Determination of Generic Dimensions of Controllable Subspaces and Its Application

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Abstract—**The controllable subspace and its dimension of a structured linear system vary as a function of the free parameters. However, the dimension is stable in the sense that it takes, for almost any system parameters, some maximal constant which is the generic rank of the controllability matrix. In this paper, this maximal constant is called the generic dimension of the controllable subspace. Two simple methods for determining generic dimensions of controllable subspaces are derived. As an application, the results are applied to the determination of system types of linear multivariable unity feedback systems.**

I. Introduction

Since Lin introduced the concept of structural controllability [1], there appeared many papers on the subject extending Lin's single-input results to the multiinput case or giving more elegant proofs to the structural controllability theorems [2]–[5]. At this point, however, considerations in these researches are mainly directed to determining whether or not a given system is structurally controllable. In order for the concept to be utilized in synthesis, e.g., robust synthesis of linear feedback systems, more work is needed.

This paper aims at extending some of the previous results to include the case where a system is structurally uncontrollable. In such a case, the controllable subspace and its dimension vary as a function of the system parameters. The dimension, however, is stable in the sense that it takes, for almost any system parameters, its maximal constant value which equals the generic rank of the controllability matrix. In this paper, this maximal constant value is called the generic dimension of the controlla-

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ble subspace. Two simple methods for determining the generic dimensions of controllable subspaces are derived.

As an application, the above results are used to analyze the steady-state characteristics of linear feedback systems. In our earlier paper [6] and also in [7], as an extended concept of system type in the classical servomechanism theory, the concept of type $[l_1,\cdots,l_m]$ was introduced. In this paper, the concept of structural system type is defined and a systematic method for determining types is given.

II. Preliminaries

Notations and Definitions

Consider the linear time-invariant control system

$$\dot{x}(t)=Ax(t)+Bu(t) \tag{1}$$

with $x(t)\in R^n$, $u(t)\in R^m$, and assume that A and B are structured matrices, i.e., their elements are either fixed (zero) or nonfixed (indeterminate) and the indeterminate parameters are unrelated. In the following, (1) is called a structured system and is denoted by (A,B). $R^{n_A}\times R^{n_B}$ denotes the associated parameter space where $n_A(n_B)$ indicates the number of nonfixed entries of $A(B)$. $\tilde{A}(\tilde{B})$ denotes a matrix that is obtained from $A(B)$ by fixing its indeterminates at some particular values. $\tilde{A}$ will be identified with the corresponding data point $p_{\tilde{A}}\in R^{n_A}$. This identification allows us to use such expressions as $\tilde{A}\in S$ when S is a subset of R^{n_A}. The structured system (A,B) is said to be irreducible if there exists no permutation matrix P such that

$$PAP^{-1}=\begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22}\end{pmatrix}, PB=\begin{pmatrix} 0 \\ B_2\end{pmatrix} \tag{2}$$

with A_{11} of order $k\times k$, $1\le k<n$. Otherwise, (A,B) is said to be reducible.

Let M be an $n\times l$ $(n<l)$ structured matrix. $M_{i_1,\cdots,i_s}$ denotes an $s\times(l-n+s)$ submatrix of M formed by striking out all except rows $i_1,\cdots,i_s$ and all except columns $i_1,\cdots,i_s,n+1,n+2,\cdots,l$ $(1\le i_1<i_2<\cdots<i_s\le n)$. $M^{j_1,\cdots,j_t}$, on the other hand, denotes another submatrix of M which is formed from columns $j_1,\cdots,j_t$ of M and of order $n\times t$ $(1\le j_1<\cdots<j_t\le l)$.

Example: Let $A=\begin{pmatrix}0 & x\\ 0 & 0\end{pmatrix}$, $B=\begin{pmatrix}x\\0\end{pmatrix}$, and let $M=[A\quad B]$. Then $M_1=[0\quad x]$, $M_2=[0\quad 0]$, and $M^{1,2}=A$.

