Topological vertex and its applications

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Abstract

We give two new results on Gromov–Witten invariants of Calabi–Yau threefolds. The first one is a computation of local Gromov–Witten invariants of cubic surfaces at all genera. We give an explicit formula for the generating function of these invariants in a closed form. The second one is on the flop invariance of Gromov–Witten invariants of toric Calabi–Yau threefolds. We prove transformation formulas for generating functions of Gromov–Witten invariants on general toric Calabi–Yau threefolds under flops. Both results are based on the theory of the topological vertex. We present proofs of these two results together with required background on Gromov–Witten invariants.

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Part I
Overview and Background

1 Overview

1.1 Introduction

Let X be a smooth projective Calabi–Yau threefold, i.e. complex three-dimensional algebraic manifold whose canonical bundle is trivial. Around 1991, string theorists made a new prediction on "numbers" of algebraic curves in X of genus zero. It had been known that a certain generating series F_X^0 whose coefficients are these numbers has a physical meaning (the "prepotential" or the "genus zero part of the free energy"). The mirror symmetry identified F_X^0 as a specific combination of hypergeometric series. For mathematical understanding of this phenomenon, an appropriate definition of "numbers" of curves was needed. Nowadays, these numbers are rigorously defined and called the Gromov-Witten invariants. See [12, 32] for reviews of these developments. The Gromov–Witten invariants of X is defined by counting the number of maps from curves to X as follows. The moduli space of stable maps $\overline{\mathcal{M}}_{g,0}(X,\beta)$ into X from the genus g curves and the prescribed homology class (called the degree) $\beta \in H_2(X,\mathbb{Z})$ of the images has virtual dimension zero. The Gromov–Witten invariant $N_{g,\beta}(X)$ of X with the genus g and the degree β is defined by the integral of the unit cohomology class 1 against the 0-dimensional virtual fundamental class of $\overline{\mathcal{M}}_{g,0}(X,\beta)$. The generating function F_X which sums $N_{g,\beta}(X)$ over all g and β with the weight $\lambda^{2g-2}Q^\beta$ (here λ and Q are parameters) is called the free energy and $Z_X := \exp F_X$ is called the partition function.

In this paper, we study the partition functions of toric Calabi–Yau threefolds [1]. The partition function Z_X of a toric Calabi–Yau threefold X can be written in an explicit form by the theory of "topological vertex" developed by Aganagic–Klemm–Mariño–Vafa [2]. They found an algorithm to write down a formula for Z_X from the toric fan of X, which is as follows. First, one assigns a certain amplitude to each vertex of the dual graph of the toric fan of X as a building block. Then the algorithm tells the gluing rule of vertex amplitudes between two vertices which are joined by an edge of the graph. The gluing rule was motivated by a geometric idea of localization (see [16]). On the other hand, the vertex amplitude, from which the name of the theory comes [2], has a combinatorial expression in terms of special values of skew-Schur functions. It was obtained by using a certain duality between closed A-model topological string theory (Gromov–Witten theory) and Chern–Simons theory. See [40, 63] for expositions on these developments. Although the argument based on a string duality is not mathematically rigorous, a mathematical theory for the topological vertex was developed later by Li–Liu–Liu–Zhou [59]. In their approach, a combinatorial expression for the vertex amplitude comes from so-called Hodge integrals over moduli spaces of stable curves.

The topological vertex is an efficient tool to study Gromov–Witten invariants of toric Calabi–Yau threefolds. So far, many applications have been given: proofs of a conjecture of Nekrasov [69] on geometric engineering by Iqbal–Kahsani-Poor [36, 37], Eguchi–Kanno [22, 23] and Zhou [83], proofs of the Gopakumar–Vafa conjecture for toric Calabi–Yau threefolds by Konishi [48, 49] and Peng [75], and so on. Furthermore, it also shows us a surprising connection between string theory and some statistical model. By relating the combinatorial expression of the topological vertex to three dimensional partitions, Okounkov–Reshetikhin–Vafa [72] proposed a duality between topological A-model strings on Calabi–Yau threefolds and a statistical mechanical model of crystal melting. Inspired by this work, Maulik–Nekrasov–Okounkov–Pandharipande [64] proposed a conjecture which states the equivalence between Gromov–Witten theory and Donaldson–Thomas theory (which concerns integration over moduli of sheaves on threefolds) for Calabi–Yau threefolds. As an evidence, they proved the conjecture for local toric del Pezzo surfaces [3] based on the topological vertex. This conjecture was generalized to general projective threefolds later in [65].

The aim of this paper is to present two new applications of the topological vertex. The first main result gives explicit formulas for the partition functions of local Gromov–Witten invariants of del Pezzo surfaces S_d of degree $d = 3, 4, 5$, which are non-toric [4].

Theorem A . We have explicit formulas for the partition functions of local Gromov–Witten invariants of non-toric del Pezzo surfaces S_3, S_4 and S_5.

Formulas for toric del Pezzo surfaces ($d \geq 6$) was obtained by Zhou [82] (see also [1, 14, 15, 35]). The exact form of our formula is given in Theorems 1.4 and 6.8. Our main idea is to use the deformation invariance of local Gromov–Witten invariants and reduce the computations to those of $-K$-nef toric surfaces whose local Gromov–Witten invariants can be evaluated by the topological vertex formula (Theorems 1.3 and 4.6).

[1] A Calabi–Yau threefold X is toric if it contains the algebraic torus $(\mathbb{C}^*)^3$ as a dense open subset and the action of $(\mathbb{C}^*)^3$ on itself extends to X. Particular examples are total spaces of the canonical line bundles of toric surfaces, which are called *local toric surfaces*.

[2] The vertex amplitude itself is also called the topological vertex.

[3] A nonsingular algebraic surface S is called a del Pezzo surface if its canonical bundle K_S is negative.

[4] For a del Pezzo surface S, the positive integer $d = c_1(K_S)^2$ is called the degree of S. It is known that del Pezzo surfaces are classified by their degrees which take all integers from 1 to 9. A del Pezzo surface S_d of degree d is obtained as a blow up of the projective plane $\mathbb{P}^2$ at $(9 - d)$ points if $d \neq 8$ and S_8 is isomorphic to either $\mathbb{P}^2$ blown up at a point or $\mathbb{P}^1 \times \mathbb{P}^1$. In particular, S_d is toric if $d \geq 6$. See e.g.[67, §0] for a brief account of the classification of del Pezzo surfaces.

The second main result is on the flop invariance of Gromov–Witten invariants of toric Calabi–Yau threefolds. The same problem for projective Calabi–Yau threefolds was studied by Li–Ruan [55]. For related physical works, we refer to the references in *loc. cit.*.

Theorem B . Let $\phi : X \dashrightarrow X^+$ be a flop with respect to a $(-1,-1)$-curve between toric Calabi–Yau threefolds. Then Gromov–Witten invariants of X and X^+ coincide under the identification of degrees given by ϕ.

See Theorem 1.6 for a precise statement. We show the invariance of the partition functions under flops (Theorem 9.3), which implies that for Gromov–Witten invariants (Corollary 9.4). By virtue of the topological vertex and local analysis of fans of toric Calabi–Yau threefolds, the flop invariance of the partition functions is obtained from a combinatorial identity on skew-Schur functions (Theorems 1.5 and 7.9).

The above two results were obtained by joint works [50, 51] with Yukiko Konishi. The author's main contribution to the first result was the idea of using the topological vertex and the invariance of Gromov–Witten invariants under deformations to compute the local Gromov–Witten invariants of the non-toric del Pezzo surfaces which admit $-K$-nef toric degenerations. That to the second one was to prove the combinatorial identity in Theorems 1.5 which is a key to our proof of the flop invariance. In the next two sections, outlines of results and their proofs will be explained in further detail. Since we are going to address two problems which are related but different in nature, we treat them separately.

1.2 Topological vertex and local Gromov–Witten invariants of del Pezzo surfaces

The first problem is computations of local Gromov–Witten invariants of del Pezzo surfaces. These invariants have been intensively studied in physics by various methods: local mirror symmetry, the holomorphic anomaly equation, Seiberg–Witten curve technique, and so on. See e.g. [11, 44, 54] and the references therein.

