On the Irreducibility Condition in the Structural Controllability Theorem

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Abstract—It is known that the structural system (A,B) is structurally controllable if and only if the corresponding matrix $[A \quad B]$ is generically full rank and irreducible. In this paper it is shown that the irreducibility condition alone implies that every nonzero mode of (A,B) is generically controllable. This result provides an easy proof to the structural controllability theorem stated above. In addition, it is shown that the basic structure of the Jordan canonical form of (A,B) remains unaffected, in the generic sense, under the variation of the free parameters of (A,B).

I. Introduction

To present the objective of this paper, we begin by reviewing some definitions and a theorem associated with the concept of structural controllability. Thus, consider the linear time-invariant system

$$\dot{x}(t)=Ax(t)+Bu(t) \tag{1}$$

where $x(t)\in R^n$ and $u(t)\in R^r$. The matrices A, B are structured ones, i.e., their elements are either fixed zeros or independent free parameters. In this framework Lin [1] called the structured pair (A,B) structurally controllable if it is possible to fix free parameters in A, B so that (A,B) is controllable. Lin also derived graph-theoretically a necessary and sufficient condition for a system to be structurally controllable for the single-input case ($r=1$). The extension to the multiinput case was done by Shield and Pearson [2], Glover and Silverman [3], and Davison [4] in terms of matrix algebra. The result obtained by them is the following.

Theorem 1: The structured system (A,B) is structurally controllable if and only if the matrix $[A \quad B]$ has generically full rank and is irreducible. Here the generic rank of a structured matrix is defined to be the maximum rank that the matrix attains as a function of the free parameters, and the irreducibility of $[A \quad B]$ implies that there exists no permutation matrix P such that

$$PAP^{-1}=\begin{pmatrix} A_{11} & 0 \\ A_{12} & A_{21}\end{pmatrix}, \qquad PB=\begin{pmatrix} 0 \\ B_2\end{pmatrix}$$

with A_{11} of order $k\times k$, $1<k<n$.

Note that the proof of necessity is almost obvious because this is the direct consequences of the well-known controllability criterion. Therefore, the crucial point of the proofs in [1]–[4] existed in the sufficient part.

The main concern of the present paper lies also in this proof, but differ from [1]–[3] in that the system theoretical implication of the irreducibility condition on $[A \quad B]$ is sought independently from the generically full rank condition. As the result it will be shown that if $[A \quad B]$ is irreducible then every nonzero mode of the system (A,B) is controllable for all but an exceptional set of values of the free parameters of $[A \quad B]$. From this result the proof of Theorem 1 follows immediately. We believe that the present derivation is substantially simpler than any others. Furthermore, it will be shown that the structure of the Jordan canonical form of the structured system (A,B) is, generically, uniquely determined. The approach of [4] has analogous points with the present paper and [4, Lemma 12] is close to our result Theorem 2. Theorem 2, however, states more clearly about the consequence that can be obtained from the assumption of irreducibility.

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II. Notation

1) Assume that M is a structured matrix and n_M is the number of its nonzero entries. Then we associate with M the parameter space R^{n_M}. The set of n_M elements of a point $p=(p_1,\cdots,p_{n_M})'\in R^{n_M}$ specifies the values that the independent free parameters of M take. With this, we can identify a matrix whose free parameters are fixed at some values with the corresponding data point $p\in R^{n_M}$. This identification allows us to use expression $M\in S$ when S is a subset of R^{n_M}. For example, consider the structured matrix

$$M=\begin{pmatrix} x & x \\ 0 & x\end{pmatrix}.$$

Associated with this is the three-dimensional space R^3. A point $(p_1,p_2,p_3)'\in R^3$ corresponds to the matrix $\begin{pmatrix} p_1 & p_2 \\ 0 & p_3\end{pmatrix}$. If $S=\{p|p_1^2+P_2^2+p_3^2=3\}\subset R^3$ and $\tilde{M}=\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}$, then $M\in S$.

According to [3], $\tilde{M}$ denotes the matrix that can be obtained by fixing the free parameters of M at some particular values. For simplicity, however, the tilde over M will usually be omitted as long as this causes no confusion.