By $\nu(M)$ we denote the integer defined by

$$\nu(M)=\max\{s:\text{integer}\,|\,{}^\exists i_1,\cdots,i_s,\ M_{i_1,\cdots,i_s}\text{ has generically full rank}\} \tag{3}$$

where the generic rank of a structured matrix is defined to be the maximal rank that the matrix achieves as a function of its free parameters. If M is the matrix given in the above example, then $\nu(M)=1$. Note that we have, in general, the relation

$$\nu(M)\le \text{generic rank } M.$$

Actually, if the matrix A is as above, then we have $\nu(A)=0$, whereas generic rank $A=1$.

Graph G(A, B)

Associated with the structured system (A,B), the graph $G(A,B)$ is defined as the directed graph which contains $n+m$ nodes $v_1,\cdots,v_n,v_{n+1},\cdots,v_{n+m}$ and n_A+n_B oriented edges. The oriented edge (v_j,v_i) (an arrow going from v_j to v_i) corresponds in a one-to-one way to the nonzero entry c_{ij} of $[A\quad B]$. The nodes $v_{n+1},\cdots,v_{n+m}$ that correspond to the columns of B are called the origins of the graph.

III. Generic Dimensions of Controllable Subspaces

Consider (A,B) to be a structured system. Then the dimension of its controllable subspace, which equals the rank of the controllability matrix $[B\quad AB\cdots A^{n-1}B]$, varies as a function of the free parameters of A, B. The function, however, takes some constant value for all but an exceptional set of values of the free parameters. To see this, let

$$d_c=\text{generic rank}[B\quad AB\quad\cdots\quad A^{n-1}B]. \tag{4}$$

Then from the definition of generic rank, every minor of the controllability matrix with order larger than d_c is identically zero and at least one minor with order d_c is not. So that if $\psi(A,B)$ denotes the polynomial defined as the sum of squares of all minors with order d_c, the set

$$U\triangleq\{[\tilde{A}\quad\tilde{B}]\in R^{n_A}\times R^{n_B}\,|\,\psi(\tilde{A},\tilde{B})=0\}$$

is a proper variety of the parameter space. Furthermore, the dimension of the controllable subspace of $(\tilde{A},\tilde{B})$ is equal to d_c if and only if $[\tilde{A}\quad\tilde{B}]\notin U$ and, therefore, for almost any $[\tilde{A}\quad\tilde{B}]$. This observation allows us to propose the following.

Definition 1: The generic rank of the controllability matrix of a structured system is called the generic dimension of the controllable subspace. If the generic dimension is equal to n (the dimension of the whole space), then the structured system is called structurally controllable.

Before proceeding to the next section, let us recall the following results.

Lemma 1 [1]–[5]: The structured system (A,B) is structurally controllable if and only if it is irreducible and the matrix $[A\quad B]$ has generically full rank.

Lemma 2 [5]: Assume that the structured system (A,B) is irreducible. Then every mode of (A,B) that corresponds to a nonzero eigenvalue of A is generically controllable. More precisely, there exists a proper variety $V\in R^{n_A}\times R^{n_B}$ such that for any $[\tilde{A}\quad\tilde{B}]\notin V$: 1) $\tilde{A}$ has $\nu(A)$ nonzero distinct eigenvalues, and 2) all the modes that correspond to these eigenvalues are controllable, i.e., rank $[\lambda I_n-\tilde{A}\quad\tilde{B}]=n$ for any λ which is a nonzero eigenvalue of $\tilde{A}$.

Although it was not stated in [5], it is easy to see that $\tilde{A}$ cannot have more than $\nu(A)$ nonzero eigenvalues.

IV. Determination of Generic Dimensions of Controllable Subspaces

The generic dimension d_c of the controllable subspace of the structured system (A,B) can be determined by seeking the maximal order of the minors of $[B\quad AB\cdots A^{n-1}B]$ that are not identically zero. But this procedure is very awkward since it requires a large amount of computation which involves many indeterminate parameters. Instead, we give in this section two different methods. In the first method, the determination of d_c is reduced to the computation of the generic ranks of usual structured matrices which can be computed, for example, by the Hungarian method [8]. The second method is the graph-theoretic reinterpretation of the first.

