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Commuting flows and conservation laws for noncommutative Lax hierarchies

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We discuss commuting flows and conservation laws for Lax hierarchies on noncommutative spaces in the framework of the Sato theory. On commutative spaces, the Sato theory has revealed essential aspects of the integrability for wide class of soliton equations which are derived from the Lax hierarchies in terms of pseudo-differential operators. Noncommutative extension of the Sato theory has been already studied by the author and Toda, and the existence of various noncommutative Lax hierarchies are guaranteed. In this paper, we present conservation laws for the noncommutative Lax hierarchies with both space–space and space–time noncommutativities and prove the existence of infinite number of conserved densities. We also give the explicit representations of them in terms of Lax operators. Our results include noncommutative versions of KP, KdV, Boussinesq, coupled KdV, Sawada-Kotera, modified KdV equation and so on. © *2005 American Institute of Physics.* [DOI: 10.1063/1.1865321]

I. INTRODUCTION

Noncommutative (NC) extension of field theories has been studied intensively for the last several years.[1] NC gauge theories are equivalent to ordinary gauge theories in the presence of background magnetic fields and succeeded in revealing various aspects of them.[2] NC solitons especially play important roles in the study of D-brane dynamics, such as the confirmation of Sen's conjecture on tachyon condensation.[3] One of the distinguished features of NC theories is resolution of singularities. This gives rise to various new physical objects such as U(1) instantons and makes it possible to analyze singular configurations as usual.

NC extension of integrable equations such as the Korteweg–de Vries (KdV) equation[4] is also one of the hot topics.[5–37] These equations imply no gauge field and NC extension of them perhaps might have no physical picture or no good property on integrabilities. To make matters worse, the NC extension of (1+1)-dimensional equations introduces infinite number of time derivatives, which makes it hard to discuss or define the integrability. However, some of them actually possess integrable properties, such as the existence of infinite number of conserved quantities[7–9,20] and the linearizability[30,31] which are widely accepted as definition of complete integrability of equations. Furthermore, a few of them can be derived from NC (anti-)self-dual Yang–Mills (YM) equations by suitable reductions.[14,30,33] This fact may give some physical meanings and good properties to the lower-dimensional NC field equations and makes us expect that the Ward conjecture[38] still holds on NC spaces.[27] So far, however, those equations have been examined one by one. Now it is very natural to discuss their integrabilities in more general framework.

The author and Toda have studied systematic NC extension of integrable systems.[27,30,36] In the previous paper,[36] we have obtained wide class of NC Lax hierarchies which include various NC versions of soliton equations in the framework of the Sato theory.[39] On commutative spaces, the Sato theory is known to be one of the most beautiful theories of solitons and reveals essential

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aspects of the integrability, such as, the construction of exact multisoliton solutions, the structure of the solution space, the existence of infinite conserved quantities, and the hidden symmetry of them. In the Sato theory, the soliton equations are described by Lax hierarchies in terms of pseudodifferential operators.

In the present paper, we prove the existence of infinite conserved quantities for Lax hierarchies on NC spaces in the framework of the Sato theory. We show the conservation laws for them and give the explicit representations with both space–space and space–time noncommutativities. This suggests that the NC soliton equations are also completely integrable and infinite-dimensional symmetries would be hidden. Our results include wide class of NC soliton equations, such as, NC versions of Kadomtsev–Petviashvili (KP),[40] KdV, Boussinesq,[41] coupled KdV,[42] Sawada–Kotera,[43] modified KdV (mKdV) equations and so on.

II. COMMENTS ON NONCOMMUTATIVE FIELD THEORIES

NC spaces are defined by noncommutativity of the coordinates,

$$[x^i,x^j] = i\theta^{ij}, \tag{2.1}$$

where θ^{ij} are real constants and called the *NC parameters*.

NC field theories are obtained from given commutative field theories by exchange of ordinary products in the commutative field theories for *star-products*. The star-product is defined for ordinary fields on commutative spaces. On Euclidean spaces, it is explicitly given by

$$f(x)\star g(x) := f(x)\exp\left(\frac{i}{2}\overleftarrow{\partial}_i\theta^{ij}\overrightarrow{\partial}_j\right)g(x) = f(x)g(x) + \frac{i}{2}\theta^{ij}\partial_i f(x)\partial_j g(x) + \mathcal{O}(\theta^2), \tag{2.2}$$

where $\partial_i := \partial/\partial x^i$. This explicit representation is known as the *Moyal product*.[44]

The star-product possesses associativity, $f\star(g\star h)=(f\star g)\star h$, and returns back to the ordinary product in the commutative limit, $\theta^{ij}\to 0$. The modification of the product makes the ordinary spatial coordinates "noncommutative," that is, $[x^i,x^j]_\star := x^i\star x^j - x^j\star x^i = i\theta^{ij}$.

We note that the fields themselves take c-numbers values and the differentiation and the integration for them are well-defined as usual. NC field theories should be interpreted as deformed theories from commutative ones. One of the nontrivial points in the NC extension is the order of nonlinear terms. The difference between commutative equations and the NC equations arises as commutators of fields which sometimes become serious obstructions.

