

Free energy of solvation for the reference interaction site model: Critical comparison of expressions

Seiichiro Ten-no^{a)}

Graduate School of Human Informatics, Nagoya University, Chikusa-ku Nagoya 464-8601, Japan

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We investigate expressions of excess chemical potential in the reference interaction site model (RISM) integral equation theory. In addition to the previous expressions from the Gaussian density fluctuation theory and from the extended RISM (XRISM) theory, we examine a new free energy functional from the distributed partial wave expansion of molecular correlation functions, using the embedded site model and alcohols with different parameter sets. The results clearly show that the free energy of solvation in the XRISM theory includes a serious error, which is related to the number of interaction sites and the geometry of a solute molecule. © 2001 American Institute of Physics. [DOI: 10.1063/1.1389851]

I. INTRODUCTION

The importance of the solvation effects in chemical and biological processes is widely appreciated. During the past three decades, a lot of effort has been devoted to the development of the equilibrium theory of molecular liquids. In particular, it is proved that integral equation theories can provide a fast and rigorous statistical mechanical approach incorporating the microscopic nature in describing solvation phenomena.^{1,2} The high dimensionality of a coordinate system for describing molecular correlation functions, however, has prohibited a straightforward implementation of the theory with the exception of a very simple molecule. The direct treatment of molecular correlation functions was studied by Fries and Patey³ in terms of a powerful expansion method developed by Blum and Torruella⁴ and by Blum.⁵ The method was applied to polar fluids consisting of spherically shaped molecules.⁶ Independently, Lado proposed a spherical harmonics expansion for fluids of linear molecules and applied the theory to hard dumbbell solutions.⁷ This was later extended to a method for fluids of fully anisotropic molecules by Lado, Lomba, and Lombardero.⁸ So far, there exists no integral equation theory for molecular liquids, which is readily applicable as well as more sophisticated than the above-mentioned theories, though they include a difficulty in treating chemical reactions and molecules with high anisotropy like proteins.

There are some theories that are specifically designed to treat interaction site fluids in terms of spherically symmetric site–site interactions and site–site correlations. The theory of Chandler and Andersen,⁹ called the reference interaction site model (RISM) integral equation theory and originally derived for the liquids of rigidly connected hard spheres, motivated the later developments of approximate methods for equilibrium statistical mechanics of interaction site fluids. The extended RISM (XRISM),^{10–12} which directly inserts a simple liquid closure into the site–site Ornstein–Zernike equation, has been extensively applied to a variety of phe-

nomena including charged molecular systems. However, the XRISM method includes ad hoc elements substantially and has no straightforward way to improve the accuracy systematically. On the positive side, Ladanyi and Chandler developed explicit expansions of interaction site diagrams in terms of the site–site functions.¹³ The expansions led to the later formulation of the diagrammatically proper integral equation theory,^{14,15} the expansions of which are without diagrams absent in the exact theory. It should be noted that the bridge diagrams including intramolecular correlation functions in the theory are particularly important for quantitative results.¹⁶

Another trend in recent years is the use of fast Fourier transforms of three dimensions.^{17–20} For instance, Cortis, Rosky, and Friesner²⁰ developed a three-dimensional reduction applicable to any molecular liquid using a derivation similar to that for the one-dimensional formula, i.e., the Chandler–Andersen equation, starting from the full six-dimensional Ornstein–Zernike equation. Applying an approximate closure, like the simple liquid one in the XRISM theory, the error will be decreased, though an inadequacy similar to that in XRISM is still there.

Evaluating the above-mentioned developments, we recently formulated another type of integral equation theory in terms of the distributed partial functions of molecular correlation functions.²¹ The theory rationalizes the RISM-type expansion assuming the existence of nonspherical components in site–site functions as higher order corrections. Applying the interaction site model, the unallowed diagrams in the interaction site formalism, which include circles intersected by more than one intramolecular correlation function, are identified as *improvable* truncated expressions of bonds for molecular circles. We also demonstrated that the theory is useful to derive approximate free energy expressions and closure relations in the RISM theory.

In this paper, we extend the partial wave expansion theory for the solvation of a polyatomic molecule to derive a rational expression of excess chemical potential. First, the previous expressions of the solvation free energy are outlined

^{a)}Electronic mail: tenno@info.human.nagoya-u.ac.jp

in Sec. II. We explain the basic ideas of the partial wave expansion formalism in Sec. III. The formalism is applied to the integral equation theory for solvation in Sec. IV. The excess chemical potential expression is derived in Sec. V. The expression is compared with previous ones within the RISM framework in Secs. VI and VII. The conclusion is given in Sec. VIII.

II. FREE ENERGY EXPRESSIONS IN THE RISM THEORY

Henceforward, the Greek and Roman subscripts, α, γ, \dots and i, j, \dots , refer to the interaction sites of solute and solvent molecules, respectively. The site-site Ornstein-Zernike (SSOZ) equation for a single polyatomic solute in a solvent,

$$\rho \tilde{h}_{\alpha i}(r) = \sum_{\gamma j} \omega_{\alpha \gamma} * c_{\gamma j} * \chi_{ji}(r), \quad (1)$$

defines $c_{\gamma j}$ in the RISM theory, the so-called site-site direct correlation functions, where ρ is the number density of solvent molecules, the asterisk denotes convolution integrals, χ_{ij} are the site density pair correlation functions of the pure solvent,

$$\chi_{ij}(|\mathbf{r}_i - \mathbf{r}_j|) = \langle \delta \rho_i(\mathbf{r}_i) \delta \rho_j(\mathbf{r}_j) \rangle, \quad (2)$$

and $\omega_{\alpha \gamma}(r)$ is the probability density of finding the sites at the subscripts distant by r . For rigid-body molecules, this has the form

$$\omega_{\alpha \gamma}(r) = \frac{1}{4\pi R_{\alpha \gamma}^2} \delta(r - R_{\alpha \gamma}) \quad (3)$$

for the site-site separation, $R_{\alpha \gamma}$. We used the notation, $\tilde{\mathbf{h}}$, for the site-site correlation functions rather than the usual one, \mathbf{h} , because $\tilde{\mathbf{h}}$ includes all contributions of \mathbf{c} over the interaction sites through the intramolecular correlation functions, $\omega_{\alpha \gamma}(r)$. The site-site correlation functions in the RISM theory, $\tilde{\mathbf{h}}$ and \mathbf{c} , cannot be treated on the same footing, as will be seen later.

