Structure Analysis of Periodically Controlled Nonlinear Systems via the Stroboscopic Approach

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Abstract—It is shown that an efficient structure analysis of periodic controlled nonlinear systems is possible by use of the computer simula technique coupled with the stroboscopic approach. The existence stability of every periodic state can be determined together with dynamic behavior in the neighborhood of these periodic states. Especi the proposed technique seems to be the only way for finding out peri states of the saddle mode.

I. Introduction

Suppose that a system

$$\dot{x}=f(x,u), \qquad x(t)\in R^n,\ u(t)\in\Omega\subseteq R^r$$

is subject to a certain specific periodic control u such that

$$u(t+\tau)=u(t), \qquad \tau>0.$$

The u will be so chosen that the system (1) is optimized with respect certain objective function. Since any steady control $u(t)=\bar{u}$ (=const included in the class of controls satisfying (1b), the specific u may

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steady control. Although in the conventional system design it is preferred to choose the steady control exclusively, a proper periodic control that is not steady is often superior to the optimal steady control [1].

The system (1) will generally be so operated that the state x is held at a periodic state with period $m\tau$, $m=1,2,\cdots$, though an almost periodic state may also be adopted if it exists. Such a periodic state of (1) is not unique in general, and in addition some of these states may be unstable. Since, however, an unstable periodic state is usually stabilizable by adopting an appropriate feedback control, any periodic state corresponding to the specific periodic control can be a candidate of the optimal periodic state. Therefore, in order to optimize the system (1), it is necessary to investigate thoroughly the topological structure of the system for all the candidates of the optimal periodic control.

In this note an efficient method of structure analysis based on the stroboscopic method in the classical theory of nonlinear oscillations [2] is presented in order to satisfy the previously stated requirements.

II. Basic Concepts in the Stroboscopic Approach

It is generally necessary in finding out all the periodic states of (1) with period $m\tau$, $m=1,2,\cdots$ to integrate (1a) starting from various initial states. Although such an integration can efficiently be carried out with sufficient accuracy by use of an electronic computer if all the parameter values of the function f are specified, it will be hard to obtain a clear insight into the topological structure of the system directly from a set of calculated trajectories that are helical in shape. In such a situation it will be quite helpful to direct our attention only to sequences of equiphase sampled states $x((k+p)\tau)$, $k=\cdots,-2,-1,0,1,2,\cdots,0\leq p<1$. Since the transformation from $x((k+p)\tau)$ to $x((k+1+p)\tau)$ is clearly time-invariant (or independent of k), the state equation of the sampled system must have the form

$$x((k+1+p)\tau)=\phi(x((k+p)\tau)),\qquad k=\cdots,-1,0,1,\cdots. \tag{2}$$

Further, if the function f is well defined, there must exist a time-invariant continuous system

$$\dot{x}=\pi(x) \tag{3}$$

whose state just agrees with that of (2) at every instant $(k+p)\tau$, $k=\cdots,-1,0,1,\cdots$.

The discrete equiphase system (2) and the continuous equiphase system (3) were first introduced in the classical theory of nonlinear oscillations [2], and the technique was called the stroboscopic method because illumination of the state space by stroboscopic flashes occurring once for each period τ visualizes the dynamic behavior of system (2) and the persistence of vision guides one to "the pseudocontinuous slow motion" represented by (3).

It is evident that systems (2) and (3) have the following properties.

1) The existence of a closed trajectory $x_c(t)$ in the state space of (1) such that $x_c(t+\tau)=x_c(t)$ is equivalent to the existence of a discrete singular point (a fixed point) at $x_c(p\tau)$ in the state space of (2) or the existence of a singular point at $x_c(p\tau)$ in the state space of (3).

2) The existence of a closed trajectory $x_c^{(m)}(t)$ in the state space of (1) such that $x_c^{(m)}(t+m\tau)=x_c^{(m)}(t)$, $m=2,3,\cdots$ is equivalent to the existence of a discrete limit cycle switched cyclically between m points $x_c^{(m)}((k+p)\tau)$, $k=0,1,\cdots,m-1$ in the state space of (2) or the existence of a limit cycle $x_l(t)$ in the state space of (3) such that $x_l(t+m\tau)=x_l(t)$ and $x_l((k+p)\tau)=x_c^{(m)}((k+p)\tau)$, $k=0,1,\cdots,m-1$.

3) In the special case of u being a steady control, system (2) as well as system (3) just agrees with system (1) if the period of u is regarded as infinitesimal. Further, the closed trajectory $x_c(t)$ in 1) reduces to a singular point, while the closed trajectory $x_c^{(m)}(t)$ in 2) does to a limit cycle with period $m\tau$, where $m\tau$ must be nonzero and finite ($m\to\infty$ and $\tau\to 0$).