Here we point out a special property of the NC commutators of fields. It is convenient to introduce the following symbol:

$$P := \tfrac{1}{2}\overleftarrow{\partial}_i\theta^{ij}\overrightarrow{\partial}_j, \tag{2.3}$$

and the *Strachan product*[45]

$$f(x)\diamondsuit g(x) := f(x)\left(\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s+1)!}P^{2s}\right)g(x). \tag{2.4}$$

A commutator of fields is straightforwardly calculated as follows:

$$\begin{aligned}[f(x),g(x)]_\star &= f(x)(e^{iP}-e^{-iP})g(x) =: 2if(x)(\sin P)g(x)\\ &= -\theta^{ij}\partial_i f(x)\diamondsuit\partial_j g(x) = -\theta^{ij}\partial_i(f(x)\diamondsuit\partial_j g(x)).\end{aligned} \tag{2.5}$$

In the second line, we use the fact that $\sin P$ is the composite of P and "$P^{-1}\sin P$" and the Strachan product "$\diamondsuit$" corresponds to the latter. This derivation was first presented by Dimakis and Müller-Hoissen in order to generate infinite number of conserved densities of the NC nonlinear Schrödinger (NLS) equation,[7] the NC KdV equation,[8] and the NC extended matrix-NLS equation.[9] Here more widely, we would like to stress that *commutators of fields on NC spaces* *always appear as total derivatives in the NC directions.* This will be crucial in the derivation of conservation laws in Sec. V.

As a consequence, we can prove

$$\int \mathrm{d}^D x\, f(x)\star g(x) = \int d^D x\, f(x) g(x), \tag{2.6}$$

where the integration is taken in all NC directions.

III. NONCOMMUTATIVE LAX HIERARCHIES IN SATO'S FRAMEWORK

In this section, we derive various NC Lax equations in terms of pseudodifferential operators which include negative powers of differential operators. We note that the present discussion in this section can be applied to more general cases where the products are not necessarily the star-products but noncommutative associative products with differentiations, which has already been discussed in, e.g., Ref. 46. However, we believe that some explicit examples here are new equations and would be useful for further studies.

An Nth order (monic) pseudodifferential operator A is represented as follows:

$$A = \partial_x^N + a_{N-1}\partial_x^{N-1} + \cdots + a_0 + a_{-1}\partial_x^{-1} + a_{-2}\partial_x^{-2} + \cdots . \tag{3.1}$$

Here we introduce useful symbols,

$$A_{\geqslant r} := \partial_x^N + a_{N-1}\partial_x^{N-1} + \cdots + a_r\partial_x^r, \tag{3.2}$$

$$A_{\leqslant r} := A - A_{\geqslant r+1} = a_r\partial_x^r + a_{r-1}\partial_x^{r-1} + \cdots , \tag{3.3}$$

$$\mathrm{res}_r\, A := a_r. \tag{3.4}$$

The symbol $\mathrm{res}_{-1}\, A$ is especially called the *residue* of A.

The action of a differential operator ∂_x^n on a multiplicity operator f is formally defined as the following generalized Leibniz rule:

$$\partial_x^n \cdot f := \sum_{i\geqslant 0} \binom{n}{i} (\partial_x^i f)\partial^{n-i}, \tag{3.5}$$

where the binomial coefficient is given by

$$\binom{n}{i} := \frac{n(n-1)\cdots(n-i+1)}{i(i-1)\cdots 1}. \tag{3.6}$$

We note that the definition of the binomial coefficient (3.6) is applicable to the case for negative n, which just defines the action of negative power of differential operators. The examples are

$$\begin{aligned} \partial_x^{-1}\cdot f &= f\partial_x^{-1} - f'\partial_x^{-2} + f''\partial_x^{-3} - \cdots , \\ \partial_x^{-2}\cdot f &= f\partial_x^{-2} - 2f'\partial_x^{-3} + 3f''\partial_x^{-4} - \cdots , \\ \partial_x^{-3}\cdot f &= f\partial_x^{-3} - 3f'\partial_x^{-4} + 6f''\partial_x^{-5} - \cdots , \end{aligned} \tag{3.7}$$

where $f' := \partial f/\partial x$, $f'' := \partial^2 f/\partial x^2$ and so on, and ∂_x^{-1} in the RHS acts as an integration operator $\int^x \mathrm{d}x$.

The composition of pseudodifferential operators is also well-defined and the total set of pseudodifferential operators forms an operator algebra. For more on pseudodifferential operators and the Sato theory, see, e.g., Refs. 47–49.

Let us introduce a Lax operator as the following first-order pseudodifferential operator:

$$L = \partial_x + u_1 + u_2\partial_x^{-1} + u_3\partial_x^{-2} + u_4\partial_x^{-3} + \cdots , \tag{3.8}$$

where the coefficients $u_k(k=1,2,\ldots)$ are functions of infinite variables $(x^1,x^2,\ldots)$ with $x^1 \equiv x$,

$$u_k = u_k(x^1,x^2,\ldots). \tag{3.9}$$

The noncommutativity is arbitrarily introduced for the variables $(x^1,x^2,\ldots)$ as Eq. (2.1) here.