The expression of the solvation free energy with an assumption of a Gaussian density field (GF) in the behavior of the solvent bath,²² $\Delta \mu^{(\text{GF})}$, is given by

$$\Delta \mu^{(\text{GF})} = \Delta \mu_c - \frac{\rho}{2\beta} \sum_{\alpha i} \int 4\pi r^2 \tilde{h}_{\alpha i}(r) c_{\alpha i}(r) dr, \quad (4)$$

where $\Delta \mu_c$ is the component of a single set of site-site direct correlation functions,

$$\Delta \mu_c = - \frac{\rho}{\beta} \sum_{\alpha i} \int 4\pi r^2 c_{\alpha i}(r) dr. \quad (5)$$

With another assumption the extended RISM (XRISM) theory (where the simple liquid closure is inserted directly into the SSOZ equation), the resulting solvation free energy of the hypernetted-chain-like closure includes an additional term,²³ which is absent from the Gaussian field theory,

$$\Delta \mu^{(\text{HNC})} = \Delta \mu_c + \frac{\rho}{2\beta} \sum_{\alpha i} \int 4\pi r^2 \tilde{h}_{\alpha i}(r) [\tilde{h}_{\alpha i}(r) - c_{\alpha i}(r)] dr. \quad (6)$$

This expression reduces to the exact HNC free energy expression in the extended atom limit. The XRISM theory, however, takes no account of the molecular shape in the closure. Consequently, the resulting XRISM-HNC free energy is incorrect in the united atom limit even if the correlation functions are exact in the limit. The unphysical property of the functional form was pointed out in the total free energy case.²¹ In the following sections, we derive another expression for the free energy of solvation that is correct both in the extended and united atom limits.

III. DISTRIBUTED PARTIAL WAVE EXPANSION

Let us suppose a spatial distribution, $X(\mathbf{r} - \mathbf{r}_M)$, of simple (spherical) particles interacting with a polyatomic molecule M fixed in the space (like the electronic charge distribution, a canonical orbital of an electron, and the correlation function of a simple liquid around the molecule). We use the same notation for the molecular configuration in terms of the position vector of the molecular origin, \mathbf{r}_M (for instance, the center of mass), and a set of Euler angles, $\hat{\Omega}_M$,

$$M = \{\mathbf{r}_M, \hat{\Omega}_M\}, \quad (M = 1, 2, \dots). \quad (7)$$

The distribution can be expanded into the angular components about the position of the molecular origin as

$$X(\mathbf{r}) = \sum_{lm} X^{(lm)}(|\mathbf{r}|) Y_{lm}(\hat{\mathbf{r}}), \quad (8)$$

where $Y_{lm}(\hat{\mathbf{r}})$ denotes the spherical harmonics and $\hat{\mathbf{r}}$ is the unit vector in the direction of \mathbf{r} . However, the truncation of angular functions in practical applications loses the resolution of the distribution around atoms far from the molecular origin.

This feature can be mitigated by decomposing the distribution into partial functions around atomic sites,

$$\begin{aligned} X(\mathbf{r} - \mathbf{r}_M) &= \sum_{\alpha} X_{\alpha}(\mathbf{r} - \mathbf{r}_{\alpha}) \\ &= \sum_{\alpha} X_{\alpha}(\mathbf{r} - \mathbf{r}_M - \mathbf{R}_{\alpha M}(\hat{\Omega}_M)), \end{aligned} \quad (9)$$

where we use notations such as $\mathbf{R}_{\alpha M}(\hat{\Omega}_M)$ and $\mathbf{R}_{\alpha \gamma}(\hat{\Omega}_M)$ for the intramolecular site-site displacement vectors. For simplicity, we do not distinguish the notations for the interaction sites and the atomic (expansion) sites. The distributed partial functions are called *partial waves* (PWs). One notes the distinction between the present multicenter decomposition and the angular one around a single molecular origin. Each PW is generally nonorthogonal to the others and is further expanded into angular components about the corresponding expansion site as

$$X_{\alpha}(\mathbf{r}) = \sum_{lm} X_{\alpha}^{(lm)}(|\mathbf{r}|) Y_{lm}(\hat{\mathbf{r}}). \quad (10)$$

The advantage of employing the distributed PWs will be manifested if we suppose the expansions of molecular orbitals, i.e., the superiority of the linear combination of atomic orbitals over the one-center expansion around the molecular origin.

It is convenient to define auxiliary functions, in which the position vectors are referred to the atomic sites rather than the molecular origin,

$$\tilde{X}_\alpha(\mathbf{r}-\mathbf{r}_\alpha)=X(\mathbf{r}-\mathbf{r}_M)=\sum_\gamma X_\gamma(\mathbf{r}-\mathbf{r}_\gamma), \quad \forall \alpha. \quad (11)$$

Using the Dirac delta function in three dimension, they become

$$\begin{aligned} \tilde{X}_\alpha(\mathbf{r}) &= \sum_\gamma \int d\mathbf{r}' w_{\alpha\gamma}(\mathbf{r}-\mathbf{r}', \hat{\Omega}_1) X_\gamma(\mathbf{r}') \\ &= \sum_\gamma w_{\alpha\gamma} * X_\gamma(\mathbf{r}), \end{aligned} \quad (12)$$

where the “full” intramolecular correlation functions, \mathbf{w} , are in the form

$$w_{\alpha\gamma}(\mathbf{r}, \hat{\Omega}_M) = \delta(\mathbf{r}-\mathbf{R}_{\alpha\gamma}(\hat{\Omega}_M)). \quad (13)$$