1.2.1 Local Gromov–Witten invariants of algebraic surfaces

First we give a definition of local Gromov–Witten invariants for any algebraic surface. See §3 for details.

Let S be a nonsingular projective surface and K_S be the canonical bundle of S. We denote by $\overline{\mathcal{M}}_{g,n}(S,\beta)$ the moduli space of n-pointed stable maps into S with genus g, degree $\beta \in H_2(S,\mathbb{Z})$. Consider the following universal diagram:

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,1}(S,\beta) & \xrightarrow{\mu} & S \\ {\scriptstyle\pi}\downarrow & & \\ \overline{\mathcal{M}}_{g,0}(S,\beta) \ , & & \end{array}$$

where $\mu = ev_1$ is the evaluation at the marked point and $\pi = \pi_1$ is the map forgetting the marked point, which form the universal stable map and the universal curve respectively (cf. §2.1.1).

Definition 1.1. The local Gromov–Witten invariant $I_{g,\beta}(S)$ of S with genus g, degree β is defined as follows:

$$I_{g,\beta}(S) := \int_{[\overline{\mathcal{M}}_{g,0}(S,\beta)]^{vir}} c(-R^\bullet \pi_* \mu^* K_S) \in \mathbb{Q},$$

where $c(\cdot)$ is the total Chern class and $[\overline{\mathcal{M}}_{g,0}(S,\beta)]^{vir}$ is the virtual fundamental class of $\overline{\mathcal{M}}_{g,0}(S,\beta)$ (cf. §2.1.3).

1.2.2 Gromov–Witten invariants of local toric surfaces

Next, we introduce the Gromov–Witten invariants of local toric surfaces. The topological vertex, which is our main tool, gives a closed formula for their generating functions. See §4 for details.

Let S be a nonsingular complete toric surface, which is equipped with an action of the 2-dimensional algebraic torus $\mathbb{T} = (\mathbb{C}^*)^2$. We regard the canonical bundle K_S of S as a $\mathbb{T}$-equivariant line bundle $\mathcal{O}_S(-\sum_{i=1}^r C_i)$, where $C_1, \cdots, C_r$ are irreducible toric divisors on S. It is known that they generate $H_2(S, \mathbb{Z})$.

Definition 1.2. The Gromov–Witten invariants $N_{g,\beta}(K_S)$ of K_S with genus g, degree β is defined by the following $\mathbb{T}$-equivariant integral:

$$N_{g,\beta}(K_S) := \int_{[\overline{\mathcal{M}}_{g,0}(S,\beta)]^{vir}_{\mathbb{T}}} e_{\mathbb{T}}(-R^\bullet \pi_* \mu^* K_S),$$

where $e_{\mathbb{T}}(\cdot)$ is the $\mathbb{T}$-equivariant Euler class.

If S is a $-K$-nef (i.e. $-K_S$ is nef) toric surface then

$$N_{g,\beta}(K_S) = I_{g,\beta}(S), \tag{1.1}$$

for β satisfying $\int_\beta c_1(K_S) < 0$ [5] (cf. Lemma 4.4). Let us introduce

$$F_{K_S}(\lambda, Q) := \sum_{\beta \in H_2(S,\mathbb{Z}), \beta \neq 0} \sum_{g \geq 0} N_{g,\beta}(K_S) \lambda^{2g-2} Q^\beta,$$

where Q is a formal variable satisfying $Q^\beta Q^{\beta'} = Q^{\beta+\beta'}$, and define

$$Z_{K_S}(\lambda, Q) := \exp F_{K_S}(\lambda, Q).$$

Theorem 1.3 (Topological vertex formula). *Let s_i be the self-intersection number of C_i and $t_i := Q^{[C_i]} (1 \leq i \leq r)$. Then we have*

$$Z_{K_S}(\lambda, Q) = \sum_{(\nu^1, \ldots, \nu^r) \in \mathcal{P}^r} \prod_{i=1}^r ((-1)^{s_i} t_i)^{|\nu^i|} e^{\sqrt{-1}\lambda s_i \frac{\kappa(\nu^i)}{2}} W_{\nu^i, \nu^{i+1}}(e^{\sqrt{-1}\lambda}).$$

Here ν^i runs over all partitions and we regard $i \in \mathbb{Z}/r$. See §2.3.1 and §2.3.3 for notations on partitions and combinatorial functions. In §5, we give a proof of Theorem 1.3 which is essentially due to Zhou [82] and Liu–Liu–Zhou [60].

1.2.3 Local Gromov–Witten invariants of del Pezzo surfaces

Let S_d be a del Pezzo surface of degree d. By eq. (1.1) and Theorem 1.3, we have a closed formula for the generating functions of $I_{g,\beta}(S_d)$ for toric del Pezzo surfaces S_d ($6 \leq d \leq 9$). We extend this result for $d \geq 3$. It is enough to state the result for S_3 (see Remark 6.6). Let $p : S_3 \to \mathbb{P}^2$ be a blow-up at six points in a general position and $e_1, \cdots, e_6$ be the classes of the exceptional curves of p, l be the class of a line in $\mathbb{P}^2$ pulled back by p. They form a basis for $H_2(S_3, \mathbb{Z})$. We consider the partition function of local Gromov–Witten invariants of S_3:

$$Z^{\mathrm{loc}}_{S_3}(\lambda, Q) := \exp\Big[\sum_{\beta \in H_2(S_3,\mathbb{Z}), -K_{S_3}\cdot\beta > 0} \sum_{g \geq 0} I_{g,\beta}(S_3) \lambda^{2g-2} Q^\beta \Big].$$

[5] Both of the two conditions ($-K$-nef + $\int_\beta c_1(K_S) < 0$) are essential for the equality.

Our goal here is to obtain an explicit formula for $Z_{S_3}^{\rm loc}$. A key fact is that S_3 is deformation equivalent to a $-K$-nef (but not a del Pezzo) toric surface S_3^0, whose toric fan can be found in Figure 2 (cf. §6.2). It has nine irreducible toric divisors $C_1, \dots, C_9$. Then, by deformation invariance of local Gromov–Witten invariants for $-K$-nef surfaces (Proposition 3.7), local Gromov-Witten invariants of S_3 can be obtained from those of S_3^0 under appropriate identifications of second homology classes. Applying eq. (1.1) and Theorem 1.3 to S_3^0 and subtracting the contributions from chains of (-2)-curves which are orthogonal to the canonical divisor $K_{S_3^0}$, we obtain the following

Theorem 1.4 ([51]).

$$Z_{S_3}^{\rm loc}(\lambda, Q) = \frac{Z_{K_{S_3^0}}(\lambda, t_1, \cdots, t_9)}{\prod_{i=1,4,7} Z_{(-2,0)}(\lambda, t_i) Z_{(-2,0)}(\lambda, t_{i+1}) Z_{(-2,0)}(\lambda, t_i t_{i+1})}, \tag{1.2}$$

with the following identification of the parameters

$$\begin{aligned} &t_1 = Q^{e_2-e_5}, \quad t_2 = Q^{l-e_2-e_3-e_6}, \quad t_3 = Q^{e_6}, \quad t_4 = Q^{e_3-e_6}, \quad t_5 = Q^{l-e_1-e_3-e_4}, \\ &t_6 = Q^{e_4}, \quad t_7 = Q^{e_1-e_4}, \quad t_8 = Q^{l-e_1-e_2-e_5}, \quad t_9 = Q^{e_5}. \end{aligned} \tag{1.3}$$

Here $Q = (Q_1, \dots, Q_6, Q_7)$ is a set of formal variables and $Q^\beta = Q_1^{a_1} Q_2^{a_2} \dots Q_7^{a_7}$ for $\beta = a_1 e_1 + \cdots + a_6 e_6 + a_7 l \in H_2(S_3, \mathbb{Z})$ and $Z_{(-2,0)}(\lambda, t)$ has been defined by

$$Z_{(-2,0)}(\lambda, t) = \prod_{k=1}^{\infty} (1 - t e^{\sqrt{-1}\lambda k})^{-k} \ .$$

This is the first main result of this paper. Its proof will be given in §6. Note that an explicit formula for $Z_{K_{S_3^0}}(\lambda, t_1, \cdots, t_9)$ is given by Theorem 1.3. The change of the variables (1.3) corresponds to an identification $H_2(S_3^0, \mathbb{Z}) \cong H_2(S_3, \mathbb{Z})$. The denominator of the formula (1.2) is the contributions from (-2)-curves on S_3^0.