The Lax hierarchy is defined in Sato's framework as

$$\partial_m L = [B_m, L]_\star, \quad m = 1,2,\ldots , \tag{3.10}$$

where the action of ∂_m on the pseudodifferential operator L should be interpreted to be coefficient-wise, that is, $\partial_m L := [\partial_m, L]$ or $\partial_m \partial_x^k = 0$. The operator B_m is given by

$$B_m := (\underbrace{L\star \cdots \star L}_{m\ \text{times}})_{\geqslant r} =: (L^m)_{\geqslant r}, \tag{3.11}$$

where r is 0 for $u_1 = 0$ and 1 for $u_1 \neq 0$ as commutative cases.[50,51] The Lax hierarchy gives rise to a set of infinite differential equations with respect to infinite kinds of fields from the coefficients in Eq. (3.10) for a fixed m. Hence it contains huge amount of differential (evolution) equations for all m. The left-hand side (LHS) of Eq. (3.10) becomes $\partial_m u_k$ which shows a flow in the x^m direction.

If we set the constraint $L^l = B_l$ on the Lax hierarchy (3.10), we get an infinite set of NC (reduced) Lax hierarchies. We can easily show

$$\frac{\partial u_k}{\partial x^{Nl}} = 0, \tag{3.12}$$

for all N, k because

$$\frac{\mathrm{d}L^l}{\mathrm{d}x^{Nl}} = [B_{Nl}, L^l]_\star = [(L^l)^N, L^l]_\star = 0, \tag{3.13}$$

which implies Eq. (3.12). The reduced NC hierarchy is called the *l-reduction* of the NC KP hierarchy. This time, the constraint $L^l = B_l$ gives simple relationships which make it possible to represent infinite kind of fields $u_{l-r+1}, u_{l-r+2}, u_{l-r+3}, \ldots$ in terms of $(l-1)$ kind of fields $u_{2-r}, u_{3-r}, \ldots, u_{l-r}$ (cf. Appendix).

From now on, let us see that those equations in the Lax hierarchy contain various soliton equations with some constraints. We discuss it separately in the following two cases: $u_1 = 0(r=0)$ case and $u_1 \neq 0(r=1)$ case. Some of them are already discussed in Ref. 36. For commutative discussions, see also Ref. 52.

For $u_1 = 0$ $(r=0)$: In this case, the Lax hierarchy (3.10) is just the NC KP hierarchy which includes the NC KP equation.[16,46] Let us see it explicitly.

(i) NC KP hierarchy. The coefficients of each powers of (pseudo) differential operators in the Lax hierarchy (3.10) yield a series of infinite NC "evolution equations," that is, for $m=1$,

$$\partial_x^{1-k} \quad \partial_1 u_k = u_k', \quad k = 2,3,\ldots \Rightarrow x^1 \equiv x, \tag{3.14}$$

for $m=2$,

$$\begin{aligned} \partial_x^{-1} &\quad \partial_2 u_2 = u_2'' + 2u_3', \\ \partial_x^{-2} &\quad \partial_2 u_3 = u_3'' + 2u_4' + 2u_2\star u_2' + 2[u_2,u_3]_\star, \\ \partial_x^{-3} &\quad \partial_2 u_4 = u_4'' + 2u_5' + 4u_3\star u_2' - 2u_2\star u_2'' + 2[u_2,u_4]_\star, \end{aligned} \tag{3.15}$$

$$\partial_x^{-4} \quad \partial_2 u_5 = \cdots ,$$

and for $m=3$,

$$\partial_x^{-1} \quad \partial_3 u_2 = u_2''' + 3u_3'' + 3u_4' + 3u_2'\star u_2 + 3u_2\star u_2',$$

$$\partial_x^{-2} \quad \partial_3 u_3 = u_3''' + 3u_4'' + 3u_5' + 6u_2\star u_3' + 3u_2'\star u_3 + 3u_3\star u_2' + 3[u_2,u_4]_\star,$$

$$\partial_x^{-3} \quad \partial_3 u_4 = u_4''' + 3u_5'' + 3u_6' + 3u_2'\star u_4 + 3u_2\star u_4' + 6u_4\star u_2' - 3u_2\star u_3'' - 3u_3\star u_2'' + 6u_3\star u_3' + 3[u_2,u_5]_\star + 3[u_3,u_4]_\star,$$

$$\partial_x^{-4} \quad \partial_3 u_5 = \cdots . \tag{3.16}$$

These just imply the $(2+1)$-dimensional NC KP equation[16,46] with $2u_2 \equiv u, x^2 \equiv y, x^3 \equiv t$,

$$\frac{\partial u}{\partial t} = \frac{1}{4}\frac{\partial^3 u}{\partial x^3} + \frac{3}{4}\frac{\partial (u\star u)}{\partial x} + \frac{3}{4}\int^x dx' \frac{\partial^2 u(x')}{\partial y^2} - \frac{3}{4}\left[u, \int^x dx' \frac{\partial u(x')}{\partial y}\right]_\star . \tag{3.17}$$

The important point is that infinite kinds of fields $u_3, u_4, u_5, \ldots$ are represented in terms of one kind of field $2u_2 \equiv u$ as is seen in Eq. (3.15). This guarantees the existence of the NC KP hierarchy which implies the existence of reductions of the NC KP hierarchy. The order of nonlinear terms are determined in this way.