The convolution integrals including \mathbf{w} are easily calculated using the Rayleigh expansion of the plane wave,

$$\begin{aligned} w_{\alpha\gamma}(\mathbf{k}, \hat{\Omega}_M) &= 4\pi \sum_l (-i)^l j_l(kR_{\alpha\gamma}) \sum_m \\ &\quad \times Y_{lm}(\hat{\mathbf{k}}) Y_{lm}^*(\hat{\mathbf{R}}_{\alpha\gamma}(\hat{\Omega}_M)), \end{aligned} \quad (14)$$

where the underline denotes Fourier transforms and j_l are spherical Bessel functions. The spherical ($l=0$) components of \mathbf{w} are identical to the intramolecular correlation functions in the RISM theory,

$$\langle w_{\alpha\gamma}(\mathbf{r}, \hat{\Omega}_M) \rangle_{\hat{\Omega}_M} = \omega_{\alpha\gamma}(|\mathbf{r}|). \quad (15)$$

Each auxiliary function is also expanded with the same basis set spanned at the site. Therefore, the function is actually a projected quantity of the distribution about the site. We term the auxiliary functions simply as *projections*.

Another important relation is for the integral of a distribution product,

$$\begin{aligned} &\int d\mathbf{r} X(\mathbf{r}-\mathbf{r}_M) Y(\mathbf{r}-\mathbf{r}_M) \\ &= \sum_{\alpha\gamma} \int d\mathbf{r} X_\alpha(\mathbf{r}-\mathbf{r}_\alpha) [w_{\alpha\gamma} * Y_\gamma(\mathbf{r}-\mathbf{r}_\alpha)] \\ &= \sum_\alpha \int d\mathbf{r} X_\alpha(\mathbf{r}) \tilde{Y}_\alpha(\mathbf{r}). \end{aligned} \quad (16)$$

It should be noted that *any integral including distributed PWs with different site indices must be bridged by the full intramolecular function with the indices*. This is equivalent to the integration of one DPW and another projection with a coincident site index. Otherwise, integrals like $\int d\mathbf{r} \tilde{X}_\alpha(\mathbf{r}) \tilde{Y}_\beta(\mathbf{r})$ are physically meaningless, because such an operation is not involved in the original integration of the distributions.

IV. THE RISM–OZ EQUATION IN THE PW FORMALISM

We apply the PW formalism to the theory of molecular liquids. There are two different ways of grasping the RISM theory. One treats the SSOZ as a definition of the site–site direct correlation functions, $\mathbf{c}(r)$, with a given set of site–site correlation functions, $\tilde{\mathbf{h}}(r)$. The diagrammatically proper integral equation theory^{14,15} implies that this is not a natural basis for constructing a systematic theory of interaction site fluids. The other way regards the SSOZ as an approximation. On this footing, we clarified the systematic relationship²¹ between the RISM theory and the integral equation theory based on the molecular OZ (MOZ) equation. In this work, we derive the RISM–OZ equation for solvation as an approximate expression of MOZ using the PW formalism.

The MOZ equations for the solute–solvent and pure solvent distributions are in the forms

$$H(1,2) = C(1,2) + \rho \int d\mathbf{r}_3 \langle C(1,3) \bar{H}(3,2) \rangle_{\hat{\Omega}_3}, \quad (17)$$

$$\bar{H}(1,2) = \bar{C}(1,2) + \rho \int d\mathbf{r}_3 \langle \bar{C}(1,3) \bar{H}(3,2) \rangle_{\hat{\Omega}_3}, \quad (18)$$

where the capital letters denote correlation functions in the molecular form, and the corresponding quantities with overbars refer to the functions for the pure solvent. For uniform liquids, the correlation functions are translationally symmetric,

$$X(1,2) = X(\mathbf{r}_1 - \mathbf{r}_2, \hat{\Omega}_1, \hat{\Omega}_2), \quad (X = H, C, \dots). \quad (19)$$

We can rewrite them applying the PW expansion to both the solute and solvent coordinates as

$$X(1,2) = \sum_{ai} X_{ai}(\mathbf{r}_\alpha - \mathbf{r}_i, \hat{\Omega}_1, \hat{\Omega}_2) = \sum_{ai} X_{ai}(1^\alpha, 2^i), \quad (20)$$

where the site indices at the molecular coordinates mean that the spatial positions are specified by the sites as

$$M^\alpha = \{\mathbf{r}_\alpha, \hat{\Omega}_M\} = \{\mathbf{R}_{\alpha M}(\hat{\Omega}_M) + \mathbf{r}_M, \hat{\Omega}_M\}. \quad (21)$$

In the absence of external field, the correlation functions are also invariant with respect to the simultaneous rotations about $\hat{\Omega}_1$, $\hat{\Omega}_2$, and $\hat{\mathbf{r}}_{12}$. Then we can restrict all of the PWs to be rotationally symmetric to be expanded as

$$X_{ai}(1^\alpha, 2^i) = \sum_{\mathbf{J}} X_{ai}^{(\mathbf{J})}(|\mathbf{r}_{\alpha\gamma}|) \Phi_{\mathbf{J}}(\hat{\Omega}_1, \hat{\Omega}_2, \hat{\mathbf{r}}_{\alpha\gamma}), \quad (22)$$

where $\Phi_{\mathbf{J}}$ are the rotational invariants introduced by Blum,^{4,5}