2) By $\nu(M)$ we denote the largest order of principal minor which is not identically zero. (Here we mean by a principal minor the matrix formed from the elements of the principal minor.)

III. Irreducibility Condition on $[A \quad B]$

The main result of this section is the following.

Theorem 2: Assume that the structured matrix $[A \quad B]$ is irreducible. Then every mode of the structured system (A,B) that corresponds to the nonzero eigenvalue of A is controllable generically. More precisely, there exists a proper variety $V\subset R^{n_A}\times R^{n_B}$ such that for any $[A \quad B]\in V^c$ $(=R^{n_A}\times R^{n_B}-V)$ 1) A has $\nu(A)$ nonzero mutually distinct eigenvalues and 2) all the modes that correspond to these eigenvalues are controllable, i.e., $\operatorname{rank}[\lambda I_n-A \quad B]=n$ where λ denotes a nonzero eigenvalue of A.

To prove this theorem we need three lemmas, the first of which is well known in the theory of polynomial equations [5].

Lemma 1: Let $\varphi_1(s)$ and $\varphi_2(s)$ be polynomials in s given by

$$\varphi_i(s)=a_{i_0}s^{n_i}+a_{i_1}s^{n_i-1}+\cdots+a_{i_{n_i}} \qquad (i=1,2). \tag{2}$$

Then $\varphi_1(s)$ and $\varphi_2(s)$ have a common factor if and only if the Sylvester determinant

$$R(\varphi_1,\varphi_2)=\begin{vmatrix} a_{10} & a_{11} & \cdots & a_{1_{n_1}} & 0 & \cdots & \\ 0 & a_{10} & \cdots & a_{1_{n_1}-1} & a_{1_{n_1}} & \cdots & \\ & & \cdots & & & \cdots & \\ & & \cdots & & & a_{1n_1-1} & a_{1_{n_1}} \\ & & a_{20} & a_{22} & \cdots & a_{2n_2-1} & a_{2_{n_2}} \\ & a_{20} & a_{21} & & \cdots & & 0 \\ & & & & \cdots & & \\ a_{20} & a_{21} & & & \cdots & & \end{vmatrix} \tag{3}$$

is zero, provided a_{10} and a_{20} are not both zero.

The following lemma proves part 1) of Theorem 2.

Lemma 2: Let A be an $n\times n$ structured matrix. Assume $\nu(A)>0$. Then A has generically $\nu(A)$ nonzero eigenvalues which are mutually distinct and the other $n-\nu(A)$ eigenvalues are fixed at zero.

Proof: Expand $|sI_n-A|$ as

$$|sI_n-A|=s^n+a_1s^{n-1}+\cdots+a_n \tag{4}$$

where

$$a_k=(-1)^k \sum_{i_1<i_2<\cdots<i_k} \begin{vmatrix} a_{i_1i_1} & a_{i_1i_2} & \cdots & a_{i_1i_k} \\ a_{i_2i_1} & a_{i_2i_2} & \cdots & a_{i_2i_k} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i_ki_1} & a_{i_ki_2} & \cdots & a_{i_ki_k} \end{vmatrix} \tag{5}$$

in which some $a_{i_l i_l}$ are fixed zeros and the others are independent free parameters of A. Since every principal minor of order larger than $\nu(A)$ is zero by the definition of $\nu(A)$, $a_{\nu(A)+1}=\cdots=a_n=0$ by (5). Therefore, (4) becomes

$$|sI_n-A|=s^{n-\nu(A)}\varphi_A(s) \tag{6}$$

where

$$\varphi_A(s)=s^{\nu(A)}+a_1(A)s^{\nu(A)-1}+\cdots+a_{\nu(A)}(A). \tag{7}$$

Thus, to prove the lemma, it suffices to show that $\varphi_A(s)$ has generically $\nu(A)$ nonzero mutually distinct zeros. In order to do this, define the set in R^{n_A} by

$$V_1=\{A\in R^{n_A} | a_{\nu(A)}=0 \text{ or } R(\varphi_A,\varphi_A')=0\} \tag{8}$$

where φ_A' is the derivative of φ_A with respect to s. Note that V_1 is a variety, for in view of (3) and (5) $a_{\nu(A)}$ and $R(\varphi_A,\varphi_A')$ are polynomials of the independent free parameters of A. Note further that $A\in V_1^c(=R^{n_A}-V_1)$ if and only if all the zeros of φ_A are nonzero and mutually distinct. This is obvious by (7) and Lemma 1.