(ii) NC KdV hierarchy (2-reduction of the NC KP hierarchy). Taking the constraint $L^2 = B_2 =: \partial_x^2 + u$ for the NC KP hierarchy, we get the NC KdV hierarchy. This time, the following NC Lax hierarchy,

$$\frac{\partial u}{\partial x^m} = [B_m, L^2]_\star , \tag{3.18}$$

include neither positive nor negative power of (pseudo) differential operators for the same reason as the commutative case (see, e.g., Ref. 53) and gives rise to the mth KdV equation for each m. For example, the NC KdV hierarchy (3.18) becomes the $(1+1)$-dimensional NC KdV equation[8] for $m=3$ with $x^3 \equiv t$,

$$\dot{u} = \tfrac{1}{4}u''' + \tfrac{3}{4}(u\star u)' , \tag{3.19}$$

and the $(1+1)$-dimensional fifth NC KdV equation[23] for $m=5$ with $x^5 \equiv t$,

$$\dot{u} = \tfrac{1}{16}u''''' + \tfrac{5}{16}(u\star u''' + u'''\star u) + \tfrac{5}{8}(u'\star u' + u\star u\star u)' , \tag{3.20}$$

where $\dot{u} := \partial u / \partial t$.

(iii) NC Boussinesq hierarchy (3-reduction of the NC KP hierarchy). The 3-reduction $L^3 = B_3$ yields the NC Boussinesq hierarchy which includes the $(1+1)$-dimensional NC Boussinesq equation[23] with $t \equiv x^2$,

$$\ddot{u} = \tfrac{1}{3}u'''' + (u\star u)'' + ([u, \partial_x^{-1}\dot{u}]_\star)' , \tag{3.21}$$

where $\ddot{u} := \partial^2 u / \partial t^2$ and $\partial_x^{-1} = \int^x dx$.

(iv) NC coupled KdV hierarchy (4-reduction of the NC KP hierarchy). The hierarchy includes the $(1+1)$-dimensional NC coupled KdV equation $t \equiv x^3$,

$$\dot{u} = \tfrac{1}{4}u''' + \tfrac{3}{4}(u\star u)' + \tfrac{3}{4}(\omega - \phi^2)' - \tfrac{3}{4}[u, \phi']_\star , \tag{3.22}$$

and the other two equations with respect to three kinds of fields u, ω, and ϕ, which are determined by Eqs. (3.15) and (3.16). The x^2-dependence of the fields is absorbed by the fields ω, ϕ. In this way, we can generate infinite set of the l-reduced NC hierarchies. If we take other set-up, we can get many other hierarchies.

(v) NC Sawada–Kotera hierarchy (3-reduction of the NC BKP hierarchy). The NC version of BKP hierarchy[54] is obtained from the NC KP hierarchy by the constraint that the constant terms of B_m for $m=1,3,5,\ldots$ should vanish. The 3-reduction of the NC BKP hierarchy includes the (1+1)-dimensional NC Sawada–Kotera equation with $t\equiv x^5$, $u\equiv 3u_2$,

$$\dot{u}+\frac{1}{9}u''''' + \frac{5}{9}u'''\star u + \frac{5}{9}u''\star u' + \frac{5}{9}u\star u'\star u = 0, \tag{3.23}$$

which is new.

For $u_1\neq 0(r=1)$: On commutative spaces, this situation generates modified KP (mKP) hierarchy and its reductions. On NC spaces, however, the existence of them is not always guaranteed. For the NC KP hierarchy, infinite kinds of fields are described by one kind of x^2-flow equations (3.15). However, this time the flow equation becomes

$$\begin{aligned} \partial_x^0 \quad & \partial_2 u_1 = u_1'' + 2u_2' + 2u_1\star u_1' + 2[u_1,u_2]_\star, \\ \partial_x^{-1} \quad & \partial_2 u_2 = u_2'' + 2u_3' + 2u_1\star u_2' + 2[u_1,u_3]_\star, \\ \partial_x^{-2} \quad & \partial_2 u_3 = \cdots . \end{aligned} \tag{3.24}$$

Hence due to the commutator $[u_1,u_k]$, it is very hard to represent the field u_k in terms of $u_1,u_2,\ldots,u_{k-1}$. The same is true of other flows. That is why the existence of NC modified KP hierarchy is nontrivial.

Some reduced hierarchies are obtained from constraint conditions.

(i) NC mKdV hierarchy (2-reduction of the "NC mKP hierarchy"). This time, the 2-reduction constraint $L^2=B_2$ makes it possible to represent infinite kinds of fields $u_2,u_3,\ldots$ in terms of one kind of field $2u_1\equiv v$. The NC mKdV hierarchy includes the (1+1)-dimensional NC mKdV equation for $m=3$ with $t_3\equiv t$,

$$\dot{v} = \frac{1}{4}v''' - \frac{3}{8}v\star v'\star v + \frac{3}{8}[v,v'']_\star. \tag{3.25}$$

(ii) NC Burgers hierarchy.[30] This is obtained by an irregular reduction. Setting the constraint $L_{\leqslant -1}=0$ or $L=:\partial_x+v$, the Lax hierarchy (3.10) yields the NC Burgers hierarchy which includes neither positive nor negative power of differential operator. For $m=2$, the hierarchy becomes the (1+1)-dimensional NC Burgers equation with $t\equiv x^2$,

$$\dot{v} = [B_2,L]_\star = [\partial_x^2 + 2v\partial_x, \partial_x + v]_\star = v'' + 2v\star v'. \tag{3.26}$$

The NC Burgers equation is linearizable and easily solved via NC Cole–Hopf transformation.[30,31] In the linearization, the order of the nonlinear term plays crucial roles. This order is automatically realized from Sato's framework.

