On a Time-Domain Characterization of the Numerator Polynomials of the Smith McMillan form

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Abstract—**A short and direct new proof is given to Moore and Silverman's theorem that the set of transmission polynomials of a linear system is equal to the set of numerator polynomials in the Smith McMillan form of its transfer function.**

INTRODUCTION

Recently [1], Moore and Silverman proved the following result [1]. Let

$$\dot{x}=Ax+Bu \tag{1a}$$

$$y=Cx \tag{1b}$$

be a time-invariant controllable observable linear dynamical system with $x\in R^n$, $u\in R^r$, and $y\in R^m$. The subspaces ν and φ are defined to be the maximal (A, B) invariant subspace and the maximal controllability subspace, respectively, contained in ker C. The map F is such that $(A+BF)\nu\subset\nu$, and A_t is defined to be the map induced by $(A+BF)$ in ν/φ.

The transfer function matrix of (1) is written as

$$H(s)=C(sI-A)^{-1}B \tag{2}$$

and its Smith McMillan form is represented by

$$\Gamma(s)=M(s)H(s)N(s)=\text{diag}[\epsilon_1(s)/\gamma_1(s),\cdots,\epsilon_q(s)/\gamma_q(s),\cdots,0], \tag{3}$$

where $M(s)$ and $N(s)$ are unimodular polynomial matrices, i.e., their determinants are nonzero constants.

The theorem proved by Moore and Silverman is the following.

Theorem: The set of invariant factors of A_t, transmission polynomials, is equal to the set of numerator polynomials in the Smith McMillan form of $H(s)$.

The purpose of this correspondence is to exhibit a short proof of this significant result which is considerably simpler than Moore and Silverman's.

PROOF OF THE THEOREM

Morse [2] proved that by a transformation of the form

$$(C,A,B)\mapsto(HCT^{-1},T(A+BF+KC)T^{-1},TBG), \tag{4}$$

where the matrices H, T, and G are nonsingular, (C,A,B) can be reduced to a unique canonical form $(\tilde{C},\tilde{A},\tilde{B})$, where

$$\tilde{C}=\begin{bmatrix}0&0&C_3&0\\0&0&0&C_4\end{bmatrix},$$

$$\tilde{A}=\begin{bmatrix}A_1&0&0&0\\0&A_2&0&0\\0&0&A_3&0\\0&0&0&A_4\end{bmatrix},\qquad \tilde{B}=\begin{bmatrix}0&0\\B_2&0\\0&0\\0&B_4\end{bmatrix}. \tag{5}$$

In the above, the matrices (A_2,B_2), (C_3,A_3), (C_4,A_4,B_4) are completely determined, respectively, by the three set of integers I_2, I_3, and I_4, A_1 is uniquely determined by the set I_1 of the transmission polynomials, i.e., invariant factors of A_t, and the invariant polynomials of A_1 coincide with the elements of I_1 [2]. The special structures of the matrices A_i, B_i, C_i should be referred to [2].

Write the system matrix [3] corresponding to $(\tilde{C},\tilde{A},\tilde{B})$ as

$$\begin{bmatrix}sI-\tilde{A}&\tilde{B}\\\tilde{C}&0\end{bmatrix}. \tag{6}$$

Then, regarding (6) as a singular pencil of matrices [5], it is easy to see that the set I_1 of the invariant polynomials of A_1 coincides with the set of the finite elementary divisors of the pencil (6), and I_2, I_3, and I_4 in [2] correspond to the set of minimal indices for the column, the set of minimal indices for the row, and the set of the infinite elementary devisors, respectively. This result can be verified by direct computation on (6), or even by simple comparison of the two canonical forms, the one derived by a transformation of the form (4), the other one derived by a strict equivalent transformation of pencils [4].

Define similarly the system matrix corresponding to (C,A,B) as

$$\begin{bmatrix}sI-A&B\\-C&0\end{bmatrix}, \tag{7}$$

which is relatively (left and right) prime since (A,B,C) is observable and controllable by assumption. Now by (4), the system matrices (6) and (7) are related by

$$\begin{bmatrix}sI-\tilde{A}&\tilde{B}\\-\tilde{C}&0\end{bmatrix}=\begin{bmatrix}T&-K\\0&H\end{bmatrix}\begin{bmatrix}sI-A&B\\C&0\end{bmatrix}\begin{bmatrix}T^{-1}&0\\-F&G\end{bmatrix}. \tag{8}$$

Since the matrices H, T, and G are nonsingular, this implies that the two pencils (6) and (7) are strictly equivalent. So, the elementary divisors and the minimal indices of these pencils coincide. Therefore, from this and the statement below (6), in order to prove the theorem, it is only necessary to show that the finite elementary divisors of (7) coincide with the numerator polynomials of the Smith McMillan form (3). But this can be verified as follows. First, recall [3, theorem 4.1] that the relatively (right and left) prime system matrix (7) whose transfer function $H(s)$ has the Smith McMillan form (3) can be reduced by a unimodular transformation into

$$\text{diag}[1,\cdots,1,\epsilon_1,\cdots,\epsilon_q,0,\cdots,0]. \tag{9}$$

The finite elementary divisors of (9) are obviously $\epsilon_1,\cdots,\epsilon_q$. Since the elementary divisors of polynomial matrices are invariant under unimodular transformations [5], $\epsilon_1,\cdots,\epsilon_q$ are also the elementary divisors of (7). Thus the theorem was proved.

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