III. Stability Analysis of Periodic States

The dynamic behavior of (1) in the vicinity of its closed trajectory $x_c(t)$ with period τ can be represented with sufficient accuracy by the linearized equation

$$\dot{\xi}=A(t)\xi \tag{4}$$

where

$$\xi(t)=x(t)-x_c(t)$$

and

$$A(t)=\frac{\partial f(x_c(t),u(t))}{\partial x}. \tag{5}$$

Let the state-transition matrix of (4) be $\Phi(t,0)$. Then one can find a constant matrix P such that

$$e^{P\tau}=\Phi((p+1)\tau,p\tau) \tag{6}$$

[3] and the discrete and the continuous equiphase system of (4) become

$$\xi((k+1+p)\tau)=\Phi((p+1)\tau,p\tau)\xi((k+p)\tau) \tag{7}$$

and

$$\dot{\xi}=P\xi, \tag{8}$$

respectively.

Under a certain limited situation, approximate analytic expressions of matrices $\Phi((p+1)\tau,p\tau)$ and P are obtainable. Expansion of $\Phi((p+1)\tau,p\tau)$ into a Neumann series yields

$$\Phi((p+1)\tau,p\tau)=I+\int_{p\tau}^{(p+1)\tau}A(t_1)\,dt_1+\int_{p\tau}^{(p+1)\tau}A(t_1)\,dt_1\int_{p\tau}^{t_1}A(t_2)\,dt_2+\cdots \tag{9}$$

[4]. Since matrix $A(t)$ is periodic, one has

$$A(t)=\bar{A}+\tilde{A}(t),\quad \bar{A}=\frac{1}{\tau}\int_{p\tau}^{(p+1)\tau}A(t)\,dt$$

and

$$\int_{p\tau}^{(p+1)\tau}\tilde{A}(t)\,dt=0. \tag{10}$$

Hence, it follows that

$$\Phi((p+1)\tau,p\tau)=e^{\bar{A}\tau}+\int_{p\tau}^{(p+1)\tau}\tilde{A}(t_1)\,dt_1\int_{p\tau}^{t_1}\tilde{A}(t_2)\,dt_2+\cdots. \tag{11}$$

If either period τ or the absolute value of each element in $\tilde{A}(t)$ is sufficiently small, (11) can be reduced to

$$\Phi((p+1)\tau,p\tau)=e^{\bar{A}\tau} \tag{12}$$

and therefore it follows that

$$P=\bar{A}. \tag{13}$$

The eigenvalues of $\Phi((p+1)\tau,p\tau)$ and those of P, which are called the characteristic multipliers and the characteristic exponents, respectively, in the Floquet theory [2], determine the stability of the closed trajectory $x_c(t)$. The stability of the closed trajectory $x_c^{(m)}(t)$, $m=2,3,\cdots$ can be analyzed by replacing $x_c(t)$ by $x_c^{(m)}(t)$ and τ by $m\tau$ in (4)–(13).

The characteristic multipliers or the characteristic exponents also determine the dynamic behavior of the system in the neighborhood of a closed trajectory. For example, a closed trajectory of a two-dimensional system can be said to be in the nodal mode, in the focal mode, or in the saddle mode when the characteristic exponents are real with the same sign, a conjugate complex pair, or real with the opposite sign, respectively.

IV. Computer-Aided Structure Analysis

It is generally impossible to determine the functions ϕ and π in explicit form except for the linear case such as is considered in Section III. Therefore, the stroboscopic method developed before the advent of the electronic computer with high efficiency was considered only in the two-dimensional state space using an approximate analytical technique such as the perturbation method. In this computerized age, however, the more extensive use of the stroboscopic method is possible with the aid of the computer simulation technique. Numerical integration of (1) and (4) can easily be carried out with sufficient accuracy, and therefore the successive equiphase points and the matrix $\Phi((p+1)\tau, p\tau)$ (or $\Phi((p+1)m\tau, pm\tau)$) are easy to determine.

Forward (backward) integration will finally converge to a stable (an unstable) closed trajectory with period $m\tau$, $m=1,2,\cdots$. Otherwise, some boundary point of the system will be encountered. Hence, most closed trajectories will be found by repeating forward and backward integrations for various initial states. An unstable closed trajectory of the saddle mode in the general sense, whose characteristic exponents include those with both positive and negative real parts is, however, difficult to find because neither forward nor backward integration can converge to such a closed trajectory. The only possible way will be to predict its existence by investigating the topological structure of the discrete equiphase system and to identify its position by determining the sequences of equiphase points with sufficient density. The existence of such a closed trajectory is likely to be overlooked unless the stroboscopic approach is used.