The present discussion is applicable to the matrix Sato theory where the fields $u_k(k=1,2,\ldots)$ are $N\times N$ matrices. For $N=2$, the Lax hierarchy includes the Ablowitz–Kaup–Newell–Segur (AKNS) system,[55] the Davey–Stewarson equation, the NLS equation and so on. (For commutative discussion, see, e.g., Ref. 48.)

NC version[23] of the Bogoyavlenskii–Calogero–Schiff (BCS) equation[56] is also derived from this framework because the Sato theory works well on the commutative BCS equation.

IV. COMMUTING FLOWS FOR NC LAX HIERARCHIES

First let us show all flows are commuting,

$$\partial_m \partial_n u_k = \partial_n \partial_m u_k \tag{4.1}$$

for any m, n, k. The derivation in this section is straightforward as the commutative case[57,53] and already discussed in a more general situation where the products are noncommutative associative products with differentiations. (See, e.g., Refs. 46, 58, and 59.)

From NC Lax equation (3.10), we get

$$\partial_m \partial_n L = [\partial_m B_n, L]_\star + [B_n, \partial_m L]_\star = [\partial_m B_n, L]_\star + [B_n, [B_m, L]_\star]_\star. \tag{4.2}$$

Hence

$$[\partial_m, \partial_n] L = [F_{mn}, L]_\star, \tag{4.3}$$

where

$$F_{mn} := \partial_m B_n - \partial_n B_m - [B_m, B_n]_\star. \tag{4.4}$$

Now we show the "zero-curvature equation" $F_{mn}=0$. We note that

$$\partial_m B_n = \partial_m (L^n)_{\geqslant r} = (\partial_m L^n)_{\geqslant r} = [B_m, L^n]_{\star \geqslant r} = -[B_m^c, L^n]_{\star \geqslant r} = -[B_m^c, B_n]_{\star \geqslant r}, \tag{4.5}$$

where the operator B_m^c is the compliment of B_m and defined by

$$B_m^c := L^m - B_m, \tag{4.6}$$

and the suffix r is equal to 0 for $u_1=0$ and 1 for $u_1 \neq 0$. Therefore we get

$$\begin{aligned} F_{mn} &= -[B_m^c, B_n]_{\star \geqslant r} + [B_n^c, B_m]_{\star \geqslant r} - [B_m, B_n]_\star = -[B_m^c, L^n - B_n^c]_{\star \geqslant r} + [L^n - B_n, B_m]_{\star \geqslant r} - [B_m, B_n]_{\star \geqslant r} \\ &= [B_m^c, B_n^c]_{\star \geqslant r} = 0, \end{aligned} \tag{4.7}$$

which implies

$$\partial_m \partial_n L = \partial_n \partial_m L. \tag{4.8}$$

Hence Eq. (4.1) is proved.

We note that the present discussion works well for arbitrary noncommutativity. Here we call the Eq. (4.7) the *NC Zakharov–Shabat equation* because reduces to the usual Zakharov–Shabat equation in the commutative limit,

$$\partial_m B_n - \partial_n B_m - [B_m, B_n]_\star = 0. \tag{4.9}$$

Of course, we can get the conjugate of the NC Zakharov–Shabat equation in terms of B_n^c,

$$\partial_m B_n^c - \partial_n B_m^c + [B_m^c, B_n^c]_\star = 0. \tag{4.10}$$

V. CONSERVATION LAWS FOR NC LAX HIERARCHIES

Here let us prove the conservation laws for NC Lax equations, which are the main results in the present paper.

First we would like to comment on conservation laws of NC field equations.[30] The discussion is basically the same as the commutative case because both the differentiation and the integration are the same as the commutative ones in the Moyal representation.

Let us suppose the conservation law,

$$\frac{\partial \sigma(t, x^i)}{\partial t} = \partial_i J^i(t, x^i), \tag{5.1}$$

where $\sigma(t,x^i)$ and $J^i(t,x^i)$ are called the *conserved density* and the *associated flux*, respectively. The conserved quantity is given by spatial integral of the conserved density,

$$Q(t) = \int_{\text{space}} \mathrm{d}^D x\ \sigma(t,x^i), \tag{5.2}$$

where the integral $\int_{\text{space}} \mathrm{d}x^D$ is taken for spatial coordinates. The proof is straightforward,

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\partial}{\partial t}\int_{\text{space}} \mathrm{d}^D x\ \sigma(t,x^i) = \int_{\text{space}} \mathrm{d}^D x\ \partial_i J_i(t,x^i) = \int_{\substack{\text{spatial}\\ \text{infinity}}} \mathrm{d}S^i\ J_i(t,x^i) = 0, \tag{5.3}$$

unless the surface term of the integrand $J_i(t,x^i)$ vanishes. The convergence of the integral is also expected because the star-product naively reduces to the ordinary product at spatial infinity due to $\partial_i \sim \mathcal{O}(r^{-1})$ where $r := |x|$.

For commutative field equations, the existence of infinite number of conserved quantities is expected to lead to infinite-dimensional hidden symmetry from Noether's theorem. For NC field equations, this would also be true and the existence of infinite number of conserved quantities would be special and meaningful, and suggest an infinite-dimensional hidden symmetry deformed from the commutative one.