Now assume that the system (A,B) is structurally uncontrollable. Then by Lemma 1, either $(A\quad B)$ is reducible or generic rank $[A\quad B]<n$ or both. In the first case, utilizing Boolean operations, for example, it is easy to find a permutation matrix P such that

$$PAP^{-1}=\begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22}\end{pmatrix}\begin{matrix}\}n_1\\ \}n_2\end{matrix}, PB=\begin{pmatrix} 0 \\ B_2\end{pmatrix}\begin{matrix}\}n_1\\ \}n_2\end{matrix} \tag{5}$$

and $(A_{22}\quad B_2)$ is irreducible. Then since

$$\text{generic rank}[B\quad AB\quad\cdots\quad A^{n-1}B] = \text{generic rank}[B_2\quad A_2B_2\quad\cdots\quad A_2^{n-1}B_2], \tag{6}$$

the problem of determining d_c for the reducible system (A,B) reduces to that for the irreducible system (A_{22},B_2). With this observation, it can be assumed at the start that the structured system (A,B) is irreducible.

Theorem 1: Let d_c be the generic dimension of the controllable subspace of the structured system (A,B). Assume that (A,B) is irreducible. Then

$$d_c=\nu([A\quad B]) \tag{7}$$

where the definition of $\nu(\cdot)$ is given by (3). In terms of the graph $G(A,B)$, (7) can be represented by

$$d_c = \max_{G\in\mathcal{G}}\{|E(G)|\} \tag{8}$$

where $\mathcal{G}$ denotes the set of subgraphs of $G(A,B)$ which is defined by

$$\mathcal{G}=\{G\subset G(A,B)|G \text{ consists of cycles and at most } m \text{ simple paths of } G(A,B). \text{ The paths start from the origin. The cycles and paths have no node in common}\}. \tag{9}$$

$|E(G)|$ denotes the number of edges contained in G.

Before the presentation of the proof, let us examine a simple example to illustrate the notations.

Example [2]: Let

$$A=\begin{bmatrix} a_1 & a_5 & a_7 & a_9 \\ a_2 & a_6 & a_8 & a_{10} \\ a_3 & 0 & 0 & 0 \\ a_4 & 0 & 0 & 0 \end{bmatrix}, B=\begin{bmatrix} b_1 & 0 \\ 0 & b_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then, generic rank $[A \quad B]=3<n$ and

$$\text{generic rank } [A \quad B]_{1,2,3}=\text{generic rank}\begin{bmatrix} a_1 & a_5 & a_7 & b_1 & 0 \\ a_2 & a_6 & a_8 & 0 & b_2 \\ a_3 & 0 & 0 & 0 & 0 \end{bmatrix}=3$$

so that $\nu([A \quad B])=3$. This shows that the structured system of this example has three generic dimensional controllable subspace. The same result can be obtained by using the graph $G(A,B)$. In fact, the graph that corresponds to the present example is shown in Fig. 1. By inspection, it is easily seen that the graph cannot have subgraphs with more than three edges such that the properties indicated on the right-hand side of (9) are satisfied. On the other hand, the subgraph indicated by the bold arrow satisfies all the required properties. Therefore, $\max_{G\in\mathcal{G}}\{|E(G)|\}=3$. This result coincides with the above result.

To prove the theorem, the next lemma is needed.

Lemma 3: Let F and G be, respectively, $n\times n$ and $n\times m$ numerical matrices, and let $p(s)$ be the monic greatest common divisor of $n\times n$ minors of $[sI-F \quad G]$. Then

$$\text{rank}[G \quad FG \quad \cdots \quad F^{n-1}G]=n-\deg p(s). \tag{10}$$

For the proof, see [9], for example.

