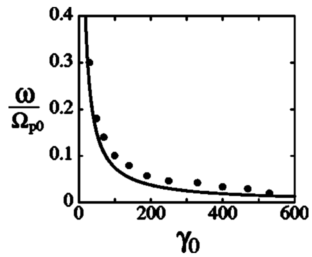

FIG. 4. Perturbation frequency vs γ_0. Here, γ_0 is the Lorentz factor averaged over oscillation period. The dots and solid line show simulation results and theoretical curve, respectively.

$\simeq B_{z0}/B_{x0}$. Also, the magnitude of v_y was much smaller than the other two components. These are consistent with the theory.

Its Lorentz factor γ almost linearly grew with time until $\omega_{pe}t \simeq 1000$, with an energy increase rate $d\gamma/d\omega_{pe}t = 0.51$. If we substitute the above value $v_y/c \simeq -0.15$ and Eq. (27) in Eq. (12), we have $d\gamma/d\omega_{pe}t = 0.51$. This is in good agreement with the observed value.

In Fig. 4 in Ref. 27, we find small oscillation of γ. Its oscillation frequency ω decreases with increasing γ. Figure 4 in this paper shows these observed frequencies as a function of γ_0, which is the Lorentz factor averaged over each oscillation period $2\pi/\omega$ and should be equal to the zeroth-order γ in the theory. The simulation results (the dots) fit well to the theoretical curve (the solid line) given by Eq. (54), where we used simulation values for $\boldsymbol{B}\cdot\boldsymbol{v}_0$ averaged over the acceleration time (from $\omega_{pe}t = 230$ to $\omega_{pe}t = 1000$).

To examine whether or not $\boldsymbol{v}_1$ is perpendicular to $\boldsymbol{U}$, we show in Fig. 5 time variation of $|(\boldsymbol{U}/U)\cdot\boldsymbol{v}_1|/\langle v_1\rangle$, where the observed $\boldsymbol{v}_1$ was given by $\boldsymbol{v}_1 = \boldsymbol{v} - \boldsymbol{v}_0$ with $\boldsymbol{v}_0$ the average velocity, and $\langle v_1\rangle$ is time-averaged $|\boldsymbol{v}_1|$. The values of $|(\boldsymbol{U}/U)\cdot\boldsymbol{v}_1|/\langle v_1\rangle$ are always much smaller than unity; shortly after the encounter with the shock wave, i.e., for $\omega_{pe}t \gtrsim 300$, they are mostly less than 0.1. Hence, $\boldsymbol{v}_1$ is nearly perpendicular to $\boldsymbol{U}$, which is in accord with the theory.

VI. SUMMARY

We have analytically studied motions of highly relativistic particles in a plasma wave propagating obliquely to an external magnetic field. First, assuming that $v\,d\gamma/dt \gg \gamma dv/dt$, we obtained the zeroth-order motion in the wave. The particle velocity is almost constant, with $v_x \simeq v_{\rm sh}$. The Lorentz factor γ linearly grows with time. When $v_{\rm sh} \sim c\cos\theta$, the velocity is nearly parallel to the external magnetic field. Then, perturbations to the zeroth-order solution are investigated. The ion perturbation is one dimensional with the oscillation frequency $\omega \sim \Omega_{i0}\gamma^{-1/2}$, while the positron perturbation is elliptic with $\omega \sim \Omega_{p0}\gamma^{-1}$. The latter has been compared with the recent simulation results, where some positrons were accelerated to $\gamma \sim 600$.

As future work, it would be interesting and important to carry out simulations on the motion of highly relativistic ions and compare with the present theory.

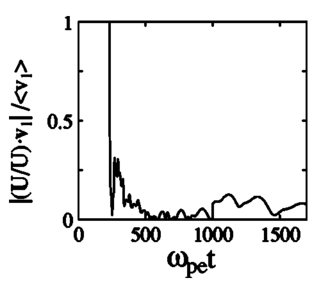

FIG. 5. Time variation of $|(\boldsymbol{U}/U)\cdot\boldsymbol{v}_1|/\langle v_1\rangle$. Here, the brackets indicate time averaging. The fact that this quantity is much smaller than unity shows that the perturbed velocity $\boldsymbol{v}_1$ is nearly perpendicular to $\boldsymbol{U}$.