Now let us prove that the variety V_1 is proper or equivalently that it is possible to choose a data point $p\in R^{n_A}$ so that the matrix A with its parameters fixed at p has $\nu(A)$ nonzero mutually distinct eigenvalues. Now by the definition of $\nu(A)$, there exists at least one $\nu(A)\times\nu(A)$ principal minor of A whose generic rank is $\nu(A)$ [see Section II, part 2)]. Let one of them be A_{11}, and set every free parameters of A except those of A_{11} to zero. Then after a permutation operation P,

$$PAP^{-1}=\begin{pmatrix} A_{11} & 0 \\ 0 & 0\end{pmatrix}\}\nu(A). \tag{9}$$

Since rank $A_{11}=\nu(A)$ (generically), A_{11} has a set of $\nu(A)$ independent nonzero entries [2]. That is, no two entries of the set lie on the same row or column of A_{11}. Let us further set all the entries of A_{11} except these $\nu(A)$ independent entries to zero. Then by Lemma 5 in the Appendix a permutation matrix Q exists such that

$$QA_{11}Q^{-1}=\begin{bmatrix} \boxed{\begin{matrix} 0 & x & 0 & \cdots & 0 \\ 0 & 0 & x & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & 0 & \cdots & x \\ x & 0 & 0 & \cdots & 0 \end{matrix}} & & & & \bigcirc \\ & \ddots & & & \\ & & \boxed{\begin{matrix} 0 & x & 0 & \cdots & 0 \\ 0 & 0 & x & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & 0 & \cdots & x \\ x & 0 & 0 & \cdots & 0 \end{matrix}} & & \\ \bigcirc & & & x & \\ & & & & \ddots \\ & & & & & x \end{bmatrix} \tag{10}$$

where x's denote independent entries. In (10), it is clearly possible to choose values for x's so that $|sI_{\nu(A)}-QA_{11}Q^{-1}|$ has $\nu(A)$ nonzero mutually distinct zeros. Therefore, if we fix values of the independent entries of A thus above, then the polynomial

$$|sI_n-A|=|sI_n-PAP^{-1}|=s^{n-\nu(A)}|sI_{\nu(A)}-QA_{11}Q^{-1}| \tag{11}$$

has $\nu(A)$ mutually distinct nonzero zeros. Thus $A\in V_1^c$. This proves the variety V_1 is proper, and therefore the proof is completed.

From this lemma we immediately obtain the following.

Corollary: Let A be a structured matrix. Suppose that there exists a data point for A such that the matrix A has a nonzero eigenvalue. Then for all but an exceptional set of values for the free parameters of A the nonzero eigenvalues of A are mutually distinct.

In the next lemma, we will show that the properties 1) and 2) in Theorem 2 are the generic ones. For this purpose, we will regard in the next lemma that the variety V_1 defined by (8) is the variety in the parameter space $R^{n_A}\times R^{n_B}$.

Lemma 3: Consider the structured matrix $[A \quad B]$ and suppose $\nu(A)>0$. Let $V_1\subset R^{n_A}\times R^{n_B}$ be the proper variety defined by (8). Then there exists a variety $V_2\subset R^{n_A}\times R^{n_B}$ such that if $[A \quad B]\in V_2$ and $[A \quad B]\in V_1^c(=R^{n_A}\times R^{n_B}-V_1)$ then at least one nonzero mode of the system (A,B) is uncontrollable, and conversely if $[A \quad B]\in V_1^c$ and some nonzero modes of (A,B) are uncontrollable, then $[A \quad B]\in V_2$.