Proof of Theorem 1: Since (A,B) is irreducible by assumption, Lemma 2 shows that there exists a proper variety V of $R^{n_A}\times R^{n_B}$ such that every nonzero mode of $(\tilde{A},\tilde{B})$ is controllable if $[\tilde{A} \quad \tilde{B}]\notin V$. Therefore, the greatest common divisor $p_{\tilde{A},\tilde{B}}(s)$ of the largest minors of $[sI-\tilde{A} \quad \tilde{B}]$ cannot have any nonzero zeros for any $[\tilde{A} \quad \tilde{B}]\notin V$, i.e., $p_{\tilde{A},\tilde{B}}(s)$ has the form s^{x}. Now consider an arbitrary largest minor of $[sI-A \quad B]$ which is represented by

$$\det\left([sI-A \quad B]^{i_1,\cdots,i_k,j_1,\cdots,j_l}\right) = a_k s^k+\cdots+a_q s^q+\cdots+a_1 s+a_0 \tag{11}$$

where

$$1\leq i_1<\cdots<i_k\leq n<j_1<\cdots<j_l\leq n+m$$

and

$$k+l=n.$$

For ease of the explanation, suppose that $i_1=1$, $i_2=2,\cdots,i_k=k$ and $j_1=n+1,\cdots,j_l=n+l$. Then by expanding the left-hand side of (11), we obtain

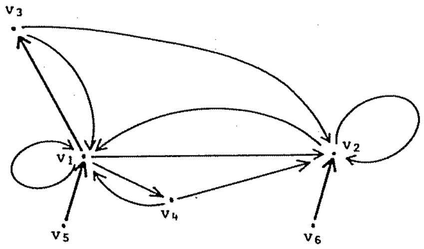

Fig. 1. Graph $G(A,B)$.

Note that the number of lower subindices of $[A \quad B]$ in the above equals $n-q$. This observation immediately proves that if

$$n-q>\nu([A \quad B]), \tag{13}$$

then

$$a_q\equiv 0, \tag{14}$$

for from (13) and by the definition of ν, the relation

$$\text{generic rank } [A \quad B]_{t_1,\cdots,t_{n-q}}<n-q$$

holds for any $1\leq t_1<t_2<\cdots<t_{n-q}\leq n$.

This result shows that the minor under consideration is divisible by $s^{n-\nu([A \quad B])}$. In the same manner, it is possible to show that the result holds for any remaining largest minor of $[sI-A \quad B]$.

Next it will be shown that at least one largest minor of $[sI-A \quad B]$ has a term of the form $a\cdot s^{n-\nu[(A \quad B)]}$. To see this, define the veriety W by

$$W=\left\{[\tilde{A} \quad \tilde{B}]\in R^{n_A}\times R^{n_B}|\Sigma a^2_{n-\nu([A \quad B])}(\tilde{A},\tilde{B})=0\right\}$$

where $a_{n-\nu([A \quad B])}$ denotes the coefficient of $s^{n-\nu([A \quad B])}$ in a largest minor of $[sI-A \quad B]$ (which is clearly a polynomial of the free parameters of A,B) and the sum is carried over all the choice of the largest minors. Then what is needed is to show that W is a proper variety. For this, recall the definition of ν again. Then there exist integers $i_1,\cdots,i_{\nu([A \quad B])}$ (for ease, they will be assumed to be $1,2,\cdots,\nu([A \quad B])$, respectively) such that

$$\text{generic rank } [A \quad B]_{1,2,\cdots,\nu([A \quad B])}=\nu([A \quad B]).$$

This shows that $[A \quad B]_{1,\cdots,\nu([A \quad B])}$ and, therefore, the structured matrix $[A \quad B]$ containes a submatrix M of order $\nu([A \quad B])\times\nu([A \quad B])$ which has generically full rank. Now in $[A \quad B]$ fix the independent free parameters except those of M to zero and fix the parameters of M so that $\det\tilde{M}\neq 0$ and denote the matrix thus obtained by $[\tilde{A} \quad \tilde{B}]$. Then it is easy to see that the minor of $[sI-\tilde{A} \quad \tilde{B}]$ which is computed from the columns containing $\tilde{M}$ and the columns $\nu([A \quad B])+1$, $\nu([A \quad B])+2,\cdots,n$ has the term $\det\tilde{M}\cdot s^{n-\nu([A \quad B])}$. This shows that $[\tilde{A} \quad \tilde{B}]\notin W$. Hence, the variety W is proper.