ACKNOWLEDGMENTS

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APPENDIX A: PERTURBATIONS OF ION MOTION

Here, we show calculations to obtain ω given by Eq. (44). For $\boldsymbol{x}$ and $\boldsymbol{v}$, we use the expansion scheme (31) and (32). The orders of γ, $d\boldsymbol{v}_0/dt$, and $\boldsymbol{v}_0\cdot\boldsymbol{v}_1$ are given in Eqs. (33)–(39).

First, we discuss the expansion of γ. With the aid of Eq. (32) and the relation $\gamma = (1-\boldsymbol{v}^2)^{-1/2}$, we expand γ as

$$\gamma = \gamma_0 + \delta\gamma_0 + \epsilon'\gamma_0^3(\boldsymbol{v}_0\cdot\boldsymbol{v}_1) + \cdots, \tag{A1}$$

where $\gamma_0 = (1-\boldsymbol{v}_0^2)^{-1/2}$, and $\delta\gamma_0$ represents a small correction to γ_0, which we have obtained neglecting $d\boldsymbol{v}_0/dt$. In fact, it follows from the exact equation of motion, Eq. (7), that

$$\frac{d\gamma}{dt} = \boldsymbol{E}\cdot\boldsymbol{v}, \tag{A2}$$

while Eq. (8) which does not include $d\boldsymbol{v}_0/dt$ gives

$$v_0^2\frac{d\gamma_0}{dt} = \boldsymbol{E}\cdot\boldsymbol{v}_0. \tag{A3}$$

Equations (A2) and (A3) indicate that $d\gamma_0/dt$ differs from the true $d\gamma/dt$ by an amount $d(\gamma-\gamma_0)/dt \sim -\boldsymbol{E}\cdot\boldsymbol{v}_0/(v_0^2\gamma_0^2) \sim \Gamma^{-2}$, even if $\boldsymbol{v}_0$ is very close to the true velocity $\boldsymbol{v}$. This small difference $\delta\gamma_0$ exists even in the absence of the perturbation $\boldsymbol{v}_1$ which is considered below. The time derivative of γ is given as

$$\frac{d\gamma}{dt} = \frac{d\gamma_0}{dt} + \frac{d\delta\gamma_0}{dt} + \epsilon'\gamma_0^3\left(\boldsymbol{v}_0\cdot\frac{d\boldsymbol{v}_1}{dt}\right) + \epsilon'\gamma_0^3\left(\frac{d\boldsymbol{v}_0}{dt}\cdot\boldsymbol{v}_1\right) + \epsilon' 3\gamma_0^2\frac{d\gamma_0}{dt}(\boldsymbol{v}_0\cdot\boldsymbol{v}_1) + \cdots. \tag{A4}$$

It is supposed that $\delta\gamma_0$ varies with the same time scale as γ_0.

We then find the zeroth-order equation of motion as

$$\frac{d\gamma_0}{dt}\boldsymbol{v}_0 = \boldsymbol{E}(\xi_0) + \boldsymbol{v}_0\times\boldsymbol{B}(\xi_0), \tag{A5}$$

which is the same as Eq. (8).

The relation between $\delta\gamma_0$ and $\boldsymbol{v}_0$ may be given as

$$\frac{d\delta\gamma_0}{dt}\boldsymbol{v}_0+\gamma_0\frac{d\boldsymbol{v}_0}{dt}+\delta\gamma_0\frac{d\boldsymbol{v}_0}{dt}=0. \tag{A6}$$

Equation (A6) can be integrated to give

$$\boldsymbol{v}_0(t)=\boldsymbol{v}_0(0)\exp\left(-\int\frac{d\delta\gamma_0/dt}{\gamma_0+\delta\gamma_0}dt\right). \tag{A7}$$