$$\begin{aligned} \Phi_{j_1 j_2 j n_1 n_2}(\hat{\Omega}_1, \hat{\Omega}_2, \hat{\mathbf{r}}) &= \sqrt{(2j_1+1)(2j_2+1)} \\ &\quad \times \sum_{m_1 m_2 m} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} D_{m_1 n_1}^{(j_1)} \\ &\quad \times (\hat{\Omega}_1) D_{m_2 n_2}^{(j_2)}(\hat{\Omega}_2) D_{m 0}^{(j)}(\hat{\mathbf{r}}), \end{aligned} \quad (23)$$

and $\mathbf{D}^{(j)}(\hat{\Omega})$ denotes the Wigner rotation matrix. It is noted that we can use the alternative expansion of Lado^{7,8} in terms of transformations of Euler angles choosing each site–site direction as a z axis. At this stage, it is possible to derive useful formulas including the rotational invariants and \mathbf{w} . We

do not discuss this in more detail, but we just mention that the nonspherical components of \mathbf{w} do not contribute to the convolution integrals of spherically symmetric functions,

$$\int d\mathbf{r}_3 \langle X_{\alpha\gamma}^{(0)}(|\mathbf{r}_{13}|) [w_{\gamma\kappa} * Y_{\kappa\lambda}^{(0)}(3,2)] \rangle_{\hat{\Omega}_3} = X_{\alpha\gamma}^{(0)*} \omega_{\gamma\kappa} * Y_{\kappa\lambda}^{(0)}(|\mathbf{r}_{12}|), \quad (24)$$

i.e., the integration over $\hat{\Omega}_3$ is decoupled, where $X_{\alpha\gamma}^{(0)}(|\mathbf{r}|)$ and $Y_{\kappa\lambda}^{(0)}(|\mathbf{r}|)$ are the $\mathbf{J}=\mathbf{0}$ components of the distributed PWs.

The projections of the total correlation functions for the solute–solvent and pure solvent distributions are given by

$$\tilde{H}_{ai}(1^\alpha, 2^i) = \sum_{\gamma j} w_{\alpha\gamma} * H_{\gamma j} * \bar{w}_{ji}(1^\alpha, 2^i), \quad (25)$$

$$\tilde{\tilde{H}}_{ij}(1^i, 2^j) = \sum_{kl} \bar{w}_{ik} * \bar{H}_{kl} * \bar{w}_{lj}(1^i, 2^j), \quad (26)$$

where we introduced the full intramolecular correlation function for the solvent molecule, \bar{w}_{ij} . The Euler angles must be coincident for the same site indices. The projections are rewritten using the molecular OZ and the integral relation as

$$\tilde{H}_{ai}(1^\alpha, 2^i) = \sum_{\gamma j} \left\{ w_{\alpha\gamma} * C_{\gamma j} * \bar{w}_{ji}(1^\alpha, 2^i) + \rho \int d\mathbf{r}_3 \langle [w_{\alpha\gamma} * C_{\gamma j}(1^\alpha, 3^j)] \tilde{\tilde{H}}_{ji}(3^j, 2^i) \rangle_{\hat{\Omega}_3} \right\}, \quad (27)$$

$$\tilde{\tilde{H}}_{ij}(1^i, 2^j) = \sum_{kl} \left\{ \bar{w}_{ik} * \bar{C}_{kl} * \bar{w}_{lj}(1^i, 2^j) + \rho \int d\mathbf{r}_3 \langle [\bar{w}_{ik} * \bar{C}_{kl}(1^i, 3^l)] \tilde{\tilde{H}}_{lj}(3^l, 2^j) \rangle_{\hat{\Omega}_3} \right\}. \quad (28)$$

From the definition of the projections, these are equivalent to the equations called the partial OZ (POZ) equations,

$$H_{ai}(1^\alpha, 2^i) = C_{ai}(1^\alpha, 2^i) + \rho \sum_{jk} C_{\alpha j} * \bar{w}_{jk} * \bar{H}_{ki}(1^\alpha, 2^i), \quad (29)$$

$$\bar{H}_{ij}(1^i, 2^j) = \bar{C}_{ij}(1^i, 2^j) + \rho \sum_{kl} \bar{C}_{ik} * \bar{w}_{kl} * \bar{H}_{lj}(1^i, 2^j). \quad (30)$$

The double arrows over \bar{w} denote averages over the Euler angles.

The striking relationship of the projection expressions with the SSOZ is not surprising because the RISM theory was originally derived assuming that the direct correlation function is a sum of site–site ones which are orientationally independent,

$$C_{ai}(1^\alpha, 2^i) \approx c_{ai}(|\mathbf{r}_\alpha - \mathbf{r}_i|), \quad (31)$$

$$\bar{C}_{ij}(1^i, 2^j) \approx \bar{c}_{ij}(|\mathbf{r}_i - \mathbf{r}_j|). \quad (32)$$

Consequently, the chain sums of the POZ equations become

$$h_{ai}(r) = c_{ai}(r) + \rho \sum_{jk} c_{\alpha j} * \bar{w}_{jk} * \bar{h}_{ki}(r), \quad (33)$$

$$\bar{h}_{ij}(r) = \bar{c}_{ij}(r) + \rho \sum_{kl} \bar{c}_{ik} * \bar{w}_{kl} * \bar{h}_{lj}(r), \quad (34)$$

where $\bar{w}_{ij}(r)$ is the intramolecular correlation function of solvent. This is the minimum model in the angular expansion of PWs in the present theory. If we are interested in the spherical components of the projections, the POZ reduces to the SSOZ in terms of the following identities:

$$\begin{aligned} \langle \tilde{H}_{ai}(1^\alpha, 2^i) \rangle_{\hat{\Omega}_1 \hat{\Omega}_2} &= \tilde{h}_{ai}(|\mathbf{r}_\alpha - \mathbf{r}_i|) \\ &= \sum_{\gamma j} \omega_{\alpha\gamma} * h_{\gamma j} * \bar{w}_{ji}(|\mathbf{r}_\alpha - \mathbf{r}_i|), \end{aligned} \quad (35)$$

$$\begin{aligned} \langle \tilde{\tilde{H}}_{ij}(1^i, 2^j) \rangle_{\hat{\Omega}_1 \hat{\Omega}_2} &= \tilde{\tilde{h}}_{ij}(|\mathbf{r}_i - \mathbf{r}_j|) \\ &= \sum_{kl} \bar{w}_{ik} * \bar{h}_{kl} * \bar{w}_{lj}(|\mathbf{r}_i - \mathbf{r}_j|). \end{aligned} \quad (36)$$