1.3 Flop invariance of the topological vertex

The second problem is on the flop invariance of Gromov–Witten invariants of toric Calabi–Yau threefolds. In this paper, we mean by a flop a birational transformation obtained by contracting a rational curve whose normal bundle is isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, which is called a $(-1,-1)$-curve, and resolving the resulting singularity appropriately (cf. §9.1.1).

1.3.1 An identity

Let $\mathcal{P}$ be the set of partitions. For $\lambda_1, \lambda_2, \lambda_3 \in \mathcal{P}$, the topological vertex $C_{\lambda_1,\lambda_2,\lambda_3}(q) \in \mathbb{Q}(q^{\frac{1}{2}})$ is defined by certain special values of skew-Schur functions associated to $\lambda_1, \lambda_2, \lambda_3$. See §7.1 for details. Take four partitions $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and introduce:

$$Z_0'(q, Q_0) = \frac{Z_0(q, Q_0)}{Z_{(-1,-1)}(q, Q_0)} \ , \quad Z_0^{+\prime}(q, Q_0^+) = \frac{Z_0^+(q, Q_0^+)}{Z_{(-1,-1)}(q, Q_0^+)} \ , \tag{1.4}$$

where

$$\begin{aligned} Z_0(q, Q_0) &= \sum_{\mu \in \mathcal{P}} (-Q_0)^{|\mu|} C_{\lambda_1,\lambda_2,\mu^t}(q) C_{\lambda_3,\lambda_4,\mu}(q) \ , \\ Z_0^+(q, Q_0^+) &= \sum_{\mu \in \mathcal{P}} (-Q_0^+)^{|\mu|} C_{\lambda_1,\mu^t,\lambda_4}(q) C_{\lambda_3,\mu,\lambda_2}(q) \ , \end{aligned}$$

and $Z_{(-1,-1)}(q,Q) = \prod_{k=1}^{\infty}(1-Qq^k)^k$. These are formal power series in Q_0 (resp. Q_0^+). A key result to our proof of the flop invariance of Gromov–Witten invariants of toric Calabi–Yau threefolds is the following combinatorial identity.

Theorem 1.5 ([50]). *Under the identification $Q_0^+ = Q_0^{-1}$, we have*

$$Z_0^{+\prime}(q,Q_0^+) = (-Q_0)^{-(|\lambda_1|+|\lambda_2|+|\lambda_3|+|\lambda_4|)} q^{\frac{1}{2}(\kappa(\lambda_1)-\kappa(\lambda_2)+\kappa(\lambda_3)-\kappa(\lambda_4))} Z_0'(q,Q_0)\ .$$

Theorem 1.5 will be proved in §7.2. Our proof is based on the technique which has been developed in [36, 37, 22, 23, 83] to prove a conjecture of Nekrasov [69] on the equivalence between the partition function of Gromov–Witten invariants of a certain toric Calabi–Yau threefold and the Nekrasov partition function in instanton counting.

1.3.2 Flop invariance of the Gromov–Witten invariants

Let X and X^+ be toric Calabi–Yau threefolds which are birationally equivalent under a flop $\phi : X \dashrightarrow X^+$ with respect to a $(-1,-1)$-curve C_0 in X. Let $C_0^+ \subset X^+$ be the flopped $(-1,-1)$-curve. See §9.1 for details. Note that we have an isomorphism between $H_2(X,\mathbb{Z})$ and $H_2(X^+,\mathbb{Z})$ induced by ϕ.

Let us consider the partition functions of X and X^+ :

$$\begin{aligned} Z_X(\lambda,Q) &= \exp\Big[\sum_{\beta\in H_2(X,\mathbb{Z}),\beta\neq 0}\ \sum_{g\geq 0} N_{g,\beta}(X)\lambda^{2g-2}Q^{\beta}\Big], \\ Z_{X^+}(\lambda,Q^+) &= \exp\Big[\sum_{\beta\in H_2(X^+,\mathbb{Z}),\beta\neq 0}\ \sum_{g\geq 0} N_{g,\beta}(X^+)\lambda^{2g-2}(Q^+)^{\beta}\Big], \end{aligned}$$

where $N_{g,\beta}(X)$ and $N_{g,\beta}(X^+)$ are Gromov–Witten invariants of X and X^+ respectively. The topological vertex enables us to write down a combinatorial formula for $Z_X(\lambda,Q)$ and $Z_{X^+}(\lambda,Q^+)$. See §8.2 for a summary of the theory of the topological vertex. Under appropriate identifications of the expansion parameters Q and Q^+, one can show that they are equal except for factors coming from multiples of $[C_0]$ and $[C_0^+]$ (Theorem 9.3). Since a flop is a local operation, the difference between the two can only appear at the local contributions from neighborhoods of C_0 and C_0^+. If we set $q = e^{\sqrt{-1}\lambda}$, they are given by $Z_0'(q,Q_0)$ and $Z_0^{+\prime}(q,Q_0^+)$, defined in (1.4), respectively. Then, showing the equality between the two partition functions reduces to the identity in Theorem 1.5. As a result, we obtain the following

Theorem 1.6 ([50]). *For $\beta \in H_2(X,\mathbb{Z})$ which is not a multiple of $[C_0]$, we have*

$$N_{g,\beta}(X) = N_{g,\phi_*(\beta)}(X^+).$$

For a multiple of flopping curve class, we have

$$N_{g,d[C_0]}(X) = N_{g,d[C_0^+]}(X^+).$$

Theorem 1.6 is the second main result of this paper. Its proof and some applications will be given in §9.

1.4 Organization of the paper

The paper is organized as follows. The rest of the part I covers the background material. Notations introduced in §2, especially those on moduli of stable maps (§2.1) and partitions (§2.3) are used throughout the paper. Parts II and III are devoted to prove the results presented in §1.2 and §1.3 respectively. In principle, they can be read separately. More details will be explained at the beginning of each section.

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2 Background

In §2.1, a definition of Gromov–Witten invariants for nonsingular projective varieties is given. Special attention will be paid to projective Calabi–Yau threefolds. §2.2 is devoted to the virtual localization formula, which is one of the most powerful tools in the study of Gromov–Witten invariants. In §2.3, a definition of Hodge integrals and necessary formulas are given.

2.1 Gromov–Witten invariants

2.1.1 Stable maps

The notion of stable maps was introduced by Kontsevich [52]. Let X be a non-singular projective variety over $\mathbb{C}$. Stable maps into X are defined as follows. Recall that an n-pointed pre-stable curve of genus g is a tuple $(C, p_1, \cdots, p_n)$ where C is a projective, connected, reduced, nodal curve of arithmetic genus g, and $p_1, \cdots, p_n$ are distinct non-singular points on C. Nodal points on C and marked points $p_1, \cdots, p_n$ are called special points of a pre-stable curve $(C, p_1, \cdots, p_n)$.

Definition 2.1. (i) An n-pointed pre-stable map $f : (C, p_1, \cdots, p_n) \to X$ into X of genus g, degree $\beta \in H_2(X, \mathbb{Z})$ is a morphism from n-pointed genus g pre-stable curves $(C, p_1, \cdots, p_n)$ to X such that $f_*[C] = \beta$.
(ii) An infinitesimal automorphism of pre-stable map is a tangent vector field v on the domain curve C which vanishes at the special points and satisfying $df(v) = 0$.
(iii) A pre-stable map is called stable if it has no nontrivial infinitesimal automorphisms.