In order to discuss conservation laws for the NC Lax hierarchies, let us first calculate the differential of the residue of L^n following Wilson's approach:[57]

$$\partial_m\ \mathrm{res}_{-1}\ L^n = \mathrm{res}_{-1}(\partial_m L^n) = \mathrm{res}_{-1}[B_m, L^n]_\star. \tag{5.4}$$

Here we note that

$$\begin{aligned}\mathrm{res}_{-1}[f\partial_x^p, g\partial_x^q]_\star &= \binom{p}{p+q+1}(f\star g^{(p+q+1)} - (-1)^{p+q+1} g\star f^{(p+q+1)}) \\ &= \binom{p}{p+q+1}\left\{\left(\sum_{k=0}^{p+q}(-1)^k f^{(k)}\star g^{(p+q-k)}\right)' + (-1)^{p+q}[g, f^{(p+q+1)}]_\star\right\},\end{aligned} \tag{5.5}$$

where $f^{(N)} := \partial^N f/\partial x^N$. Hence we can see that on NC spaces, there is an additional term as a commutator in Eq. (5.5) which vanishes in the commutative limit. However as we saw in Sec. II, commutators of fields can be represented as total derivatives, which is very important here.

Let us describe the explicit representations of the conservation laws. From the explicit forms of the Lax pair,

$$\begin{aligned} L^n &= \partial_x^n + \sum_{l=1}^{\infty} a_{n-l}\partial_x^{n-l}, \\ B_m &= \partial_x^m + \sum_{k=1}^{m} b_{m-k}\partial_x^{m-k}, \end{aligned} \tag{5.6}$$

we can evaluate Eq. (5.4) as

$$\begin{aligned}\partial_m\ \mathrm{res}_{-1}\ L^n &= \mathrm{res}_{-1}\left[\partial_x^m + \sum_{k=1}^{m} b_{m-k}\partial_x^{m-k}, \partial_x^n + \sum_{l=1}^{\infty} a_{n-l}\partial_x^{n-l}\right]_\star \\ &= \sum_{l=n+1}^{m+n}\binom{m}{l-n-1} a_{n-l}^{(m+n-l+1)} + \sum_{k=1}^{m}\sum_{l=n+1}^{n+1+m-k}\binom{m-k}{l-n-1} \\ &\quad \times\left\{\left(\sum_{N=0}^{m+n-k-l}(-1)^N b_{m-k}^{(N)}\star a_{n-l}^{(m+n-k-l-N)}\right)' + (-1)^{m+n-k-l}[a_{n-l}, b_{m-k}^{(m+n-k-l+1)}]_\star\right\}\end{aligned}$$

$$= \left\{ \sum_{l=n+1}^{m+n} \binom{m}{l-n-1} a_{n-l}^{(m+n-l)} + \sum_{k=1}^{m} \sum_{l=n+1}^{n+1+m-k} \binom{m-k}{l-n-1} \right.$$
$$\left. \times \sum_{N=0}^{m+n-k-l} (-1)^N b_{m-k}^{(N)} \star a_{n-l}^{(m+n-k-l-N)} \right\}' - \sum_{k=1}^{m} \sum_{l=n+1}^{n+1+m-k} \binom{m-k}{l-n-1}$$
$$\times (-1)^{m+n-k-l} \theta^{ij} \partial_i \left(a_{n-l} \diamondsuit \partial_j b_{m-k}^{(m+n-k-l+1)} \right).$$

This is the generalized conservation laws for the NC Lax hierarchies. The right-hand side (RHS) contains derivatives in all NC directions. When we interpret this as conservation laws, we must specify what coordinates correspond to time and space and introduce the noncommutativities in the space–time directions only.

If we identify the coordinate x^m with time t, we get the conserved density as follows:

$$\sigma = \mathrm{res}_{-1} L^n + \theta^{im} \sum_{k=0}^{m-1} \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} \mathrm{res}_{-(l+1)} L^n \diamondsuit \partial_i \partial_x^{k-l} \mathrm{res}_k L^m, \tag{5.7}$$

for $n=1,2,\ldots$, where the suffices i must run in the space–time directions only. We can easily see that deformation terms appear in the second term of Eq. (5.7) in the case of space-time noncommutativity. On the other hand, in the case of space–space noncommutativity, the conserved density is given by the residue of L^n as the commutative case.

Let us show more explicit representations as follows.

(i) In the case that the space–time coordinates are $(x,y,t) \equiv (x^1,x^2,x^3)$. The conserved density is given by

$$\sigma = \mathrm{res}_{-1} L^n + \theta^{i3} \sum_{k=0}^{2} \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} \mathrm{res}_{-(l+1)} L^n \diamondsuit \partial_i \partial_x^{k-l} \mathrm{res}_k L^3, \tag{5.8}$$

more explicitly, for $u_1=0$ and $[t,x]=i\theta$, which includes the NC KP equation with space–time noncommutativity, the NC KdV equation and so on,

$$\sigma = \mathrm{res}_{-1} L^n - 3\theta((\mathrm{res}_{-1} L^n) \diamondsuit u_3' + (\mathrm{res}_{-2} L^n) \diamondsuit u_2'), \tag{5.9}$$

and for $u_1 \neq 0$ and $[t,x]=i\theta$, which includes the NC modified KdV equation and so on,

$$\sigma = \mathrm{res}_{-1} L^n + 3\theta((\mathrm{res}_{-1} L^n) \diamondsuit (u_2 + u_1^2)'' - (\mathrm{res}_{-2} L^n) \diamondsuit (u_2 - u_1' - u_1^2)' - (\mathrm{res}_{-3} L^n) \diamondsuit u_1'). \tag{5.10}$$