In the first order of ϵ' (or ϵ), the equation of motion takes the following form:

$$\begin{aligned}&\epsilon'\gamma_0^3\left(\frac{d\boldsymbol{v}_0}{dt}\cdot\boldsymbol{v}_1\right)\boldsymbol{v}_0+\epsilon' 3\gamma_0^2\frac{d\gamma_0}{dt}(\boldsymbol{v}_0\cdot\boldsymbol{v}_1)\boldsymbol{v}_0\\&+\epsilon'\gamma_0^3\left(\boldsymbol{v}_0\cdot\frac{d\boldsymbol{v}_1}{dt}\right)\boldsymbol{v}_0+\epsilon'\frac{d\gamma_0}{dt}\boldsymbol{v}_1+\epsilon'\frac{d\delta\gamma_0}{dt}\boldsymbol{v}_1\\&+\epsilon'\gamma_0\frac{d\boldsymbol{v}_1}{dt}+\epsilon'\delta\gamma_0\frac{d\boldsymbol{v}_1}{dt}+\epsilon'\gamma_0^3(\boldsymbol{v}_0\cdot\boldsymbol{v}_1)\frac{d\boldsymbol{v}_0}{dt}\\&=\epsilon\frac{d\boldsymbol{E}}{d\xi_0}x_1+\epsilon\boldsymbol{v}_0\times\frac{d\boldsymbol{B}}{d\xi_0}x_1+\epsilon'\boldsymbol{v}_1\times\boldsymbol{B}(\xi_0).\end{aligned} \tag{A8}$$

By virtue of assumptions (33)–(39), we can neglect some of the terms in Eq. (A8). For instance, the first term on the left-hand side, $\epsilon'\gamma_0^3[(d\boldsymbol{v}_0/dt)\cdot\boldsymbol{v}_1]\boldsymbol{v}_0$, is smaller than the third term, $\epsilon'\gamma_0^3[\boldsymbol{v}_0\cdot(d\boldsymbol{v}_1/dt)]\boldsymbol{v}_0$, by a factor of $\Gamma^{-5/2}$. On the other hand, because of the assumption that $\boldsymbol{v}_1$ is nearly perpendicular to $\boldsymbol{v}_0$, Eq. (38), the third and sixth terms on the left-hand side of Eq. (A8) are the same order. We thereby find that

$$\gamma_0^3\left(\boldsymbol{v}_0\cdot\frac{d\boldsymbol{v}_1}{dt}\right)\boldsymbol{v}_0+\gamma_0\frac{d\boldsymbol{v}_1}{dt}=\frac{d\boldsymbol{E}}{d\xi_0}x_1+\boldsymbol{v}_0\times\frac{d\boldsymbol{B}}{d\xi_0}x_1. \tag{A9}$$

In the $O(\epsilon^2)$ equation, we have terms such as $3\gamma_0^5v_{x0}^2v_{x1}(dv_{x1}/dt)\boldsymbol{v}_0$, which is smaller than the terms of Eq. (A9) when

$$\epsilon<\Gamma^{-7/2}. \tag{A10}$$

We suppose that ϵ is so small that (A10) is satisfied.

Assuming that $\boldsymbol{v}_1$ and $\boldsymbol{x}_1$ vary with time as $\exp(-i\omega t)$, we obtain from Eq. (A9) the following relations:

$$\begin{aligned}&\left[-i\omega\gamma_0^3v_{x0}^2-i\omega\gamma_0-\left(\frac{dE_x}{d\xi_0}+v_{y0}\frac{dB_z}{d\xi_0}-v_{z0}\frac{dB_y}{d\xi_0}\right)\frac{1}{(-i\omega)}\right]v_{x1}\\&+(-i\omega\gamma_0^3v_{x0}v_{y0})v_{y1}+(-i\omega\gamma_0^3v_{x0}v_{z0})v_{z1}=0,\end{aligned} \tag{A11}$$

$$\begin{aligned}&(-i\omega\gamma_0^3v_{x0}v_{y0})v_{x1}+(-i\omega\gamma_0^3v_{y0}^2-i\omega\gamma_0)v_{y1}\\&+(-i\omega\gamma_0^3v_{y0}v_{z0})v_{z1}=0,\end{aligned} \tag{A12}$$

$$\begin{aligned}&(-i\omega\gamma_0^3v_{x0}v_{z0})v_{x1}+(-i\omega\gamma_0^3v_{y0}v_{z0})v_{y1}\\&+(-i\omega\gamma_0^3v_{z0}^2-i\omega\gamma_0)v_{z1}=0,\end{aligned} \tag{A13}$$