We denote by $\overline{\mathcal{M}}_{g,n}(X, \beta)$ the moduli space of n-pointed genus g stable maps of degree β into X. The existence of the fine moduli spaces $\overline{\mathcal{M}}_{g,n}(X, \beta)$ of stable maps as Deligne–Mumford stacks and their properness was stated in [52] and proved in [6] (see also [28] together with [30, Appendix A]). These spaces come equipped with several natural morphisms. For each of the n marked points, there is an evaluation map

$$ev_i : \overline{\mathcal{M}}_{g,n}(X, \beta) \to X$$

which takes $(f; C, p_1, \cdots, p_n) \in \overline{\mathcal{M}}_{g,n}(X, \beta)$ to $f(p_i) \in X$. There is the map

$$\pi_{n+1} : \overline{\mathcal{M}}_{g,n+1}(X, \beta) \to \overline{\mathcal{M}}_{g,n}(X, \beta)$$

which forgets the last marked point (and stabilizes if necessary). The forgetful morphism π_{n+1} and the evaluation morphism ev_{n+1} form the universal curve and the universal stable map respectively:

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,n+1}(X,\beta) & \xrightarrow{ev_{n+1}} & X \\ {\scriptstyle \pi_{n+1}}\Big\downarrow & & \\ \overline{\mathcal{M}}_{g,n}(X,\beta)\ . & & \end{array} \tag{2.1}$$

2.1.2 Deformation theory of stable maps

Following [30, Appendix B], we summarize the obstruction theory on the moduli space $\overline{\mathcal{M}} := \overline{\mathcal{M}}_{g,n}(X,\beta)$ of stable maps into X. Let us set $\mathcal{U} := \overline{\mathcal{M}}_{g,n+1}(X,\beta)$ and consider the universal curve $\pi = \pi_{n+1} : \mathcal{U} \to \overline{\mathcal{M}}$ and the universal stable map $\mu = ev_{n+1} : \mathcal{U} \to X$ (cf.(2.1)). The stability condition and the projectivity of X imply that there exists a two-term complex

$$E_\bullet = [E_0 \to E_1] \tag{2.2}$$

of vector bundles over $\overline{\mathcal{M}}$ which satisfies the following condition.

Define coherent sheaves $\mathcal{T}^1$, $\mathcal{T}^2$ on $\overline{\mathcal{M}}$ by the following exact sequence:

$$0 \longrightarrow \mathcal{T}^1 \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \mathcal{T}^2 \longrightarrow 0. \tag{2.3}$$

Then $\mathcal{T}^1$ and $\mathcal{T}^2$ fit into the following exact sequence:

$$\begin{aligned} 0 &\longrightarrow R^0\pi_*\underline{Hom}(\Omega_\pi(P),\mathcal{O}_\mathcal{U}) \longrightarrow R^0\pi_*\mu^*T_X \longrightarrow \mathcal{T}^1 \\ &\longrightarrow R^1\pi_*\underline{Hom}(\Omega_\pi(P),\mathcal{O}_\mathcal{U}) \longrightarrow R^1\pi_*\mu^*T_X \longrightarrow \mathcal{T}^2 \longrightarrow 0, \end{aligned} \tag{2.4}$$

where Ω_π is the sheaf of π-relative differentials on $\mathcal{U}$ and P is the divisor of marked points.

The exact sequence (2.4) is the tangent–obstruction sequence of the deformation theory of stable maps which consists of two parts:
(i) Deformation of maps $f : C \to X$ from a fixed domain curve C,
(ii) Deformation of the n-pointed domain curve $(C, p_1, \cdots, p_n)$.
The tangent and obstruction spaces for (i) are $H^0(C, f^*T_X)$ and $H^1(C, f^*T_X)$ respectively. For (ii), $\mathrm{Ext}^0(\Omega_C(P),\mathcal{O}_C)$ and $\mathrm{Ext}^1(\Omega_C(P),\mathcal{O}_C)$ are the space of infinitesimal automorphisms and the tangent space respectively (there is no obstruction space). This explains the geometrical meaning of (2.4). We call $\mathcal{T}^1$ and $\mathcal{T}^2$ the tangent sheaf and the obstruction sheaf respectively. The complex $E_\bullet$ is called the (dual) *perfect obstruction theory* on $\overline{\mathcal{M}}$.

From (2.4), we can compute the virtual (or expected) dimension vdim $\overline{\mathcal{M}}_{g,n}(X,\beta)$ of the moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$. By Riemann–Roch theorem, it is given by

$$\mathrm{vdim}\,\overline{\mathcal{M}}_{g,n}(X,\beta) = -K_X\cdot\beta + (\dim X - 3)(1-g) + n. \tag{2.5}$$

If there are no obstructions, the moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is smooth of expected dimension. This is the case when $H^1(C, f^*T_X)$ vanishes at all points in the moduli spaces. However, in general, $\overline{\mathcal{M}}_{g,n}(X,\beta)$ may be singular, non-reduced, reducible, and of impure dimension. It may have a component whose dimension is bigger than the expected dimension.

2.1.3 Virtual fundamental class

We want to define Gromov–Witten invariants of X as integrals over $\overline{\mathcal{M}}_{g,n}(X,\beta)$. However, this space is not so nice as we mentioned earlier. In addition, it does not behave well under deformations of X. For instance, $\overline{\mathcal{M}}_{g,n}(X,\beta)$ does not necessarily deform in a flat family under a flat deformations of X. To overcome these problems, the virtual fundamental class was introduced: $\overline{\mathcal{M}}_{g,n}(X,\beta)$ carries a natural class

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir} \in A_{\text{vdim}}(\overline{\mathcal{M}}_{g,n}(X,\beta)),$$

in the expected dimensional Chow group of rational coefficients. It has good properties including nice behaviors under deformations of X. In Gromov–Witten theory, all integrals are taken against the virtual fundamental class. It is constructed from the perfect obstruction theory $E_\bullet$ (cf.(2.2)). For example, if the moduli space is smooth of expected dimension then the virtual fundamental class is equal to the ordinary one. If the moduli space is smooth but not of expected dimension, that is the case when obstruction sheaf $\mathcal{T}^2$ is a vector bundle, then the virtual fundamental class is given by

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir} = e(\mathcal{T}^2) \cap [\overline{\mathcal{M}}_{g,n}(X,\beta)],$$

where $e(\cdot)$ is the Euler class. The construction is quite subtle in general. This was the main technical point in defining Gromov–Witten invariants which was achieved by Behrend–Fantechi [7, 8] and Li–Tian [56] in algebraic category, and by Fukaya–Ono [25], Li–Tian [57], and Siebert [76] in symplectic category. Proofs of the coincidence of these constructions in two different categories were given by Li–Tian [58] and Siebert [77].

The Gromov–Witten invariants of X are defined by integrals of the following form:

$$\langle \gamma_1, \cdots, \gamma_n \rangle^X_{g,\beta} := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} \prod_{i=1}^n ev_i^* \gamma_i,$$

where $\gamma_i \in H^*(X,\mathbb{Z})$. The geometrical meaning of this invariant can be explained as follows. Take n homology cycles Γ_i on X ($i = 1, \cdots, n$). Then Gromov–Witten invariant $\langle \gamma_1, \cdots, \gamma_n \rangle^X_{g,\beta}$ for $\gamma_i = \text{PD}[\Gamma_i]$, where $\text{PD}[\Gamma_i]$ represents the cohomology class which is dual to the homology class $[\Gamma_i]$ under the Poincaré duality, is a virtual counting of n-pointed stable maps into X of genus g, degree β which intersect with the cycle Γ_i at the image of the i-th marked point. Note however that Gromov–Witten invariants are rational numbers since the virtual fundamental class is a rational class.

The Gromov–Witten invariants are invariant under smooth deformations of X (see, e.g. [56, Theorem 4.2]). A meaning of this statement can be explained as follows. Since smooth families of varieties are locally trivial in the C^∞-topology, there are canonical isomorphisms between nearby fibers of the family. With respect to this identification, the Gromov–Witten invariants are independent of the choice of a fiber.