Thus, it has been proved that if $[\tilde{A} \quad \tilde{B}]\notin W\cup V$, the monic greatest common divisor of the largest minors of $[sI-\tilde{A} \quad \tilde{B}]$ is $s^{n-\nu([A \quad B])}$. Therefore, by Lemma 3 we have

$$\text{rank}[\tilde{B} \quad \tilde{A}\tilde{B} \quad \cdots \quad \tilde{A}^{n-1}\tilde{B}]=n-\left(n-\nu([A \quad B])\right)=\nu([A \quad B]),$$
$$\text{for } \forall[\tilde{B} \quad \tilde{A}]\notin W\cup V. \tag{15}$$

This is the desired result (7).

Equation (8) is obtained by the following observation. $[A \quad B]_{i_1,\cdots,i_{\nu([A \quad B])}}$ has generically full rank if and only if there exists a set of $\nu([A \quad B])$ independent entries of $[A \quad B]_{i_1,\cdots,i_{\nu([A \quad B])}}$. If we set every entry of $[A \quad B]$ to zero except these $\nu([A \quad B])$ independent entries and write the graph corresponding to the matrix thus obtained,

$$a_q=(-1)^{k-q}\sum_{1\leq j_1<\cdots<i_{k-q}\leq k}\det\left\{\left([A \quad B]_{i_1,\cdots,i_{k-q,k+1,\cdots,n}}\right)^{1,2,\cdots,k-q,n-q+1,\cdots,n-q+l}\right\}, \qquad q=0,1,\cdots,k. \tag{12}$$

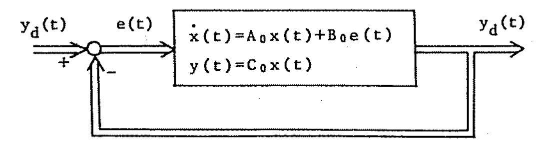

Fig. 2. m-input, m-output unity feedback system. The structured matrices corresponding to C_0, A_0, B_0 are, respectively, denoted by C, A, B.

then we have a subgraph of $G(A, B)$ which possesses all the required properties on the right-hand side of (9).

Corollary: For almost any $[\tilde{A} \quad \tilde{B}] \in R^{n_A} \times R^{n_B}$, the system $(\tilde{A}, \tilde{B})$ has $n-\nu([A \quad B]) = n - \max\{|E(G)|\}$ uncontrollable zero modes. This result is true even if (A, B) is reducible.

Proof: If the system (A, B) is irreducible, by the proof to Theorem 1 for any $[\tilde{A} \quad \tilde{B}] \notin V \cup W$ the monic greatest common divisor of the largest minors of $[sI - \tilde{A}, \tilde{B}]$ is $s^{n-\nu([A \quad B])}$. Thus, the result follows. When the system (A, B) is reducible, we can suppose that the system is represented as on the right-hand side of (5) where (A_{22}, B_2) is irreducible. In this case, for almost any $[\tilde{A}_{22} \quad \tilde{B}_2] \in R^{n_A} \times R^{n_B}$, the system $(\tilde{A}_{22}, \tilde{B}_2)$ has $n_2 - \nu([A_{22} \quad B_2])$ uncontrollable zero modes. Clearly, any zero eigenvalues of $\tilde{A}_{11}$ give uncontrollable zero modes of $(\tilde{A}, \tilde{B})$. Thus, we have

$$\begin{aligned}\text{the number of the uncontrollable zero modes}\\ &= n_2 - \nu([A_{22} \quad B_2]) + n_1 - \nu(A_{11})\\ &= n - \nu([A \quad B]).\end{aligned}$$

V. Structural System Type

In this section, we use the results in the previous section to analyze the steady-state characteristics of linear servomechanisms.