where we have used the relations

$$\frac{dE_y}{d\xi_0}-v_{x0}\frac{dB_z}{d\xi_0}=0, \tag{A14}$$

$$\frac{dE_z}{d\xi_0}+v_{x0}\frac{dB_y}{d\xi_0}=0, \tag{A15}$$

which result from Faraday's law. The condition for a nontrivial solution of Eqs. (A11)–(A13) reads as

$$\begin{aligned}&\left[-i\omega\gamma_0^3v_{x0}^2-i\omega\gamma_0+\frac{1}{i\omega}\left(\frac{dE_x}{d\xi_0}+v_{y0}\frac{dB_z}{d\xi_0}-v_{z0}\frac{dB_y}{d\xi_0}\right)\right](-i\omega\gamma_0^3v_{y0}^2-i\omega\gamma_0)(-i\omega\gamma_0^3v_{z0}^2-i\omega\gamma_0)+2i\omega^3v_{x0}^2v_{y0}^3v_{z0}^2\gamma_0^9\\&+\omega^2v_{x0}^2v_{z0}^2\gamma_0^6(-i\omega\gamma_0^3v_{y0}^2-i\omega\gamma_0)+\left[-i\omega\gamma_0^3v_{x0}^2-i\omega\gamma_0+\frac{1}{i\omega}\left(\frac{dE_x}{d\xi_0}+v_{y0}\frac{dB_z}{d\xi_0}-v_{z0}\frac{dB_y}{d\xi_0}\right)\right]\\&\times\omega^2v_{y0}^2v_{z0}^2\gamma_0^6+\omega^2v_{x0}^2v_{y0}^2\gamma_0^6(-i\omega\gamma_0^3v_{z0}^2-i\omega\gamma_0)=0.\end{aligned} \tag{A16}$$

On account of the relations $v_{x0}=v_{\rm sh}$ and $v_{x0}^2+v_{y0}^2+v_{z0}^2=1-\gamma_0^{-2}$, we obtain Eqs. (44) and (46).

APPENDIX B: PERTURBATIONS OF POSITRON MOTION

Here, we discuss perturbations of positron motion and calculate ω given by Eq. (54). As in the case of ions, we use the expansion scheme (31) and (32), and (A1)–(A4) for $\boldsymbol{x}$, $\boldsymbol{v}$, and γ. For the inner product $\boldsymbol{v}_0\cdot\boldsymbol{v}_1$, we assume Eq. (38). For $d\boldsymbol{v}_1/dt$, however, we assume that $d\boldsymbol{v}_1/dt\sim\Gamma^{-1}\boldsymbol{v}_1$, (48), and $\epsilon'\sim\Gamma^{-1}\epsilon$, (49); which are different from the ion ordering (36).

The zeroth-order equation of motion is then given by Eq. (8). The relation between $\delta\gamma_0$ and $\boldsymbol{v}_0$ is found as Eq. (A6).

In the first order of ϵ' (or ϵ), we obtain

$$\begin{aligned}&\epsilon'\gamma_0^3\left(\frac{d\boldsymbol{v}_0}{dt}\cdot\boldsymbol{v}_1\right)\boldsymbol{v}_0+\epsilon' 3\gamma_0^2\frac{d\gamma_0}{dt}(\boldsymbol{v}_0\cdot\boldsymbol{v}_1)\boldsymbol{v}_0\\&+\epsilon'\gamma_0^3\left(\boldsymbol{v}_0\cdot\frac{d\boldsymbol{v}_1}{dt}\right)\boldsymbol{v}_0+\epsilon'\frac{d\gamma_0}{dt}\boldsymbol{v}_1+\epsilon'\frac{d\delta\gamma_0}{dt}\boldsymbol{v}_1\\&+\epsilon'\gamma_0\frac{d\boldsymbol{v}_1}{dt}+\epsilon'\delta\gamma_0\frac{d\boldsymbol{v}_1}{dt}+\epsilon'\gamma_0^3(\boldsymbol{v}_0\cdot\boldsymbol{v}_1)\frac{d\boldsymbol{v}_0}{dt}\\&=\epsilon\frac{d\boldsymbol{E}}{d\xi_0}x_1+\epsilon\boldsymbol{v}_0\times\frac{d\boldsymbol{B}}{d\xi_0}x_1+\epsilon'\boldsymbol{v}_1\times\boldsymbol{B}(\xi_0).\end{aligned} \tag{B1}$$