Correspondingly, the site density pair correlation functions of the solvent become

$$\chi_{ij}(r) = \rho [\bar{w}_{ij}(r) + \rho \tilde{\tilde{h}}_{ij}(r)]. \quad (37)$$

It is apparent that the spherical decomposition of the total correlation function is inappropriate at the overlap region.² We introduced the orientational dependence in the PWs such that the present decomposition is justified. The theory therefore focuses on the angular projections around the sites as the RISM theory does only on the site–site radial functions. Within the PW decomposition formalism, the RISM-type truncation, i.e., the minimum basis in the angular functions, is exact only for the molecules in the united and extended atom limits as well as for the embedded site model, which is discussed later.

V. FREE ENERGY OF SOLVATION

The free energy of solvation in the molecular integral equation theory is derived using the hypernetted chain (HNC) approximation as

$$\begin{aligned} \Delta\mu^{(\text{MOZ})} &= \frac{\rho}{2\beta} \int d\mathbf{r}_{12} \langle H^2(1,2) - H(1,2)C(1,2) \\ &\quad - 2C(1,2) \rangle_{\hat{\Omega}_1 \hat{\Omega}_2}. \end{aligned} \quad (38)$$

Applying the chain sum identity of the total correlation function, one can see that the second term cancels with a part of the first. Substituting the correlation functions in the PW form and using the integral relation, we obtain the free energy functional,

$$\begin{aligned} \Delta\mu^{(\text{PW})} &= \frac{\rho}{2\beta} \sum_{ai} \int d\mathbf{r}_{ai} \langle \tilde{H}_{ai}(1^\alpha, 2^i) H_{ai}(1^\alpha, 2^i) \\ &\quad - \tilde{H}_{ai}(1^\alpha, 2^i) C_{ai}(1^\alpha, 2^i) - 2C_{ai}(1^\alpha, 2^i) \rangle_{\hat{\Omega}_1 \hat{\Omega}_2}. \end{aligned} \quad (39)$$

The spherical expansion limit of the direct correlation functions leads to the functional applicable to the RISM theory,

$$\Delta\mu^{(\text{PW})} \approx \Delta\mu_c + \frac{\rho}{2\beta} \sum_{ai} \int 4\pi r^2 \tilde{h}_{ai}(r) [h_{ai}(r) - c_{ai}(r)] dr. \quad (40)$$

The expression, $\Delta\mu^{(\text{PW})}$, is valid both in the extended and united atom limits, if the correlation functions are exact in the limits. The free energy expression from the GF theory, $\Delta\mu^{(\text{GF})}$, does not satisfy this requirement in both of the limits, whereas the expression of the XRISM–HNC theory, $\Delta\mu^{(\text{HNC})}$, is incorrect only in the latter situation. This does not mean that the functional of the XRISM theory, $\Delta\mu^{(\text{HNC})}$, is superior to $\Delta\mu^{(\text{GF})}$, as we will see later.

For practical calculations, it is convenient to define the functions concerning the pure solvent distribution,

$$\bar{\eta}_{ij}(r) = \rho \sum_k (\bar{\omega}_{ik} + \rho \tilde{h}_{ik})^* \bar{c}_{kj}(r). \quad (41)$$

All of the necessary quantities appear in the standard RISM theory. The functions are related to the site density pair correlation ones as

$$\chi_{ij}(r) = \rho \left[\bar{\omega}_{ij}(r) + \sum_k \bar{\eta}_{ik}^* \bar{\omega}_{kj}(r) \right]. \quad (42)$$

Then, the free energy of solvation from the PW theory becomes

$$\Delta\mu^{(\text{PW})} \approx \Delta\mu_c + \frac{\rho}{2\beta(2\pi)^3} \sum_{ai} \int 4\pi k^2 \times \left[\tilde{h}_{ai}(k) \sum_j c_{aj}(k) \bar{\eta}_{ji}(k) \right] dk. \quad (43)$$

This expression bypasses the explicit treatment of the asymptotic behavior of $h_{ai}(r)$ in the real space, which can be slightly complex in the solvation of charged molecules and molecular ions.

VI. EMBEDDED SITE MODEL

We examine the free energy expressions using the embedded site model illustrated in Fig. 1. In this model, noninteracting dummy sites, $X1, X2, \dots, Xn$, are involved in an atomic molecule, O , where n is the number of the dummy sites. The model is immersed as a solute in a liquid consisting of the same solvent atoms without embedded sites. We assume that all of the dummy sites are separated from the atomic center at a distance, R_{OX} . The pair correlation function of the main atomic site suffices for the description of the system, and the exact site–site correlation functions in the RISM theory have the following forms:

$$\mathbf{c}(r) = (c_{OO}(r) \ 0 \ 0 \ \dots), \quad (44)$$

$$\tilde{\mathbf{h}}(r) = (h_{OO}(r) \ \omega_{XO}^* h_{OO}(r) \ \omega_{XO}^* h_{OO}(r) \ \dots). \quad (45)$$

In the PW terminology, the total correlation functions in the partial forms are zero except for the central site,

$$\mathbf{h}(r) = (h_{OO}(r) \ 0 \ 0 \ \dots). \quad (46)$$

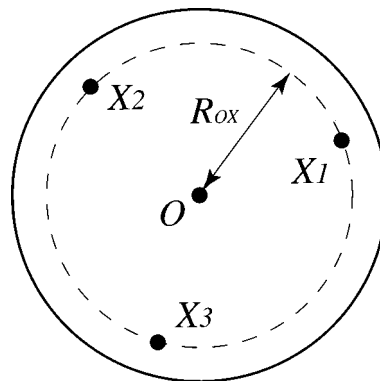


FIG. 1. Schematic representation of the embedded site model. The indices O and X denote the central site of the atomic molecule and the embedded dummy sites, respectively.