In our earlier paper [6] and also in [7], the classical concept of system type was extended to linear multivariable servomechanisms. Specifically, consider the stable unity feedback system in Fig. 2 in which C_0, A_0, B_0 are, respectively, the $m \times n$, $n \times n$, $n \times m$ usual numerical matrices. In [6], the system was called type $[l_1 \quad \cdots \quad l_m]$ if l_i is the maximal of k_i such that the system can track (with zero steady error) inputs of the form

$$y_d(t) = \begin{bmatrix} \dfrac{a_1 t^{k_1 - 1}}{(k_1 - 1)!} \\ \vdots \\ \dfrac{a_m t^{k_m - 1}}{(k_m - 1)!} \end{bmatrix}, \qquad (k_i > 0) \tag{16}$$

where $a_i t^{-1}/(-1)! \equiv 0$ if $k_i = 0$ and the a_i's are arbitrary real constants.

Now to exhibit our present motivation, let us consider the following example. In Fig. 2, let

$$C_0 = [-1, 2], \; A_0 = -\begin{pmatrix} 0.5 & 0.4 \\ 0.25 & 0.2 \end{pmatrix}, \; B_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{17}$$

Then the transfer function of the open-loop system is $2s + 1.4/s(s+0.7)$. Therefore, this stable unity feedback system can track step inputs with zero steady-state error so that the system is type 1. However, this conclusion is valid only when the numerical values in (17) are considered to be fixed. Even small variation of the values makes the system different from type 1. On the other hand, if the open-loop system is represented

$$C_0 = [-1 \quad 2], \; A_0 = -\begin{pmatrix} 0 & 1 \\ 0 & 0.7 \end{pmatrix}, \; B_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and the zeros of A_0 can be considered to be fixed by coordinatization, then irrespective of the variation of the other parameters, the feedback system remains type 1. These two examples suggest that the system types of feedback systems can be determined according to the structures of the open-loop systems. So let us propose the following.

Definition 2: Suppose that the unity feedback system of Fig. 2 is stable. Let C, A, B be the structured matrix corresponding, respectively, to C_0, A_0, B_0 (see Remark below). Finally, assume that, as long as the closed loop is stable, for all but an exceptional set of values of $\tilde{C}, \tilde{A}, \tilde{B}$, the feedback system is type $[l_1, \cdots, l_m]$, i.e., the l_i's are integers that do not depend on the specific values of the free parameters. Then the feedback system is said to be type $[l_1, \cdots, l_m]$ structurally.

Remark [2]: The zero elements of the structured matrices should be considered to be the ones that are fixed by coordinatization (e.g., the time derivative of position is velocity) or by the absence of physical connections between certain parts of the system.

Observe that two examples of this section are, respectively, type 0 and type 1 structurally. The determination of the structural system type, in the general case, is an easy application of the previous results. In fact, we have the following.

Theorem 2: Assume that (A, B) is structurally controllable and (C, A) is structurally observable (i.e., (A', C') is structurally controllable). Let

$$l_i = n - \nu([A' \quad C'_{\bar{i}}]) \qquad (i = 1, \cdots, m), \tag{18}$$

$$= n - \max_{G \in \mathcal{G}_i} \{|E(G)|\} \tag{19}$$

where the prime denotes the transpose, $C_{\bar{i}}$ denotes the submatrix formed from C by striking out row i, and $\mathcal{G}_i$ is the set of subgraphs of $G(A', C'_{\bar{i}})$ which is similarly defined as in (9). Then the stable unity feedback system of Fig. 2 is type $[l_1, \cdots, l_m]$ structurally.

Proof: Suppose that the open-loop system is characterized by $(\tilde{C}, \tilde{A}, \tilde{B})$, and let

$$\underset{(n\times n)}{\tilde{F}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & & \vdots \\ 0 & 0 & \cdots & & 1 \\ 0 & 0 & \cdots & & 0 \end{bmatrix}, \quad \underset{(m\times n)}{\tilde{H}} = \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 1 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{n} \leftarrow i \tag{20}$$

where F and H are structured matrices whose fixed zeros coincide with the zeros of (20). Then, applying the results of [6] and [7], the feedback system is type $[\tilde{l}_1, \cdots, \tilde{l}_m]$ where the $\tilde{l}_i$'s are determined by

$$\tilde{l}_i = 2n - \operatorname{rank}\left\{ \begin{pmatrix} \tilde{C}' \\ \tilde{H}' \end{pmatrix} \begin{pmatrix} \tilde{A}' & 0 \\ 0 & \tilde{F}' \end{pmatrix} \begin{pmatrix} \tilde{C}' \\ \tilde{H}' \end{pmatrix} \cdots \begin{pmatrix} \tilde{A}' & 0 \\ 0 & \tilde{F}' \end{pmatrix}^{2n-1} \begin{pmatrix} \tilde{C}' \\ \tilde{H}' \end{pmatrix} \right\}.$$