We can neglect some of the terms in Eq. (B1). The magnitudes of $x_1 d\boldsymbol{E}/d\xi_0$ and $x_1 d\boldsymbol{B}/d\xi_0$ are small if the wavelength is greater than the magnitude of x_1 ($\sim c/\Omega_{p0}$). For the magnetosonic wave, for instance, we have

$$x_1 \frac{dB}{d\xi_0} \sim \frac{B(c/\Omega_{p0})}{(c/\omega_{pi})}, \tag{B2}$$

because the characteristic width of nonlinear oblique magnetosonic waves is $\sim c/\omega_{pi}$. From the ordering of $d\boldsymbol{v}_0/dt$, Eq. (34), and $d\boldsymbol{v}_1/dt$, Eq. (48), we see that $|\boldsymbol{v}_1 \cdot (d\boldsymbol{v}_0/dt)| \ll |\boldsymbol{v}_0 \cdot (d\boldsymbol{v}_1/dt)|$. Also, we assume that

$$|\boldsymbol{v}_1 d\gamma_0/dt| \ll |\gamma_0 d\boldsymbol{v}_1/dt|. \tag{B3}$$

This will be examined later. We thus obtain the equation to solve:

$$\gamma_0^3 \left(\boldsymbol{v}_0 \cdot \frac{d\boldsymbol{v}_1}{dt} \right) \boldsymbol{v}_0 + \gamma_0 \frac{d\boldsymbol{v}_1}{dt} = \boldsymbol{v}_1 \times \boldsymbol{B}. \tag{B4}$$

Assuming that $\boldsymbol{v}_1$ varies with time as $\exp(-i\omega t)$, we put Eq. (B4) into the form of the following equations:

$$(-i\omega\gamma_0^3 v_{x0}^2 - i\omega\gamma_0)v_{x1} + (-i\omega\gamma_0^3 v_{x0}v_{y0} - B_z)v_{y1} + (-i\omega\gamma_0^3 v_{x0}v_{z0} + B_y)v_{z1} = 0, \tag{B5}$$

$$(-i\omega\gamma_0^3 v_{x0}v_{y0} + B_z)v_{x1} + (-i\omega\gamma_0^3 v_{y0}^2 - i\omega\gamma_0)v_{y1} + (-i\omega\gamma_0^3 v_{y0}v_{z0} - B_x)v_{z1} = 0, \tag{B6}$$

$$(-i\omega\gamma_0^3 v_{x0}v_{z0} - B_y)v_{x1} + (-i\omega\gamma_0^3 v_{y0}v_{z0} + B_x)v_{y1} + (-i\omega\gamma_0^3 v_{z0}^2 - i\omega\gamma_0)v_{z1} = 0. \tag{B7}$$

The condition for a nontrivial solution of Eqs. (B5)–(B7) gives

$$\begin{aligned} &-(i\omega\gamma_0^3 v_{x0}^2 + i\omega\gamma_0)(i\omega\gamma_0^3 v_{y0}^2 + i\omega\gamma_0)(i\omega\gamma_0^3 v_{z0}^2 + i\omega\gamma_0) - (i\omega\gamma_0^3 v_{x0}v_{y0} + B_z)(i\omega\gamma_0^3 v_{y0}v_{z0} + B_x)(i\omega\gamma_0^3 v_{x0}v_{z0} + B_y) \\ &\quad + (-i\omega\gamma_0^3 v_{x0}v_{z0} + B_y)(-i\omega\gamma_0^3 v_{x0}v_{y0} + B_z)(-i\omega\gamma_0^3 v_{y0}v_{z0} + B_x) - (\omega^2\gamma_0^6 v_{x0}^2 v_{z0}^2 + B_y^2)(i\omega\gamma_0^3 v_{y0}^2 + i\omega\gamma_0) \\ &\quad - (\omega^2\gamma_0^6 v_{y0}^2 v_{z0}^2 + B_x^2)(i\omega\gamma_0^3 v_{x0}^2 + i\omega\gamma_0) - (\omega^2\gamma_0^6 v_{x0}^2 v_{y0}^2 + B_z^2)(i\omega\gamma_0^3 v_{z0}^2 + i\omega\gamma_0) = 0. \end{aligned} \tag{B8}$$

Using the relation $v_{x0}^2 + v_{y0}^2 + v_{z0}^2 = 1 - \gamma_0^{-2}$, we obtain Eqs. (54)–(56).