2.1.4 Gromov–Witten invariants of Calabi–Yau threefolds

Let X be a nonsingular projective threefold. Then the virtual dimensions of the moduli spaces $\overline{\mathcal{M}}_{g,0}(X,\beta)$ do not depend upon the genus:

$$\text{vdim}\, \overline{\mathcal{M}}_{g,0}(X,\beta) = -K_X \cdot \beta.$$

We say that X is a Calabi–Yau threefold if its canonical bundle K_X is trivial. Then $-K_X \cdot \beta = 0$ for any β. For a Calabi–Yau threefold X, the genus g, degree β Gromov-Witten invariant of

X is defined as

$$N_{g,\beta}(X) := \langle 1 \rangle^X_{\beta,0} = \int_{[\overline{\mathcal{M}}_{g,0}(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Q}. \tag{2.6}$$

We consider the following generating function of Gromov–Witten invariants $N_{g,\beta}(X)$ of X in all genera and all nonzero degrees:

$$F_X(\lambda, Q) = \sum_{\beta \neq 0} \sum_{g=0}^{\infty} N_{g,\beta}(X) \lambda^{2g-2} Q^\beta. \tag{2.7}$$

$F_X(\lambda, Q)$ is called the free energy of X and its exponential

$$Z_X(\lambda, Q) := \exp F_X(\lambda, Q), \tag{2.8}$$

is called the partition function of X.

From considerations in M-theory, Gopakumar–Vafa [29] proposed a remarkable conjecture about integrality structures of the free energy of a Calabi–Yau threefold X.

Definition 2.2. Define the Gopakumar–Vafa invariants $n^g_\beta(X) \in \mathbb{Q}$ by the following formula:

$$F_X(\lambda, Q) = \sum_{\beta \neq 0} \sum_{g=0}^{\infty} \sum_{k=1}^{\infty} \frac{n^g_\beta(X)}{k} \Big(2 \sin \frac{k\lambda}{2}\Big)^{2g-2} Q^{k\beta}. \tag{2.9}$$

Matching the coefficients of the two series yields equations determining $n^g_\beta(X)$ recursively in terms of $N_{g,\beta}(X)$.

Conjecture 2.3 (Gopakumar–Vafa conjecture). *For a Calabi–Yau threefold X, $n^g_\beta(X) \in \mathbb{Z}$ for any $g \geq 0$ and $\beta \in H_2(X, \mathbb{Z})$. Moreover for any fixed β, $n^g_\beta(X) = 0$ for all but finite g.*

Gopakumar–Vafa's original argument was that for each curve class $\beta \in H_2(X, \mathbb{Z})$ and genus g, there is an integer $n^g_\beta(X)$ counting certain BPS states in M-theory and the formula (2.9) holds for a Calabi–Yau threefold X. According to their physical argument, the number of BPS states should be defined in terms of the moduli space of D-branes on X. See [33, 43] for approaches for this problem.

2.2 Virtual localization

The virtual localization formula (Theorem 2.5) due to Kontsevich [52] and Graber–Pandharipande [30] is presented. We shall use it in §3.2.2 and §5.

2.2.1 Equivariant homology groups

First, we review equivariant homology groups. We work with $\mathbb{Q}$-coefficients. Let X be a nonsingular irreducible algebraic variety equipped with an action of a linear algebraic group G (over $\mathbb{C}$). The G-equivariant homology groups of X are defined by the so-called Borel construction. Consider the universal principal bundle $EG \to BG$ of G. Here EG is a contractible space with a free G-action (these conditions determine EG uniquely up to homotopy) and $BG = EG/G$. Consider $EG \times X$ on which G acts freely and the quotient $X_G = (EG \times X)/G$, the homotopy quotient of X by G. Roughly speaking, the G-equivariant (co)homology groups are defined via ordinary (co)homology groups of X_G.

More precisely, we work with finite-dimensional approximations of the classifying space $EG \to BG$. For each $N \geq 1$, there exists a nonsingular irreducible variety EG_N with G-action which has the following properties:

(i) $H^i(EG_N) = 0$ for $i = 1, \ldots, 2N$.

(ii) A principal G-bundle quotient

$$EG_N \to BG_N := EG_N/G$$

exists.

Let $X_{G,N} := (EG_N \times X)/G$. For each n, we define the n-th G-equivariant homology group of X as follows. Take N such that $n < 2N$ and define

$$H_n^G(X) := H^{BM}_{n-2\dim G+2\dim EG_N}(X_{G,N}),$$

where the RHS is the Borel–Moore homology group (see e.g. [26, §19.1]). We also define the n-th equivariant cohomology group of X as

$$H_G^n(X) := H^n(X_{G,N}),$$

where the RHS is the ordinary cohomology group. These are independent of the choice of EG_N. If $n > 2\dim X$, then $H_n^G(X) = 0$. On the other hand, $H_n^G(X)$ may be non-zero for $n < 0$. By the Poincaré duality, we have

$$H_G^n(X) \cong H^G_{2\dim X-n}(X).$$

The projection map $p : X_{G,N} \to BG_N$ is a flat map with the fiber X, which makes $H_G^*(X)$ into a module over the graded ring $H^*(BG) = H_G^*(pt)$. The inclusion i of a fiber of p gives a morphism

$$i^* : H_*^G(X) \to H_*(X),$$

via the Poincaré duality isomorphism. The top dimensional map

$$i^* : H^G_{2\dim X}(X) \to H_{2\dim X}(X),$$

is an isomorphism. The G-equivariant fundamental class $[X]_G$ is defined by $i^*[X]_G = [X]$, where $[X]$ is the usual fundamental class of X.

Let $V \to X$ be a G-equivariant vector bundle. Consider $V_G := (EG\times V)/G$. Then $V_G \to X_G$ is a vector bundle. Define the G-equivariant Euler class $e_G(V)$ of V as

$$e_G(V) := e(V_G) \in H_G^{2\operatorname{rk} V}(X),$$

where the RHS is the ordinary Euler class (i.e. the Chern class of the top degree).

2.2.2 Classical localization theorem

In this section, we consider an action of an algebraic torus $\mathbb{T} = (\mathbb{C}^*)^r$ on a nonsingular irreducible algebraic variety X. First, we give a concrete description of $EG_N \to BG_N$ for $G = \mathbb{T} = (\mathbb{C}^*)^r$. Let us consider

$$E\mathbb{T}_N := (\mathbb{C}^{N+1} \setminus \{0\}) \times \cdots \times (\mathbb{C}^{N+1} \setminus \{0\}) \ (r \text{ times}),$$

equipped with the $\mathbb{T}$-action defined by

$$(t_1, \ldots, t_r) \cdot (v_1, \ldots, v_r) = (t_1^{-1}v_1, \ldots, t_r^{-1}v_r).$$

Then $E\mathbb{T}_N$ satisfies the conditions (i) and (ii) above and

$$B\mathbb{T}_N = \mathbb{P}^N \times \cdots \times \mathbb{P}^N \ (r \text{ times}).$$

In the limit $N \to \infty$, the map $E\mathbb{T} \to B\mathbb{T}$ is given by

$$(\mathbb{C}^\infty \setminus \{0\})^r \to (\mathbb{P}^\infty)^r.$$

We have $H^*_{\mathbb{T}}(pt) = H^*(B\mathbb{T}) \cong \mathbb{Q}[\alpha_1, \dots, \alpha_r]$, where α_i is given by $c_1(\mathcal{O}(1))$ of the i-th factor, hence $\deg \alpha_i = 2$. We denote this graded ring by $R_{\mathbb{T}}$.

Let $\mathrm{Ch}(\mathbb{T})$ be the character group of $\mathbb{T}$. For each $\rho \in \mathrm{Ch}(\mathbb{T})$, we have a 1-dimensional representation $\mathbb{C}_\rho$ of $\mathbb{T}$, hence the corresponding line bundle $L_\rho := (\mathbb{C}_\rho)_{\mathbb{T}}$ over $B\mathbb{T}$. We have a homomorphism $w : \mathrm{Ch}(\mathbb{T}) \to H^2(B\mathbb{T})$ defined by $\rho \mapsto -c_1(L_\rho)$. For $\rho \in \mathrm{Ch}(\mathbb{T})$, we call $w(\rho)$ the *weight* of ρ. Note that if $\rho_i \in \mathrm{Ch}(\mathbb{T})$ is the character defined by $\rho_i(t_1, \dots, t_r) = t_i$ for $(t_1, \dots, t_r) \in \mathbb{T}$, then we have $w(\rho_i) = \alpha_i$.