Thus, from the form of the above formula and from the discussions of Section III, the structural system type of the system can be determined by

$$l_i = 2n - \text{generic rank}\left\{ \begin{pmatrix} C' \\ H' \end{pmatrix} \begin{pmatrix} A' & 0 \\ 0 & F' \end{pmatrix} \begin{pmatrix} C' \\ H' \end{pmatrix} \cdots \begin{pmatrix} A' & 0 \\ 0 & F' \end{pmatrix}^{2n-1} \begin{pmatrix} C' \\ H' \end{pmatrix} \right\}. \tag{21}$$

The second term can be easily evaluated by Theorem 1. By the assumption of structural observability, (A', C') is irreducible and (F', H') is clearly irreducible. So that the structured system $\left(\begin{pmatrix} A' & 0 \\ 0 & F' \end{pmatrix}, \begin{pmatrix} C' \\ H' \end{pmatrix} \right)$ is irreducible. Thus, applying Theorem 1, we obtain

$$\text{the second term of (21)} = \nu\left(\begin{pmatrix} A' & 0 \\ 0 & F' \end{pmatrix} \quad \begin{pmatrix} C' \\ H' \end{pmatrix} \right). \tag{22}$$

Furthermore, from the specific structure of F and H, it is easy to verify that the last is equal to

$$n + \nu([A' \quad C'_{\bar{i}}]). \tag{23}$$

Thus, from (21)–(23) we obtain

$$l_i = 2n - (n + \nu([A' \quad C'_{\bar{i}}])) = n - \nu([A' \quad C'_{\bar{i}}]).$$

This is the desired result (18). Equation (19) can similarly be verified as in the proof of Theorem 1.

VI. Conclusions

In this paper, we have introduced the notion of the generic dimension of controllable subspaces, and employed it in the study of structural controllability. In addition, we have derived two techniques for determining the generic dimension. The results have been used to study the steady-state characteristic of linear unity feedback systems. We note that the application of the results to the synthesis problem of servomechanisms seem to be very promising. It will also be interesting to inquire what the corresponding concepts mean in the frequency domain.

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References

[1] C. T. Lin, "Structural controllability." *IEEE Trans. Automat. Contr.*, vol. AC-19, pp. 201–208, June 1974.
[2] R. W. Shields and J. B. Pearson, "Structural controllability of multiinput linear systems," *IEEE Trans. Automat. Contr.*, vol. AC-21, pp. 203–212, Apr. 1976.
[3] K. Glover and L. M. Silverman, "Characterization of structural controllability," *IEEE Trans. Automat. Contr.*, vol. AC-21, pp. 534–537, Aug. 1976.
[4] E. J. Davison, "Connectability and structural controllability of composite systems," *Automatica*, vol. 13, pp. 109–123, 1977.
[5] S. Hosoe and K. Matsumoto, "On the irreducibility condition in the structural controllability theorem," *IEEE Trans. Automat. Contr.*, vol. AC-24, pp. 963–966, Dec. 1979.
[6] S. Hosoe and M. Ito, "On steady state characteristics of linear multivariable systems," *Memoirs of the Faculty of Eng., Nagoya Univ.*, vol. 26, no. 1, pp. 54–85, 1974.
[7] C. A. Wolfe and J. S. Meditch, "Theory of system type for linear multivariable servomechanisms," *IEEE Trans. Automat. Contr.*, vol. AC-22, Feb. 1977.
[8] C. Berge, *The Theory of Graphs*. New York: Wiley, 1962.
[9] H. H. Rosenbrock, *State Space and Multivariable Theory*. New York: Wiley-Interscience, 1970.