It should be noted that only the closure relations from the density functional theory of Chandler *et al.*²⁴ and from the PW formalism²¹ give the correct correlation functions for this model. The exact HNC free energy of solvation is in the form

$$\Delta\mu^{(\text{exact})} = \frac{\rho}{2\beta} \int 4\pi r^2 [h_{OO}^2(r) - h_{OO}(r)c_{OO}(r) - 2c_{OO}(r)] dr. \quad (47)$$

Substituting the site–site functions, we obtain the energies from the previous theories,

$$\Delta\mu^{(\text{GF})} = \Delta\mu^{(\text{exact})} - \frac{\rho}{2\beta} \int 4\pi r^2 h_{OO}^2(r) dr, \quad (48)$$

$$\Delta\mu^{(\text{HNC})} = \Delta\mu^{(\text{exact})} + \frac{n\rho}{2\beta} \int 4\pi r^2 [\omega_{XO}^* h_{OO}(r)]^2 dr, \quad (49)$$

whereas the energy of the PW formalism is explicitly identical to the exact HNC one,

$$\Delta\mu^{(\text{PW})} = \Delta\mu^{(\text{exact})}. \quad (50)$$

Although $\Delta\mu^{(\text{GF})}$ is independent of the presence of the dummy sites, it is always smaller than the exact energy by $(\rho/2\beta) \int 4\pi r^2 dr h_{OO}^2(r)$. The free energy of the XRISM/HNC is a monotonically increasing function with respect to the number of the dummy sites by $(\rho/2\beta) \int 4\pi r^2 dr [\omega_{XO}^* h_{OO}(r)]^2$, as discussed in the previous work.²¹ It should be noted that the error in $\Delta\mu^{(\text{HNC})}$ is sensitive to the molecular geometry. In the united atom limit, $R_{OX}=0$, the functions, $\omega_{XO}^* h_{OO}(r)$, become the replicas of the total correlation function, h_{OO} ; the error is the maximum. On the other hand, the discrepancy vanishes in the extended atom limit, $R_{XO} \rightarrow \infty$. This fact implies that the XRISM/HNC free energy includes a systematic error in describing the conformational changes of solute, the neighboring sites playing the function of the dummy sites; for instance, the dissociation energy is underestimated, a globular conformation is always less favorable, and so on.

In order to see the above-mentioned result numerically, we compare the free energies using methanelike models, in which the number of X sites changes from 0 to 4 in the solute

TABLE I. Parameters of methanelike models.^a

Model	1	2 ^b
σ_X (Å)	0.0	1.55
ε_X/k_B (K)	0.0	1.0

^aThe common parameters are given as $\rho=0.014 \text{ \AA}^{-3}$, $T=210.0 \text{ K}$, $\sigma_O=3.82 \text{ \AA}$, $\varepsilon_O/k_B=148.2 \text{ K}$, $R_{OX}=1.090 \text{ \AA}$, and $R_{XX}=1.775 \text{ \AA}$.

^bOnly the repulsive part is included in the potential functions concerning the X sites to avoid additional associations.

methane molecule. The potential energy parameters are given in Table I. Henceforward, the standard combination rules, $\varepsilon_{ai}=(\varepsilon_a\varepsilon_i)^{1/2}$ and $\sigma_{ai}=(\sigma_a+\sigma_i)/2$, are used for site-site interactions. We show the calculated solvation free energies from the HNC theory of simple liquids and from the XRISM-HNC theory in Table II. The results of the exact HNC correlation functions are entirely parallel to the above-given argument; $\Delta\mu^{(\text{HNC})}$ deviates considerably from the atomic limit with the number of X sites, $\Delta\mu^{(\text{GF})}$ is too low, and only the functional of the PW theory, $\Delta\mu^{(\text{PW})}$, gives the exact atomic value, 0.79 kcal/mol. It is interesting to see how accurately the correlation functions of the XRISM theory reproduce the exact free energy profiles. If we naively solve the XRISM/HNC equations for the embedded site model (model 1), the results are not so encouraging; all values deviate with the increase of the sites. However, it is known that the XRISM theory does not give correct correlation functions without using modified potential parameters for noninteracting sites. We compare the site-site radial correlation functions for the O-X pair in Fig. 2. The XRISM function for model 1 shows an unphysical leak in the probability density at short distances (model 2), the result improves significantly. The free energy profiles for model 2 almost resemble those using the exact HNC correlation functions. Finally, we plot the deviations of the free energies from $\Delta\mu_c$ in Fig. 3. We can see that the quantity, $\Delta\mu - \Delta\mu_c$, is less sensitive to the choice of the model in this particular system.

To summarize, XRISM is a semiempirical model, which might give plausible site-site correlation functions for small

TABLE II. Solvation free energies (kcal/mol) of the methanelike models.