V. Example

Consider a continuous stirred tank reactor in which an irreversible, exothermic, first-order reaction $A \rightarrow B$ is occurring. The state equation of the reactor can be written as

$$\dot{x}_1 = 1 - x_1 - ae^{-\epsilon/x_2}x_1 \tag{14a}$$

$$\dot{x}_2 = ae^{-\epsilon/x_2}x_1 - (x_2-\theta_f) - (x_2-\theta_c)u \tag{14b}$$

where

$$0 < x_1 < 1, \quad x_2 > 0, \quad u \geq 0, \quad \theta_f > 0, \quad \theta_c > 0, \quad a > 0, \quad \epsilon > 0. \tag{14c}$$

The x_1, x_2, θ_f, and θ_c are the normalized concentration of A, the normalized temperature of reactor content, the normalized feed temperature, and the normalized coolant temperature, respectively. The control u, which is proportional to the overall heat-transfer coefficient, can be varied by adjusting the coolant flow rate.

The dynamic behavior of system (14) subject to a bang-bang control

$$u = \begin{cases} u_*, & n\tau \leq t < (n+\nu)\tau \\ u^*, & (n+\nu)\tau \leq t < (n+1)\tau; \quad n=0,1,2,\cdots, \quad 0 \leq \nu \leq 1 \end{cases} \tag{15}$$

was studied by use of the stroboscopic approach in order to examine how it works. The parameter values were tentatively chosen as

$$u_* = 0.0 \quad u^* = 5.0 \quad \nu = 15/17 \quad \theta_f = 2.65 \quad \theta_c = 2.50 \quad a = 10^2e^{30} \quad \epsilon = 10^2.$$

The structure analysis of the equiphase system was carried out for various values of τ. In Fig. 1 the structure of the equiphase system is illustrated for $\tau = 1.0$. One can observe three singular points L, M, and H, which are identified through the stability analysis to be a stable focus, a saddle, and a stable node, respectively. For instance, one has

$$\Phi(\tau, 0) = \begin{bmatrix} 0.3569 & -0.1584 \\ 0.006542 & 0.2958 \end{bmatrix}$$

and

$$P = \begin{bmatrix} -1.026 & -0.4851 \\ 0.02004 & -1.213 \end{bmatrix}$$

for the point L. The eigenvalues of $\Phi(\tau,0)$ are $0.3264 \pm j0.01017$ and those of P are $-1.119 \pm j0.03116$. Therefore, one can confirm that point L is a stable focus, although it looks like a node in Fig. 1 owing to the

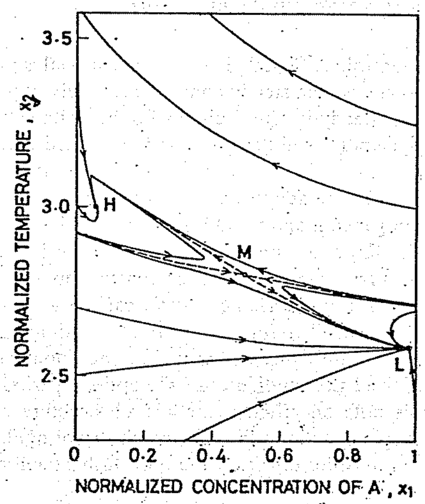

Fig. 1. Equiphase trajectories ($\tau = 1.0$). L: Stable focus. M: Saddle. H: Stable nod

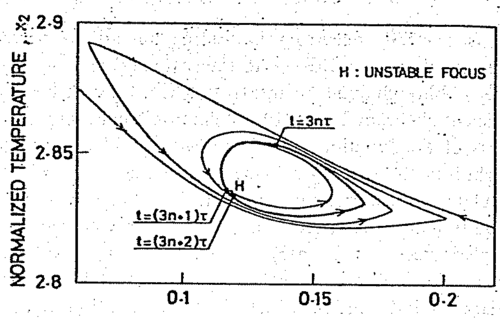

Fig. 2. Equiphase limit cycle ($\tau = 1.84$).

small imaginary part of the eigenvalues of P. If the value of τ i increased up to 1.84, point H turns into an unstable focus surrounded b a stable limit cycle, as shown in Fig. 2, while the other two points, L an M, still remain a stable focus and a saddle, respectively. As τ increased, points H and M approach each other (the limit cycle disap pears), just coincide at $\tau = 2.15$, and disappear for the value of τ larg than 2.15.

Application of the stroboscopic approach to an optimal periodic-con trol problem is currently being worked on by the authors. The techniqu could, however, be extensively applied to many other problems, too.

References

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