Proof: We will first prove the second assertion. Hence, assume $[\tilde{A} \quad \tilde{B}]\in V_1^c$ and that $\tilde{A}$ has nonzero eigenvalue λ the mode corresponding to which is an uncontrollable mode of the system $(\tilde{A},\tilde{B})$. Note that λ is distinct from any other eigenvalues of $\tilde{A}$ since $[\tilde{A} \quad \tilde{B}]\in V_1^c$ and by Lemma 2. Now by the assumption, λ satisfies

$$\varphi_{\tilde{A}}(\lambda)=\lambda^{\nu(A)}+a_1(\tilde{A})\lambda^{\nu(A)-1}+\cdots+a_{\nu(A)}(\tilde{A})=0 \tag{12}$$

and there exists a nonzero row vector x' such that

$$x'\tilde{A}=\lambda x' \tag{13}$$

$$x'\tilde{B}=0. \tag{14}$$

In (13), x' is a left eigenvector of $\tilde{A}$ and λ is a simple eigenvalue of $\tilde{A}$. Therefore, x' equals (apart from a scalar factor) to any one of the nonzero rows of

$$\mathrm{adj}(\lambda I_n-\tilde{A}). \tag{15}$$

Thus by substituting this instead of x', (14) gives

$$[\mathrm{adj}(\lambda I_n-\tilde{A})]\tilde{B}=0. \tag{16}$$

Now (12) and (16) imply that the two polynomials

$$\varphi_{\tilde{A}}(s)=s^{\nu(A)}+a_1(\tilde{A})s^{\nu(A)-1}+\cdots+a_{\nu(A)}(\tilde{A}) \tag{17}$$

and

$$\psi_{\tilde{A},\tilde{B}}(s)=\mathrm{tr}\{[\mathrm{adj}(sI_n-\tilde{A})\tilde{B}][\mathrm{adj}(sI_n-\tilde{A})\tilde{B}]'\} \tag{18}$$

have common zero λ. Therefore, if we define the set V_2 by

$$V_2=\{[A \quad B]\in R^{n_A}\times R^{n_B} | R(\varphi_A,\psi_{A,B})=0\} \tag{19}$$

ere φ_A is given by (7) and

$$\psi_{A,B}(s)=\mathrm{tr}\{[\mathrm{adj}(sI_n-A)B][\mathrm{adj}(sI_n-A)B]'\}, \tag{20}$$

en by Lemma 1, $[\tilde{A},\tilde{B}]$ belongs to V_2.

Now conversely suppose $[\tilde{A} \quad \tilde{B}]\in V_2\cap V_1^c$. Then $\varphi_{\tilde{A}}$ and $\psi_{\tilde{A},\tilde{B}}$ have a mmon zero by Lemma 1. Let the common zero be λ. Then since $\varphi_{\tilde{A}}$ is factor of det $(sI-\tilde{A})$ and $\tilde{A}\in V_1^c$, λ is a simple nonzero eigenvalue of Therefore, adj $(\lambda I-\tilde{A})$ is a nonzero matrix and any one of its nonzero ws is a left eigenvector of $\tilde{A}$. On the other hand, $\psi_{\tilde{A},\tilde{B}}(\lambda)=0$ implies 6). Therefore, these both show that (13) and (14) hold simultaneously if is replaced by any one of the nonzero rows of adj $(\lambda I-\tilde{A})$. This plies the mode corresponding to λ is uncontrollable.

Proof of Theorem 2: If $\nu(A)=0$, the assertion of the theorem is trivial. r in this case A cannot have any nonzero eigenvalues for almost any ta points. Therefore, in below it is assumed $\nu(A)>0$. Now consider the riety given by

$$V=V_1\cup V_2 \tag{21}$$

ere V_1 and V_2 are defined, respectively, by (8) and (19). We will show t V is proper. Then since $[\tilde{A} \quad \tilde{B}]\in V^c(=R^{n_A}\times R^{n_B}-V=V_1^c\cap V_2^c)$ plies $[\tilde{A} \quad \tilde{B}]\in V_1^c$ and therefore $\tilde{A}$ has $\nu(A)$ nonzero simple eigenvals by Lemma 1 and in this case every nonzero mode of $(\tilde{A},\tilde{B})$ is ntrollable if and only if $[\tilde{A} \quad \tilde{B}]\in V_2^c$ by Lemma 3, this yields the claim the theorem.