In order to have a physical picture of this oscillation, let us discuss this in a slightly different manner. Scalar multiplication of Eq. (B4) by $\boldsymbol{v}_0$ gives

$$\gamma_0^3 \left(\boldsymbol{v}_0 \cdot \frac{d\boldsymbol{v}_1}{dt} \right) = \boldsymbol{B} \cdot (\boldsymbol{v}_0 \times \boldsymbol{v}_1), \tag{B9}$$

where the relation $\boldsymbol{v}_0 \cdot (\boldsymbol{v}_1 \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\boldsymbol{v}_0 \times \boldsymbol{v}_1)$ was used. Equation (B9) suggests that the vector parallel to $\boldsymbol{v}_0$,

$$\boldsymbol{v}_{1\parallel} = (\boldsymbol{v}_1 \cdot \boldsymbol{v}_0)\boldsymbol{v}_0/v_0^2, \tag{B10}$$

is much smaller in magnitude than the vector perpendicular to $\boldsymbol{v}_0$,

$$\boldsymbol{v}_{1\perp} = \boldsymbol{v}_1 - (\boldsymbol{v}_1 \cdot \boldsymbol{v}_0)\boldsymbol{v}_0/v_0^2. \tag{B11}$$

Using $\boldsymbol{v}_{1\parallel}$ and $\boldsymbol{v}_{1\perp}$, we write Eq. (B4) as

$$\gamma_0^3 v_0 \frac{dv_{1\parallel}}{dt} \boldsymbol{v}_0 + \gamma_0 \left(\frac{d\boldsymbol{v}_{1\parallel}}{dt} + \frac{d\boldsymbol{v}_{1\perp}}{dt} \right) = (\boldsymbol{v}_{1\parallel} + \boldsymbol{v}_{1\perp}) \times \boldsymbol{B}. \tag{B12}$$

The parallel component is the same as Eq. (B9):

$$\frac{dv_{1\parallel}}{dt} = \frac{(\boldsymbol{v}_{1\perp} \times \boldsymbol{B}) \cdot \boldsymbol{v}_0}{\gamma_0^3 v_0}. \tag{B13}$$

If $\boldsymbol{v}_{1\perp}$ is known, then $\boldsymbol{v}_{1\parallel}$ will be found. The perpendicular motion $\boldsymbol{v}_{1\perp}$ is the dominant perturbation. Because of the difference in the directions of $\boldsymbol{B}$ and $\boldsymbol{v}_0$, $\boldsymbol{v}_{1\perp}$ causes the oscillation of $v_{1\parallel}$ with a much smaller amplitude.

We now consider the perpendicular component of Eq. (B12):

$$\gamma_0 \frac{d\boldsymbol{v}_{1\perp}}{dt} = \boldsymbol{v}_{1\perp} \times \boldsymbol{B} - \frac{[(\boldsymbol{v}_{1\perp} \times \boldsymbol{B}) \cdot \boldsymbol{v}_0]\boldsymbol{v}_0}{v_0^2}, \tag{B14}$$

where $\boldsymbol{v}_{1\parallel}$ on the right-hand side of Eq. (B12) was neglected because $|\boldsymbol{v}_{1\parallel}| \ll |\boldsymbol{v}_{1\perp}|$. We introduce a new coordinate system (x', y', z'), where the z' axis is parallel to $\boldsymbol{v}_0$, and the x' axis is in the plane determined by $\boldsymbol{B}$ and $\boldsymbol{v}_0$; the x' axis is parallel to the vector $(\boldsymbol{B} \cdot \boldsymbol{v}_0)\boldsymbol{v}_0/v_0^2 - \boldsymbol{B}$. Hence, $\boldsymbol{B}$ and $\boldsymbol{v}_{1\perp}$ may be written as $\boldsymbol{B} = (B_{x'}, 0, B_{z'})$ and $\boldsymbol{v}_{1\perp} = (v_{x'}, v_{y'}, 0)$ (see Fig. 6). The quantities $B_{x'}$ and $B_{z'}$ are given as