Number of X sites	0	1	2	3	4
Exact					
$\Delta\mu^{(\text{HNC})}$	0.79	1.18	1.58	1.97	2.37
$\Delta\mu^{(\text{GF})}$	-0.59	-0.59	-0.59	-0.59	-0.59
$\Delta\mu^{(\text{PW})}$	0.79	0.79	0.79	0.79	0.79
$\Delta\mu_c$	2.68	2.68	2.68	2.68	2.68
Model 1					
$\Delta\mu^{(\text{HNC})}$		0.93	1.09	1.29	1.57
$\Delta\mu^{(\text{GF})}$		-0.83	-1.05	-1.26	-1.41
$\Delta\mu^{(\text{PW})}$		0.57	0.37	0.17	0.02
$\Delta\mu_c$		2.46	2.24	2.03	1.82
Model 2					
$\Delta\mu^{(\text{HNC})}$		1.19	1.60	2.01	2.45
$\Delta\mu^{(\text{GF})}$		-0.65	-0.69	-0.70	-0.66
$\Delta\mu^{(\text{PW})}$		0.73	0.70	0.70	0.75
$\Delta\mu_c$		2.67	2.69	2.73	2.86

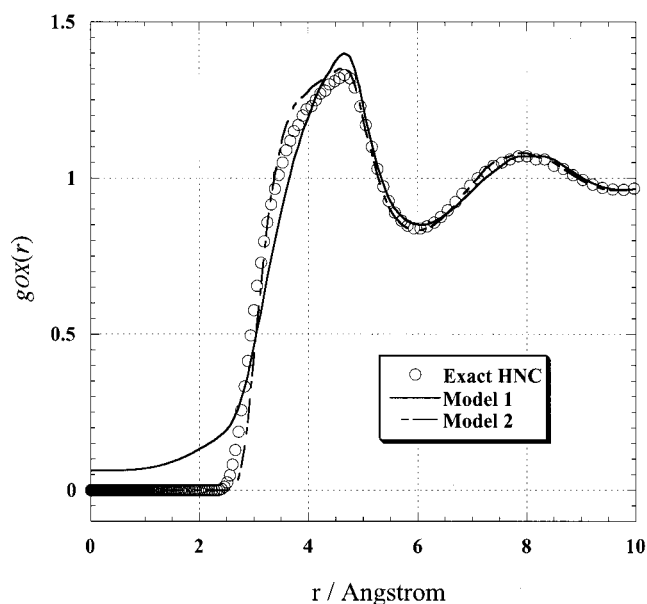


FIG. 2. Site-site correlation function for the O-X pair of the embedded site model.

molecules, only if we introduce some ad hoc potential parameters to prevent unphysical leaks in distributions. Even under such a condition, the free energy functional, $\Delta\mu^{(\text{HNC})}$, will give incorrect results, because the functional is formulated with the assumption of the XRISM/HNC theory. The ad hoc parameters cannot reproduce the entire sets of correlation functions over the charging scheme, which are implicitly included in the excess chemical potential. The error becomes more prominent as the number of sites increases. Although the expression from the Gaussian density field theory, $\Delta\mu^{(\text{GF})}$, does not show such an unphysical increase, the free

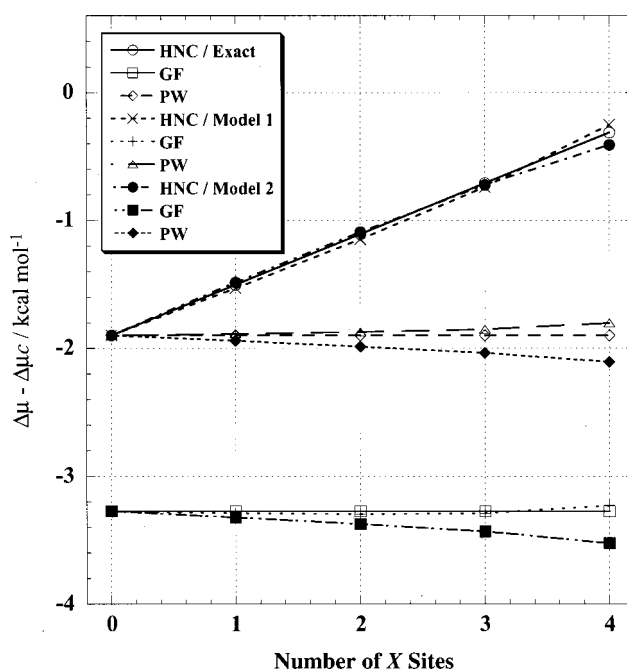


FIG. 3. Distinct components of the free energies of solvation as functions of the number of dummy sites.

TABLE III. Free energy of solvation (kcal/mol) using the OPLS sets.^a

Solute	Number of sites	$\Delta\mu^{(\text{HNC})}$	$\Delta\mu^{(\text{GF})}$	$\Delta\mu^{(\text{PW})}$	Expt.
Methanol	3	-2.20	-10.04	-4.09	-5.10
	6	1.08	-10.18	-4.01	
Ethanol	4	0.97	-9.77	-3.27	-5.0
	9	6.36	-10.52	-3.41	
1-propanol	5	5.59	-8.22	-1.19	(-4.3) ^b
	12	13.64	-10.30	-2.38	
2-propanol	5	5.29	-8.71	-1.77	-4.8
	12	12.90	-11.35	-3.38	

^aThe upper and lower numbers are the results from the OPLS united and all-atom models, respectively.

^bExtrapolated value using the experimental free energy change from ethanol to 1-propanol in Ref. 33.

energy of solvation is underestimated due to the missing term which is present in the PW functional, $\Delta\mu^{(\text{PW})}$.