(ii) In the case that the space–time coordinates are $(x,t) \equiv (x^1,x^2)$ with $[t,x]=i\theta$. The conserved density is given by

$$\sigma = \mathrm{res}_{-1} L^n - \theta \sum_{k=0}^{1} \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} \mathrm{res}_{-(l+1)} L^n \diamondsuit \partial_i \partial_x^{k-l} \mathrm{res}_k L^2, \tag{5.11}$$

more explicitly, for $u_1=0$, which includes the NC Boussinesq equation and so on,

$$\sigma = \mathrm{res}_{-1} L^n + 2\theta(\mathrm{res}_{-1} L^n) \diamondsuit u_2', \tag{5.12}$$

and for $u_1 \neq 0$:

$$\sigma = \mathrm{res}_{-1} L^n + 2\theta((\mathrm{res}_{-1} L^n) \diamondsuit u_1'' - (\mathrm{res}_{-2} L^n) \diamondsuit u_1'). \tag{5.13}$$

We note that for space–space noncommutativity, conserved quantities (not densities) are all the same as commutative ones because of Eq. (2.6). This is consistent with the present results, of course. Furthermore, for l-reduced hierarchies, the conserved densities (5.7) become trivial for

$n=Nl$ $(N=1,2,\ldots)$. The NC Burgers hierarchy is obtained by a "1-reduction" and contains no negative power of differential operators. Hence we cannot generate any conserved density for the NC Burgers equation in the present approach. This is considered to suggest that the NC Burgers equation is not a conservative system but a dispersive system as a commutative case.

We have one comment on conserved densities for the one-soliton configuration. One soliton solutions can always reduce to the commutative ones because $f(t-x)\star g(t-x)=f(t-x)g(t-x)$.[8,30] Hence the conserved densities are not deformed in the NC extension.

The present discussion is applicable to the NC matrix Sato theory, including the NC AKNS system, the NC Davey–Stewarson equation, the NC NLS equation, and the NC BCS equation.

VI. CONCLUSION AND DISCUSSION

In the present paper, we showed that the existence of an infinite number of conserved densities for a wide class of NC Lax hierarchies and obtained the explicit representations of them for both space–space and space–time noncommutativities. This suggests that NC soliton equations are completely integrable and infinite-dimensional symmetries would be hidden, which would be considered as some deformed affine Lie algebras.

In order to reveal what the hidden symmetry is, we must first study NC extension of Hirota's bilinearization.[60] This could be realized as a simple generalization of the Cole–Hope transformation whose extension to NC spaces are already successful in Refs. 30 and 31. Hirota's bilinearization leads to the theory of tau-functions which is essential in the discussion of the Lie algebraic structure of symmetry of the solution space.[47,54,61,62] After submission of the present paper, progress has been reported in, e.g., Refs. 63–65.

Our results guarantee that NC extension of soliton theories would be actually fruitful and worth studying. There are many further directions, such as, the study of relation to q-deformations of integrable systems, NC extension of the r-matrix formalism,[48,66] the inverse scattering method and the Bäcklund transformation, and so on. NC extension of the Ward conjecture[38] (see also Ref. 67) would be also very interesting.[27] Some NC equations are actually derived from NC (anti-)self-dual YM equations by reduction[14,30,33] and embedded[15,17,68] in $N=2$ string theories.[69] This guarantees that NC soliton equations would have physical meanings and might be helpful to understand new aspects of the corresponding string theory.

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APPENDIX: MISCELLANEOUS FORMULAS

We present explicit calculations of L^n for $n=1$, 2, 3, 4, 5 up to some order of the pseudodifferential operator ∂_x. We can read reduction conditions, e.g., $L^l=B_l$, and the explicit representations of $\text{res}_r\, L^n$ and B_m.

For $u_1=0(r=0)$:

$$L = \partial_x + u_2\partial_x^{-1} + u_3\partial_x^{-2} + u_4\partial_x^{-3} + u_5\partial_x^{-4} + u_6\partial_x^{-5} + \cdots ,$$

$$L^2 = \partial_x^2 + 2u_2 + (2u_3 + u_2')\partial_x^{-1} + (2u_4 + u_3' + u_2\star u_2)\partial_x^{-2} + (2u_5 + u_4' + u_2\star u_3 + u_3\star u_2 - u_2\star u_2')\partial_x^{-3}$$
$$+ (2u_6 + u_5' + u_2\star u_4 + u_4\star u_2 + u_3\star u_3 - u_2\star u_3' - 2u_3\star u_2' + u_2\star u_2'')\partial_x^{-4} + \cdots ,$$

$$L^3=\partial_x^3+3u_2\partial_x+3(u_3+u_2')+(3u_4+3u_3'+u_2''+3u_2\star u_2)\partial_x^{-1}+(3u_5+3u_4'+u_3''+3u_2\star u_3+3u_3\star u_2 + u_2'\star u_2-u_2\star u_2')\partial_x^{-2}+(3u_6+3u_5'+u_4''+3u_2\star u_4+3u_4\star u_2+3u_3\star u_3+u_2\star u_2\star u_2+u_2'\star u_3 - u_2\star u_3'+u_3'\star u_2-4u_3\star u_2'-u_2'\star u_2'+u_2\star u_2'')\partial_x^{-3}+\cdots,$$