Let $X^{\mathbb{T}}$ be the $\mathbb{T}$-fixed point set and $\iota : X^{\mathbb{T}} \hookrightarrow X$ be the inclusion. We have the push-forward morphism

$$\iota_* : H^{\mathbb{T}}_*(X^{\mathbb{T}}) \to H^{\mathbb{T}}_*(X).$$

Let $R^+_{\mathbb{T}}$ be the multiplicative closed subset of $R_{\mathbb{T}}$ which consists of all the homogeneous elements of positive degree and the unity. Let $Q_{\mathbb{T}} := (R^+_{\mathbb{T}})^{-1} R_{\mathbb{T}} \subset \mathbb{Q}(\alpha_1, \dots, \alpha_r)$ be the localized ring. Then the localization theorem of Atiyah–Bott [3] states that

$$\iota_* : H^{\mathbb{T}}_*(X^{\mathbb{T}}) \otimes Q_{\mathbb{T}} \to H^{\mathbb{T}}_*(X) \otimes Q_{\mathbb{T}}$$

is an isomorphism.

Next, we give explicit localization formula for the equivariant fundamental class. Let $X^{\mathbb{T}} = \cup X_i$ be the decomposition into irreducible components. By [39], smoothness of X implies that of X_i. Let N_i be the normal bundle to X_i in X. Since $\mathbb{T}$ acts on N_i with non-zero weights, the $\mathbb{T}$-equivariant Euler class $e_{\mathbb{T}}(N_i)$ is invertible in $H^*_{\mathbb{T}}(X_i) \otimes Q_{\mathbb{T}}$ (see e.g.[21, Proposition 4]). The localization formula for the $\mathbb{T}$-equivariant fundamental class $[X]_{\mathbb{T}}$ is given by

$$[X]_{\mathbb{T}} = \iota_* \sum_i \frac{[X_i]}{e_{\mathbb{T}}(N_i)} \quad \in H^{\mathbb{T}}_*(X) \otimes Q_{\mathbb{T}} \; . \tag{2.10}$$

This formula enables us to express integrals over X as a sum of contributions from $\mathbb{T}$-fixed point components. For example, let A be a $\mathbb{T}$-equivariant vector bundle whose rank is equal to $\dim X$ and A_i be the pull-back of A to X_i. Then we have

$$\int_X e(A) = \sum_i \int_{X_i} \frac{e_{\mathbb{T}}(A_i)}{e_{\mathbb{T}}(N_i)}, \tag{2.11}$$

which is called Bott's residue formula [9].

There is an algebraic version $A^{\mathbb{T}}_*(X)$ of the equivaraint homology group, the equivariant Chow group of X with $\mathbb{Q}$-coefficients defined by Edidin–Graham [20]. A construction of the equivariant Chow groups similar to that in §2.2.1 has been given in *loc.cit.*. Moreover, in [21], it has been shown that the localization theorem and the formula (2.10) hold in the localized equivariant Chow group $A^{\mathbb{T}}_*(X) \otimes Q_{\mathbb{T}}$. See [53] for an extension of these results to Deligne–Mumford stacks.

2.2.3 Virtual localization formula

Let X be a non-singular projective variety equipped with an action of an algebraic torus $\mathbb{T}$. A $\mathbb{T}$-action on X induces that on $\overline{\mathcal{M}} := \overline{\mathcal{M}}_{g,n}(X, \beta)$ by translating maps:

$$t \cdot [f] = [t \cdot f], \quad t \in \mathbb{T}, \; [f] \in \overline{\mathcal{M}}.$$

It was Kontsevich [52] who proposed to compute Gromov–Witten invariants of such X by using localization formula with respect to the above $\mathbb{T}$-action on $\overline{\mathcal{M}}$. However, integrals in Gromov–Witten theory are taken against the virtual fundamental class $[\overline{\mathcal{M}}]^{vir}$ of $\overline{\mathcal{M}}$. Therefore, we need a localization formula for the virtual fundamental class. This was established by Graber–Pandharipande [30]. Here we summarize their results.

Recall that we have the perfect obstruction theory $E_\bullet = [E_0 \to E_1]$ which is a two-term complex of vector bundles on $\overline{\mathcal{M}}$. It defines the virtual fundamental class $[\overline{\mathcal{M}}]^{vir}$ of $\overline{\mathcal{M}}$ (cf.§2.1.2, 2.1.3). Now if there is a $\mathbb{T}$-action on $\overline{\mathcal{M}}$ we can take the complex $E_\bullet$ so that it is $\mathbb{T}$-equivariant. Then it defines the equivariant virtual fundamental class $[\overline{\mathcal{M}}]^{vir}_{\mathbb{T}}$ in the equivariant Chow group $A^{\mathbb{T}}_{\mathrm{vdim}\overline{\mathcal{M}}}(\overline{\mathcal{M}})$ of expected dimension.

Let $\overline{\mathcal{M}}^{\mathbb{T}} = \cup_i \overline{\mathcal{M}}_i$ be the fixed point components. Let $E_{\bullet,i}$ be the restriction of $E_\bullet$ to $\overline{\mathcal{M}}_i$. The complex $E_{\bullet,i}$ may be decomposed by $\mathbb{T}$-characters:

$$E_{\bullet,i} = E^f_{\bullet,i} \oplus E^m_{\bullet,i},$$

where the first summand is the $\mathbb{T}$-fixed part corresponding to the trivial character and the second summand is the $\mathbb{T}$-moving part corresponding to non-trivial characters. The $\mathbb{T}$-fixed part $E^f_{\bullet,i}$ defines the virtual fundamental class $[\overline{\mathcal{M}}_i]^{vir}$ of $\overline{\mathcal{M}}_i$. The normal 'bundle' to $\overline{\mathcal{M}}_i$ is defined as follows.

Definition 2.4. The virtual normal bundle N_i^{vir} to $\overline{\mathcal{M}}_i$ is defined to be the $\mathbb{T}$-moving part $E^m_{\bullet,i}$. The equivariant Euler class of $N_i^{vir} = E^m_{0,i} \to E^m_{1,i}$ is defined to be

$$e_{\mathbb{T}}(N_i^{vir}) = \frac{e_{\mathbb{T}}(E^m_{0,i})}{e_{\mathbb{T}}(E^m_{1,i})}. \tag{2.12}$$

Then the virtual localization formula is stated as follows.

Theorem 2.5 (Graber–Pandharipande).

$$[\overline{\mathcal{M}}]^{vir}_{\mathbb{T}} = \iota_* \sum_i \frac{[\overline{\mathcal{M}}_i]^{vir}}{e_{\mathbb{T}}(N_i^{vir})} \quad \in A^{\mathbb{T}}_{\mathrm{vdim}\,(\overline{\mathcal{M}})}(\overline{\mathcal{M}}) \otimes Q_{\mathbb{T}}. \tag{2.13}$$

Bott's formula is also valid in this setting;

$$\int_{[\overline{\mathcal{M}}]^{vir}} e(A) = \sum_i \int_{[\overline{\mathcal{M}}_i]^{vir}} \frac{e_{\mathbb{T}}(A_i)}{e_{\mathbb{T}}(N_i^{vir})}. \tag{2.14}$$

2.3 Hodge integrals

If we consider stable maps to a point, we recover the notion of stable curves introduced in [13, 45]. We denote by $\overline{\mathcal{M}}_{g,n}$ the moduli space of stable curves of genus g with n marked points. It is an irreducible projective variety of dimension $3g-3+n$. Since there are no obstructions to deformations of curves, it follows that the virtual fundamental class of $\overline{\mathcal{M}}_{g,n}$ is the usual fundamental class. Hodge integrals are integrals of natural cohomology classes over $\overline{\mathcal{M}}_{g,n}$. They arise naturally in Gromov–Witten theory. See [24] for more information. The main result in this section is a formula for two-partition Hodge integrals (Theorem 2.6) due to Liu–Liu–Zhou [60], which will be used in §5.