Now to the contrary to what is to be proved, we will assume that V is t proper and find a contradiction. Since $V^c=V_1^c\cap V_2^c$ and V_1 was eady shown to be proper, the assumption implies V_2 is not proper. erefore, by Lemma 3, we can equivalently assume that for any $B]\in V_1^c$ the system (A,B) has at least one uncontrollable mode that rresponds to a simple nonzero eigenvalue of A. Since the uncontrollae mode of $(A \quad B)$ is in general a function of the free parameters of A, will be denoted by $\lambda(A)$.

Now take an appropriate permutation matrix P_1 such that the strucred matrices $P_1AP_1^{-1}$ and P_1B are, respectively, in the form

$$P_1AP_1^{-1}=\begin{pmatrix} A_{11} & A_{12}\\ A_{\alpha 1} & A_{\alpha 2}\end{pmatrix},\qquad P_1B=\begin{pmatrix}0\\ B_\alpha\end{pmatrix} \tag{22}$$

ere B_α has no zero rows. Since (A,B) has the uncontrollable mode $A)$, there exists n-dimensional row vector $(x_1',x_\alpha')\neq 0$ (which is in neral a function of the free parameters of $[A \quad B]$) such that

$$(x_1',x_\alpha')\begin{pmatrix} A_{11} & A_{12}\\ A_{\alpha 1} & A_{\alpha 2}\end{pmatrix}=\lambda(A)(x_1',x_\alpha') \tag{23-1}$$

$$(x_1',x_\alpha')\begin{pmatrix}0\\ B_\alpha\end{pmatrix}=0 \qquad \text{for } [A \quad B]\in V_1^c. \tag{23-2}$$

ote that the first of these implies (x_1',x_α') is a left eigenvector of A for e simple eigenvalue $\lambda(A)$. Therefore, (x_1',x_α') is uniquely determined part from a scalar factor) by A so that it is independent of B. From is and from the fact that (23-2) holds for any $[A \quad B]\in V_1^c$ and hence r any values of the free parameters of B_α, x_α' must be identically zero. erefore, by substituting this into (23-1), we have

$$x_1'A_{11}=\lambda(A)x_1' \tag{24-1}$$

$$x_1'A_{12}=0 \tag{24-2}$$

r any $[A \quad B]\in V_1^c$. Observe that these are the same relations with 3-1), (23-2) except that in this case $\lambda(A)$ may happen to be a multiple genvalue of A_{11}. By the corollary to Lemma 2, however, there exists a oper variety W_1 in the parameter space of A_{11} such that if $A_{11}\in W_1^c$ ery nonzero eigenvalue of A_{11} is simple. This variety W_1 can be nsidered to be the proper variety in $R^{n_A}\times R^{n_B}$. Therefore, if we take $_1\cup V_1$ instead of V_1 at the start, $\lambda(A)$ is a simple eigenvalue of both A d A_{11}.

Now perform the permutation P_2 that puts the nonzero entries A_{12} to the final rows:

$$P_2A_{11}P_2^{-1}=\begin{pmatrix} A_{22} & A_{23}\\ A_{\beta 1} & A_{\beta 2}\end{pmatrix},\qquad P_2A_{12}=\begin{pmatrix}0\\ A_\beta\end{pmatrix} \tag{25}$$

where A_β has no zero rows. Then (24) gives

$$(x_2',x_\beta')\begin{pmatrix} A_{22} & A_{23}\\ A_{\beta 1} & A_{\beta 2}\end{pmatrix}=\lambda(A)(x_2',x_\beta') \tag{26-1}$$