$$B_{x'} = -[B^2 - (\boldsymbol{B} \cdot \boldsymbol{v}_0)^2/v_0^2]^{1/2}, \tag{B15}$$

$$B_{z'} = \boldsymbol{B} \cdot \boldsymbol{v}_0/v_0. \tag{B16}$$

We then have an equation for $(v_{x'}, v_{y'})$:

$$\gamma_0 \frac{d}{dt}(v_{x'}, v_{y'}) = (v_{y'}B_{z'}, -v_{x'}B_{z'}). \tag{B17}$$

Hence, we find the motion perpendicular to $\boldsymbol{v}_0$ as

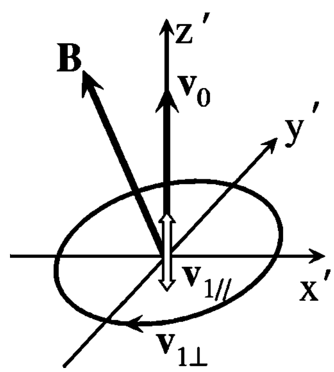

FIG. 6. Schematic diagram of the coordinate system (x', y', z'). The z' axis is parallel to $\boldsymbol{v}_0$. The x' axis, $\boldsymbol{B}$, and $\boldsymbol{v}_0$ are in the same plane. Also, $\boldsymbol{v}_{1\parallel}$ is along the z' axis, and $\boldsymbol{v}_{1\perp}$ is in the (x', y') plane.

$$(v_{x'},v_{y'})=v'_{\perp}\left[\cos\left(-\frac{B_{z'}t}{\gamma_0}+\eta\right),\ \sin\left(-\frac{B_{z'}t}{\gamma_0}+\eta\right)\right],\tag{B18}$$

where $v'_{\perp}$ and η are constant.

Since $(\boldsymbol{v}_{1\perp}\times\boldsymbol{B})\cdot\boldsymbol{v}_0=-v_0v_{y'}B_{x'}$, Eq. (B13) becomes

$$\frac{dv_{1\parallel}}{dt}=-\frac{B_{x'}v'_{\perp}}{\gamma_0^3}\sin\left(-\frac{B_{z'}t}{\gamma_0}+\eta\right).\tag{B19}$$

We thus obtain the motion parallel to $\boldsymbol{v}_0$,

$$v_{1\parallel}=-\frac{B_{x'}v'_{\perp}}{B_{z'}\gamma_0^2}\cos\left(-\frac{B_{z'}t}{\gamma_0}+\eta\right).\tag{B20}$$

The amplitude of $v_{1\parallel}$ is a factor of $\sim\gamma_0^{-2}$ times as small as that of $\boldsymbol{v}_{1\perp}$. The frequency in this approximate solution agrees with the one given by Eq. (54) in the limit of $\gamma_0\to\infty$.

In the above calculations, we have assumed Eq. (B3). This assumption is met if v_{y0} is much smaller than v_{z0}. In fact, in this case, Eq. (12) indicates that $d\gamma_0/dt\ll 1$. On the other hand, Eq. (54) shows that $\omega\sim B/\gamma_0$. We thus find that $|i\omega\gamma_0|\gg d\gamma_0/dt$; Eq. (B3) is satisfied. We note that in the simulations[27] where positrons were accelerated to $\gamma\sim 600$ with this mechanism, v_{y0} was always much smaller than v_{z0}.

If $v_{y0}\sim v_{z0}$ were realized, however, $d\gamma_0/dt$ would become of the order of unity. We would then find that $|i\omega\gamma_0|\sim d\gamma_0/dt$. In this case, we should solve the following equation:

$$\gamma_0^2(\boldsymbol{v}_0\cdot\boldsymbol{v}_1)\left(3\frac{d\gamma_0}{dt}-i\omega\gamma_0\right)\boldsymbol{v}_0+\left(\frac{d\gamma_0}{dt}-i\omega\gamma_0\right)\boldsymbol{v}_1=\boldsymbol{v}_1\times\boldsymbol{B}.\tag{B21}$$

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