2.3.1 Partitions

First, we introduce some notations related to partitions, which will be used throughout the paper. Let μ be a partition, i.e. a non-increasing sequence of positive integers $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{l(\mu)} > 0)$. The number $l(\mu)$ is called the length of μ and $|\mu| := \mu_1 + \cdots + \mu_{l(\mu)}$ is called the weight of μ. The automorphism group $\mathrm{Aut}(\mu)$ is the subgroup of the permutation group of $\{\mu_1, \ldots, \mu_{l(\mu)}\}$ which preserves μ, i.e. permutes μ_i and μ_j only if $\mu_i = \mu_j$. Its order is equal to $\prod_{k\geq 1} m_k(\mu)!$, where $m_k(\mu) := \#\{i \mid \mu_i = k\}$.

We denote by $\mathcal{P}$ the set of partitions. For $\mu \in \mathcal{P}$, let χ_μ denote the character of the irreducible representation of symmetric group $S_{|\mu|}$ indexed by μ, and let C_μ denote the conjugacy class of $S_{|\mu|}$ indexed by μ. By the orthogonality of characters of S_d, where $d = |\mu| = |\nu|$, we have

$$\sum_{\substack{\sigma\in\mathcal{P} \\ |\sigma|=d}} \frac{\chi_\mu(C_\sigma)\chi_\nu(C_\sigma)}{z_\sigma} = \delta_{\mu\nu}, \tag{2.15}$$

where

$$z_\mu = |\mathrm{Aut}(\mu)| \cdot \prod_{i=1}^{l(\mu)} \mu_i \ .$$

For $\mu \in \mathcal{P}$, we define the integer

$$\kappa(\mu) := |\mu| + \sum_{i=1}^{l(\mu)} \mu_i(\mu_i - 2i), \tag{2.16}$$

which is always even.

Let $x = (x_1, x_2, \ldots)$ be a set of variables. For $\mu \in \mathcal{P}$, let $s_\mu(x)$ be the Schur function associated to μ. See [62] for a definition. It is known that the set of Schur functions $\{s_\mu(x) \mid \mu \in \mathcal{P}\}$ is a $\mathbb{Z}$-basis of the ring of symmetric functions. The *skew Schur function* $s_{\mu/\lambda}(x)$ for $\mu, \lambda \in \mathcal{P}$ is defined in terms of Schur functions as follows:

$$s_{\mu/\lambda}(x) = \sum_\nu c^\mu_{\lambda\nu} s_\nu(x) \ ,$$

where $c^\mu_{\lambda\nu}$ are integers (called the *Littlewood-Richardson coefficients*) defined by

$$s_\lambda(x)s_\nu(x) = \sum_\mu c^\mu_{\lambda\nu} s_\mu(x) \ .$$

2.3.2 Hodge integrals

There are natural cohomology classes on $\overline{\mathcal{M}}_{g,n}$ called the λ and ψ classes. Let $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the universal curve and ω_π be the relative dualizing sheaf. The Hodge bundle $\mathbb{E} := \pi_*\omega_\pi$ is the rank g vector bundle over $\overline{\mathcal{M}}_{g,n}$ whose fiber at $(C, p_1, \ldots, p_n)$ is given by $H^0(C, \omega_C)$. The λ*-classes* are the Chern classes of $\mathbb{E}$:

$$\lambda_i := c_i(\mathbb{E}) \in H^{2i}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

Let $s_i : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$ be the section of π which corresponds to the i-th marked point. Let $\mathbb{L}_i := s_i^*\omega_\pi$ be the line bundle over $\overline{\mathcal{M}}_{g,n}$ whose fiber at $(C, p_1, \ldots, p_n)$ is the cotangent line $T^*_{p_i}C$. The ψ*-classes* are the first Chern classes of $\mathbb{L}_i$:

$$\psi_i := c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

The *Hodge integrals* are intersection numbers of ψ-classes and λ-classes:

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{j_1} \cdots \psi_n^{j_n} \lambda_1^{k_1} \cdots \lambda_g^{k_g} \in \mathbb{Q}.$$

In Gromov–Witten theory of threefolds, Hodge integrals of particular types often appear. For h ($1 \le h \le 3$), let $(\mu^1, \cdots, \mu^h)$ be an h-tuple of partitions. Let us consider the integrals of the following type:

$$\int_{\overline{\mathcal{M}}_{g,l(\mu^1)+\cdots+l(\mu^h)}} \frac{\Lambda_g^\vee(u_1)\Lambda_g^\vee(u_2)\Lambda_g^\vee(u_3)}{\prod_{j=0}^{h-1}\prod_{i=1}^{l(\mu^{j+1})}(1-\mu_i^{j+1}\psi_{l(\mu^j)+i})},$$

where we set $\mu^0 = \emptyset$ (the empty partition) and

$$\Lambda_g^\vee(u) := u^g - \lambda_1 u^{g-1} + \cdots + (-1)^g \lambda_g.$$

By definition, the integral is the combination of Hodge integrals obtained by expanding the integrand as a series in the ψ and λ classes and then integrating the terms of the appropriate cohomological degree. They are often referred to as *one-, two-, three-partition triple Hodge integrals* for $h = 1, 2, 3$ respectively. If the parameters in the numerator satisfies $u_1+u_2+u_3 = 0$, these are called *special (or Calabi–Yau) triple Hodge integrals*. For example, the integrals that arise in the computation of the Gromov–Witten invariants of a general toric Calabi–Yau threefold are three-partition special triple Hodge integrals.

Let us consider the simplest cases, one-partition Hodge integrals for $g = 0$. In this case $\Lambda_0^\vee(u) = 1$ and

$$\int_{\overline{\mathcal{M}}_{0,l(\mu)}} \frac{1}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)} = |\mu|^{l(\mu)-3}, \tag{2.17}$$

for $l(\mu) \ge 3$ (see e.g.[52, Lemma in §3.3.2]). We use the equation (2.17) to extend the definition to the case of $l(\mu) < 3$. In that case, $\overline{\mathcal{M}}_{0,l(\mu)}$ is formally defined to be a point.

2.3.3 Formula for two-partition Hodge integrals

We shall encounter the following two-partition special triple Hodge integrals in §5, where we will study Gromov–Witten invariants of local toric surfaces. Given $\mu^+, \mu^- \in \mathcal{P}$, define

$$\begin{aligned} G_{g,\mu^+,\mu^-}(\tau) =& \frac{-\sqrt{-1}^{l(\mu^+)+l(\mu^-)}}{|\mathrm{Aut}(\mu^+)||\mathrm{Aut}(\mu^-)|}(\tau(\tau+1))^{l(\mu^+)+l(\mu^-)-1} \\ &\cdot \prod_{i=1}^{l(\mu^+)} \frac{\prod_{a=1}^{\mu_i^+-1}(\mu_i^+\tau + a)}{(\mu_i^+-1)!} \prod_{j=1}^{l(\mu^-)} \frac{\prod_{a=1}^{\mu_j^--1}(\frac{\mu_j^-}{\tau}+a)}{(\mu_j^--1)!} \\ &\cdot \int_{\overline{\mathcal{M}}_{g,l(\mu^+)+l(\mu^-)}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(\tau)\Lambda_g^\vee(-\tau-1)}{\prod_{i=1}^{l(\mu^+)}(1-\mu_i^+\psi_i)\prod_{j=1}^{l(\mu^-)}\tau(\tau-\mu_j^-\psi_{l(\mu^+)+j})}. \end{aligned} \tag{2.18}$$

We give a formula for $G_{g,\mu^+,\mu^-}(\tau)$ which will be used in §5. To state the formula, we introduce some generating functions.