$$(x_2',x_\beta')\begin{pmatrix}0\\ A_\beta\end{pmatrix}=0 \tag{26-2}$$

where $(x_2',x_\beta')=x_1'P_2^{-1}$. These equations hold for any $[A \quad B]\in(W_1\cup V_1)^c$ and hence for almost any values of the free parameters of A_{11}, A_{12}. In addition, $\lambda(A)$ is a simple eigenvalue of A_{11}. Therefore, by the same reasoning used to derive (24), we have $x_\beta'=0$. Thus,

$$x_2'A_{22}=\lambda(A)x_2' \tag{27-1}$$

$$x_2'A_{23}=0 \tag{27-2}$$

for $[A \quad B]\in(W_1\cup V_1)^c$.

Observe that the above argument can be repeated at any finite times. Therefore, we finally arrive at the stage where

$$x_N'A_{NN}=\lambda(A)x_N' \tag{28-1}$$

$$x_N'A_{NN+1}=0 \tag{28-2}$$

for any $[A \quad B]\in(V_1\cup W_1\cup\cdots\cup W_{N-1})^c$ and either A_{NN+1} has no zero rows or $A_{NN+1}=0$. The first case is impossible since this implies $x_N'=0$ and by reversing the derivation we have $(x_1',x_\alpha')=0$ which contradicts to $(x_1',x_\alpha')\neq 0$. Thus, $A_{NN+1}=0$. That is, denoting the composite permutation by P, we have

$$PAP^{-1}=\left[\begin{array}{cccccc|c} A_{NN} & 0 & 0 & & \cdots & 0 & \\ A_{\gamma 1} & A_{\gamma 2} & A_{\gamma} & 0 & \cdots & 0 & \\ & \ddots & \ddots & \ddots & & \vdots & \\ \hline & A_{\beta 1} & & & A_{\beta 2} & A_\beta & \\ \hline & & A_{\alpha 1} & & & A_{\alpha 2} & \end{array}\right],\qquad PB=\begin{bmatrix}0\\0\\ \vdots\\ 0\\ \hline B_\alpha\end{bmatrix}. \tag{29}$$

This is the desired contradiction, for $[A \quad B]$ was assumed to be irreducible.

IV. Other Results

1) Proof of Theorem 1: The necessary part is obviously true. Sufficiency: Since $[A \quad B]$ has generically full rank by the assumption, there exists a proper variety W such that

$$\mathrm{rank}[A \quad B]=n \qquad \text{for } \forall[A \quad B]\in W^c=R^{n_A}\times R^{n_B}-W. \tag{30}$$

This and the irreducibility of $[A \quad B]$ implies that (A,B) is structurally controllable. For if

$$[A \quad B]\in(V\cup W)^c,$$

every nonzero mode of (A,B) is controllable by Theorem 2 and zero modes are controllable by (30).

2) Theorem 3: Assume that the structured system (A,B) is structurally controllable. Then for almost every choice of values of the free parameters of (A,B), the Jordan canonical form of the system takes the form

$$\hat{A}=\mathrm{diag}[A_0,A_1,\cdots,A_s] \tag{31}$$

$$\hat{B}=[B_0',B_1',\cdots,B_s']' \tag{32}$$

where B_0 and B_i are of order $\nu(A)\times r$ and $\nu_i\times r$, respectively, and

$$\underset{(\nu(A)\times\nu(A))}{A_o}=\mathrm{diag}[c_1,\cdots,c_{\nu(A)}] \tag{33}$$

$$\underset{(\nu_i\times\nu_i)}{A_i}=\begin{pmatrix}0 & 1 & & \\ & & \ddots & \\ & & & 1\\ 0 & & \cdots & 0\end{pmatrix}\quad (\nu_i>1),\quad A_i=0\ (\nu_i=1) \tag{34}$$

in which 1) the numbers s and ν_i $(i=1,\cdots,s)$ are uniquely determined by (A,B) without depending on the chosen values of the free parameter of (A,B), 2) every rows of B_0 is nonzero, 3) the last rows of the matrices of $B_1,\cdots,B_s$ are linearly independent, and 4) complex numbers c_i are nonzero and mutually distinct.