VII. OPLS UNITED-ATOM AND ALL-ATOM MODELS

In the applications of the XRISM theory, different authors numerically concluded that the functional $\Delta\mu^{(\text{HNC})}$ is less accurate than $\Delta\mu^{(\text{GF})}$.²⁵⁻²⁷ Their results are now understandable. In this section, we apply the free energy functional to more realistic systems. Lee and Maggiora²⁷ applied the expressions $\Delta\mu^{(\text{HNC})}$ and $\Delta\mu^{(\text{GF})}$ to various kinds of molecules in aqueous solution using the OPLS parameter set.^{28,29} We reevaluate the theoretical free energies for the alcohols, methanol, ethanol, 1-propanol, and 2-propanol, using the TIP3P solvent model,³⁰ assigning additional parameters for the hydrogen sites, $\sigma_{\text{H}}=1.0 \text{ \AA}$ and $\epsilon_{\text{H}}=25.0 \text{ K}$. The van der Waals radius is somewhat larger than that used by Lee and Maggiora but gives site-site correlation functions in better agreement with a simulation result. Other parameters are the same as used by the previous authors.²⁷ We also examine the effect of alkyl hydrogen using the OPLS all-atom parameter set³¹ instead of the united-atom one.²⁸ For correlation functions, the XRISM-HNC method is used on the conditions $\rho=0.03334 \text{ \AA}^{-3}$ and $T=298 \text{ K}$. The solvent molecular geometries are optimized at the restricted Hartree-Fock level using the 6-31G* basis set.³²

We show the results in Table III. The experimental solvation free energies are nearly constant with the increase of the methyl group.²⁷ The deviations from the experimental values are plotted in Fig. 4. The results of the expressions $\Delta\mu^{(\text{GF})}$ and $\Delta\mu^{(\text{PW})}$ are almost parallel and approximately reproduce the relative feature of the experimental energies. On the other hand, the curve $\Delta\mu^{(\text{HNC})}$ has no correlation with the others but seems to have close resemblance to the shape of the number of solute sites. This result means that the terms, which differ among the free energy expressions, are the dominating elements for these weakly charged systems, and $\Delta\mu^{(\text{HNC})}$ includes fatal errors for such applications. The result will not be the case for the solution of ionic molecules, in which the energy constitutions are different. The free energy perturbation (FEP) calculations³³ based on molecular dynamics simulations with the OPLS united-atom and TIP3P water models yield the free energy differences

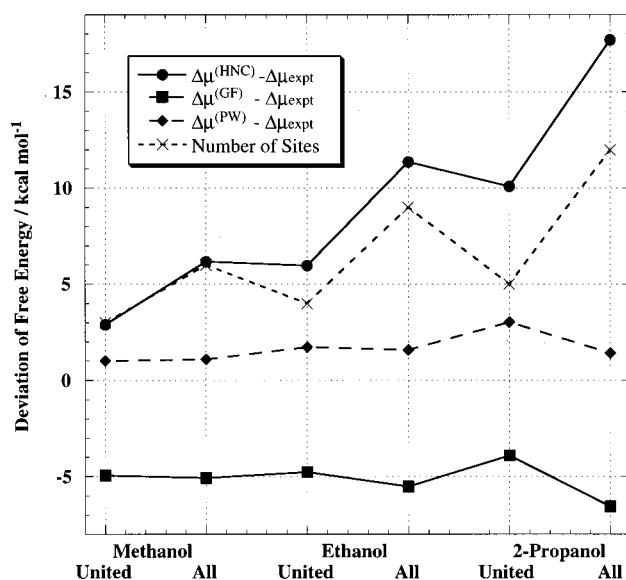


FIG. 4. Excess chemical potentials for methanol, ethanol, and 2-propanol in aqueous solution. The numbers of the interaction sites of the solute molecules are plotted in kcal/mol per site.

2.19 ± 1.29 and 0.66 ± 0.17 kcal/mol for the transformations, methanol \rightarrow ethanol, and ethanol \rightarrow 1-propanol, respectively. The $\Delta\mu^{(\text{HNC})}$ increments, 3.17 and 4.62 kcal/mol, are too large even for the united-atom models, in comparison with the results of $\Delta\mu^{(\text{GF})}$, 0.27 and 1.55 kcal/mol, and of $\Delta\mu^{(\text{PW})}$, 0.82 and 2.08 kcal/mol. Although the free energy expression, $\Delta\mu^{(\text{PW})}$, is more sophisticated than $\Delta\mu^{(\text{GF})}$, the absolute energies of $\Delta\mu^{(\text{PW})}$ are always higher than the experimental ones. We should, however, remember that the calculation assumes the XRISM-HNC correlation functions, viz. they are not optimized variationally with respect to $\Delta\mu^{(\text{PW})}$ but to $\Delta\mu^{(\text{HNC})}$. Moreover, the results are based on the minimum basis expression of $\Delta\mu^{(\text{PW})}$, in which only the spherically symmetric component of the site-site functions are used. It is therefore the argument for the obvious failure of $\Delta\mu^{(\text{HNC})}$ in neutral and weakly charged systems that is the minimum conclusion in this section.

VIII. CONCLUSION

The PW formalism provides a systematic integral equation theory combining RISM and the integral equation theory based on MOZ. The RISM-OZ corresponds to the extreme of spherically symmetric expansion of partial functions, whereas MOZ is another extreme of POZ, in which only one expansion site per molecule is used. The latter method cannot be applied to a situation involving large separations of intraatomic sites. The use of PWs along with full intramolecular correlation functions will resolve this drawback by maintaining the propriety of the theory of molecular liquids. The present comparison of the expressions for excess chemical potential manifested that the assumption of the XRISM theory is inadequate, especially for the quantity.

Finally, we mention some related applications of the previous energy expressions. For the combined RISM integral equation theory and quantum chemical solvation model,^{34,35}

so called the RISM–SCF method, an analytical energy gradient method³⁶ was proposed. The free energy expression is restricted to be the sum of the electronic energy of a solute molecule and the XRISM/HNC excess chemical potential, $\Delta\mu^{(\text{HNC})}$, due to the solute–solvent interaction. So the results are reliable for relative energies only when the differences in molecular geometry and site–site correlation functions are sufficiently small. It should be noted that the original RISM–SCF method^{34,35} is free from such a restriction, and any free energy expression can be chosen as used in the applications.^{37,38} More recently, another free energy expression³⁹ was derived for the integral equation theory based on the three-dimensional reduction of the MOZ equation. The derivation also includes the assumption of the XRISM-like closure in the reduced molecular coordinates. The resulting functional will also behave as $\Delta\mu^{(\text{HNC})}$ for neutral systems though the site effects will be less than the XRISM case.

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