$$L^4=\partial_x^4+4u_2\partial_x^2+(4u_3+6u_2')\partial_x+(4u_4+6u_3'+4u_2''+6u_2\star u_2)+(4u_5+6u_4'+4u_3''+u_2'''+6u_2\star u_3 + 6u_3\star u_2+4u_2'\star u_2+2u_2\star u_2')\partial_x^{-1}+(4u_6+6u_5'+4u_4''+u_3'''+6u_2\star u_4+6u_4\star u_2+6u_3\star u_3 + 4u_2\star u_2\star u_2+4u_2'\star u_3+2u_2\star u_3'+4u_3'\star u_2-4u_3\star u_2'-u_2'\star u_2'+u_2''\star u_2+u_2\star u_2'')\partial_x^{-2}+\cdots,$$

$$L^5=\partial_x^5+5u_2\partial_x^3+5(u_3+2u_2')\partial_x^2+5(u_4+2u_3'+2u_2''+2u_2\star u_2)\partial_x+5(u_5+2u_4'+2u_3''+u_2'''+2u_2\star u_3 + 2u_3\star u_2+2u_2'\star u_2+2u_2\star u_2')+(5u_6+10u_5'+10u_4''+5u_3'''+u_2''''+10u_2\star u_4+10u_4\star u_2 + 10u_3\star u_3+10u_2\star u_2\star u_2+10u_2'\star u_3+10u_2\star u_3'+10u_3'\star u_2+5u_2'\star u_2'+5u_2''\star u_2+5u_2\star u_2'')\partial_x^{-1} +\cdots.$$

For $u_1\neq 0(r=1)$:

$$L=\partial_x+u_1+u_2\partial_x^{-1}+u_3\partial_x^{-2}+u_4\partial_x^{-3}+u_5\partial_x^{-4}+u_6\partial_x^{-5}+\cdots,$$

$$L^2=\partial_x^2+2u_1\partial_x+(2u_2+u_1'+u_1^2)+(2u_3+u_2'+u_1\star u_2+u_2\star u_1)\partial_x^{-1}+(2u_4+u_3'+u_1\star u_3+u_3\star u_1 + u_2\star u_2-u_2\star u_1')\partial_x^{-2}+(2u_5+u_4'+u_1\star u_4+u_4\star u_1+u_2\star u_3+u_3\star u_2-2u_3\star u_1'-u_2\star u_2' + u_2\star u_1'')\partial_x^{-3}+\cdots,$$

$$L^3=\partial_x^3+3u_1\partial_x^2+3(u_2+u_1'+u_1\star u_1)\partial_x+(3u_3+3u_2'+3u_1''+3u_1\star u_2+3u_2\star u_1+u_1'\star u_1+2u_1\star u_1' + u_1\star u_1\star u_1)+(3u_4+3u_3'+u_2''+3u_1\star u_3+3u_3\star u_1+3u_2\star u_2+u_1'\star u_2+2u_1\star u_2'+u_2'\star u_1 - 2u_2\star u_1'+u_1\star u_1\star u_2+u_1\star u_2\star u_1+u_2\star u_1\star u_1)\partial_x^{-1}+\cdots,$$

$$L^4=\partial_x^4+4u_1\partial_x^3+(4u_2+6u_1'+6u_1\star u_1)\partial_x^2+(4u_3+6u_2'+4u_1''+6u_1\star u_2+6u_2\star u_1+4u_1'\star u_1+8u_1\star u_1' + 4u_1\star u_1\star u_1)\partial_x+(4u_4+6u_3'+4u_2''+u_1'''+6u_1\star u_3+6u_3\star u_1+6u_2\star u_2+4u_1'\star u_2+6u_1\star u_2' + 4u_2'\star u_1-2u_2\star u_1'+2u_1''\star u_1+2u_1\star u_1''+3u_1'\star u_1'+4u_1\star u_1\star u_2+4u_1\star u_2\star u_1+4u_2\star u_1\star u_1 + u_1'\star u_1\star u_1+2u_1\star u_1'\star u_1+3u_1\star u_1\star u_1'+u_1\star u_1\star u_1\star u_1)+\cdots,$$

$$L^5=\partial_x^5+5u_1\partial_x^4+5(u_2+2u_1'+2u_1\star u_1)\partial_x^3+5(u_3+2u_2'+2u_1''+2u_1\star u_2+2u_2\star u_1+2u_1'\star u_1+4u_1\star u_1' + 2u_1\star u_1\star u_1)\partial_x^2+(5u_4+10u_3'+10u_2''+5u_1'''+10u_1\star u_3+10u_3\star u_1+10u_2\star u_2+10u_1'\star u_2 + 20u_1\star u_2'+10u_2'\star u_1+4u_2\star u_1'+6u_1''\star u_1+15u_1'\star u_1'+11u_1\star u_1''+10u_1\star u_1\star u_2+10u_1\star u_2\star u_1 + 10u_2\star u_1\star u_1+5u_1'\star u_1\star u_1+10u_1\star u_1'\star u_1+15u_1\star u_1\star u_1'+5u_1\star u_1\star u_1\star u_1)\partial_x+\cdots.$$

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