Proof: By Lemma 2 it immediately follows that if $A \in V_1^c$ the Jordan canonical form of A has the form (31), (33), and (34). Assertion 4) is also an immediate consequence of Lemma 2. Assertions 2) and 3) are the restatement of the structurally controllability assumption. Lastly, assertion 1) comes from the fact that the numbers s and ν_i are determined by making use of the formulas

$$s=n-\operatorname{rank}A(g) \tag{35}$$

and

the numbers of the Jordan blocks of the form (34) with order $\nu_i \times \nu_i$

$$=\begin{cases}\operatorname{rank}A^{\nu_i-1}(g)-2\operatorname{rank}A^{\nu_i}(g)+\operatorname{rank}A^{\nu_i+1}(g) & (1<\nu_i<n-1)\\ \operatorname{rank}A^{\nu_i-1}(g)-\operatorname{rank}A^{\nu_i}(g) & (\nu_i=n)\end{cases} \tag{36}$$

where rank $A^j(g)$ denotes the generic rank of A^j. These formulas are obtained by an easily provable lemma.

Lemma 4: Let A be an usual $n \times n$ numerical matrix. Assume λ is an eigenvalue of A. Let n_i denote the number of Jordan blocks corresponding to λ with order $i \times i$ or larger. Then

$$n_i=\operatorname{rank}(A-\lambda I_n)^{i-1}-\operatorname{rank}(A-\lambda I_n)^i \qquad (i=1,\cdots,n). \tag{37}$$

Proof: Omitted.

V. Conclusions

A simple proof to the structural controllability theorem was derived. This was done through the comprehensive study on the irreducibility condition of the structured matrix $[A \quad B]$. Through this study it was also shown that the fundamental structure (i.e., the Jordan canonical form) of the structured system (A,B) remains (generically) unaffected by the parameter variations. Other problems of interest which are under investigations are to obtain the algorithm for the computation of rank $A^j(g)$ where A is a structured matrix, and to determine the dimension of the controllability subspace of (A,B) in the case (A,B) is not structurally controllable.

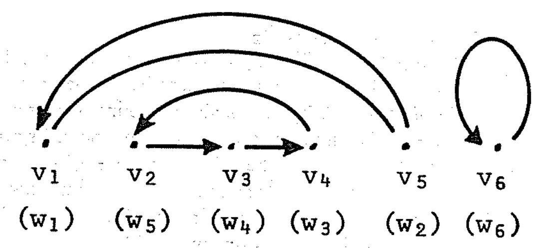
Appendix

Lemma 5: If A_{11} is a $\nu \times \nu$ matrix with every column and row having one and only one nonzero entry, then a permutation matrix Q exists such that $QA_{11}Q^{-1}$ has the form (10).

Proof: The simplest way to the proof will perhaps be graph-theoretic one. That is, write the graph corresponding to A_{11} [1]. It is then easy to see that the graph is separated into several cycles. The numbers of the cycles give the numbers of the nonzero submatrices of (10). The numbers of nodes in each cycles determine the order of the nonzero submatrix of $QA_{11}Q^{-1}$ in which Q is obtained by relabeling the nodes. We illustrate this by the following example. The extension to the general case is straightforward. Let

$$A_{11}=\begin{bmatrix}0&0&0&0&\alpha_1&0\\0&0&0&\alpha_2&0&0\\0&\alpha_3&0&0&0&0\\0&0&\alpha_4&0&0&0\\\alpha_5&0&0&0&0&0\\0&0&0&0&0&\alpha_6\end{bmatrix}.$$

Associated with this is the graph (see [1])



This graph contains three cycles. Thus, A can be reduced by a permutation into a block diagonal matrix with three nonzero blocks. To obtain Q, relabel the nodes as indicated in the brackets. If w_i corresponds to v_j, set $q_{ij}=1$. Otherwise set $q_{ij}=0$. Then Q is given by $Q=(q_{ij})$. In this case

$$Q=\begin{bmatrix}1&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&1&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&1\end{bmatrix}.$$

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