We first define generating function of two-partition Hodge integrals. Introduce variables

$$p^+ = (p_1^+, p_2^+, \ldots), \quad p^- = (p_1^-, p_2^-, \ldots).$$

Given a partition μ, define

$$p_\mu^\pm = p_{\mu_1}^\pm \cdots p_{\mu_{l(\mu)}}^\pm$$

Define generating functions

$$\begin{aligned} G_{\mu^+,\mu^-}(\lambda;\tau) &= \sum_{g=0}^{\infty} \lambda^{2g-2+l(\mu^+)+l(\mu^-)} G_{g,\mu^+,\mu^-}(\tau) \ , && (2.19) \\ G(\lambda;p^+,p^-;\tau) &= \sum_{(\mu^+,\mu^-)\neq(\emptyset,\emptyset)} G_{\mu^+,\mu^-}(\lambda;\tau) p^+_{\mu^+} p^-_{\mu^-} \ , \\ G^\bullet(\lambda;p^+,p^-;\tau) &= \exp(G(\lambda;p^+,p^-;\tau)) \\ &= \sum_{\mu^+,\mu^-} G^\bullet_{\mu^+,\mu^-}(\lambda;\tau) p^+_{\mu^+} p^-_{\mu^-} \ . && (2.20) \end{aligned}$$

$G^\bullet_{\mu^+,\mu^-}(\lambda;\tau)$ is the "disconnected" [6] version of $G_{\mu^+,\mu^-}(\lambda;\tau)$.

We next define generating functions of of symmetric group representations. Let $q = e^{\sqrt{-1}\lambda}$ and write

$$q^{-\rho} = (q^{\frac{1}{2}}, q^{\frac{3}{2}}, \ldots).$$

Introduce

$$W_{\mu,\nu}(q) := (-1)^{|\mu|+|\nu|} q^{\frac{\kappa(\mu)+\kappa(\nu)}{2}} \sum_{\eta\in\mathcal{P}} s_{\mu/\eta}(q^{-\rho}) s_{\nu/\eta}(q^{-\rho}) \ , \tag{2.21}$$

and define

$$R^\bullet_{\mu^+,\mu^-}(\lambda;\tau) = \sum_{\substack{\nu^+,\,\nu^- \\ |\nu^\pm|=|\mu^\pm|}} \frac{\chi_{\nu^+}(C_{\mu^+})}{z_{\mu^+}} \frac{\chi_{\nu^-}(C_{\mu^-})}{z_{\mu^-}} e^{\sqrt{-1}(\kappa(\nu^+)\tau+\kappa(\nu^-)\tau^{-1})\lambda/2} W_{\nu^+,\nu^-}(q) \ . \tag{2.22}$$

The following identity was conjectured by Zhou [81] and proved by Liu–Liu–Zhou [60].

Theorem 2.6 (Liu–Liu–Zhou).

$$G^\bullet_{\mu^+,\mu^-}(\lambda;\tau) = R^\bullet_{\mu^+,\mu^-}(\lambda;\tau). \tag{2.23}$$

See [61] and the references therein for various aspects of the formula and related works. A formula for three-partition special triple Hodge integrals can be found in [59].

Part II
Topological vertex and local Gromov–Witten invariants

3 Local Gromov–Witten invariants of surfaces

In §3.1, local Gromov–Witten invariants of algebraic surfaces are introduced. We discuss the case of $-K$-nef surfaces in detail in §3.2.

[6] See [71] for a meaning and discussions of disconnected version.

3.1 Local Gromov–Witten invariants of surfaces

Let S be a non-singular projective surface and K_S be its canonical bundle. Let

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,1}(S,\beta) & \xrightarrow{\mu} & S \\ {\scriptstyle\pi}\big\downarrow & & \\ \overline{\mathcal{M}}_{g,0}(S,\beta) & & \end{array} \tag{3.1}$$

be the universal curve and the universal map over the moduli space of stable maps to S (cf.(2.1)).

Lemma 3.1. *There exists a two term complex $[K_0 \to K_1]$ of vector bundles on $\overline{\mathcal{M}}_{g,0}(S,\beta)$ whose cohomology sheaves are $R^0\pi_*\mu^*K_S$ and $R^1\pi_*\mu^*K_S$.*

Proof. See [7, Proposition 5]. $\square$

Note that the difference $[K_1] - [K_0]$ as an element of the Grothendieck group of vector bundles on $\overline{\mathcal{M}}_{g,0}(S,\beta)$ is independent of a choice of the complex.

Definition 3.2. (i) We define

$$c(-R^\bullet\pi_*\mu^*K_S) := c([K_1] - [K_0]) = \frac{c(K_1)}{c(K_0)},$$

where $c(\cdot)$ is the total Chern class defined on the Grothendieck group of vector bundles (cf. [26, Example 3.2.7.]).
(ii) For $g \geq 0$ and $\beta \in H_2(S,\mathbb{Z})$, the *local Gromov–Witten invariant* $I_{g,\beta}(S)$ of S with the genus g and the degree β is defined by

$$I_{g,\beta}(S) := \int_{[\overline{\mathcal{M}}_{g,0}(S,\beta)]^{vir}} c(-R^\bullet\pi_*\mu^*K_S) \in \mathbb{Q}.$$

A geometrical meaning of local Gromov–Witten invariants can be explained as follows. The total space of the bundle $K_S \to S$ can be regarded as a neighborhood of S smoothly embedded into a Calabi–Yau threefold X. Under some assumptions, the moduli space of stable maps $\overline{\mathcal{M}}_{g,0}(X,\iota_*\beta)$ (where $\iota : S \hookrightarrow X$ is the given imbedding) has a union of connected components which is isomorphic to $\overline{\mathcal{M}}_{g,0}(S,\beta)$. Ideally, local Gromov–Witten invariants can be seen as local contributions to Gromov–Witten invariants of the ambient space X from maps into S. See [74] for details.

Remark 3.3. The following statement seems to be well-known to specialists: the local Gromov–Witten invariants $I_{g,\beta}(S)$ is invariant under smooth deformations of S in the sense that is explained in the last paragraph in §2.1.3. However, because of the lack of appropriate references, we shall prove this statement under some assumptions in Proposition 3.7.

3.2 The case of $-K$-nef surfaces

3.2.1 $-K$-nef surfaces

Let S be a non-singular projective surface whose anti-canonical bundle $-K_S$ is nef, i.e. $-K_S \cdot [C] \geq 0$ for any irreducible curve C. We call such a surface a *$-K$-nef surface.*

Lemma 3.4. *Let S be a $-K$-nef surface. Then for any $g \in \mathbb{Z}_{\geq 0}$ and $\beta \in H_2(X,\mathbb{Z})$ such that $\int_\beta c_1(K_S) < 0$, we have*

$$R^0\pi_*\mu^*K_S = 0.$$

Proof. Let $(C, f) \in \overline{\mathcal{M}}_{g,0}(S, \beta)$. By the $-K$-nef condition, f^*K_S has at most constant sections on each irreducible component of C. On the other hand, the negativity assumption $\int_\beta c_1(K_S) < 0$ implies that there exists an irreducible component of C on which the degree of f^*K_S is negative. Therefore, sections of f^*K_S on C must be zero. □

By Lemma 3.4, $R^1\pi_*\mu^*K_S$ forms a vector bundle over $\overline{\mathcal{M}}_{g,0}(S, \beta)$ of rank $(1-g)(\dim S - 3) - \int_\beta c_1(K_S) = \mathrm{vdim}\, \overline{\mathcal{M}}_{g,0}(S, \beta)$. We have

$$I_{g,\beta}(S) = \int_{[\overline{\mathcal{M}}_{g,0}(S,\beta)]^{vir}} e(R^1\pi_*\mu^*K_S). \tag{3.2}$$

3.2.2 Comparison with projective bundles

Let S be a non-singular projective surface and $\mathbb{P}(K_S \oplus \mathcal{O}_S)$ be the projectivization of the total space of the vector bundle $K_S \oplus \mathcal{O}_S$. This is a $\mathbb{P}^1$-bundle over S. Let $\iota : S \hookrightarrow \mathbb{P}(K_S \oplus \mathcal{O}_S)$ be the inclusion as the zero section of $K_S \subset \mathbb{P}(K_S \oplus \mathcal{O}_S)$. Although $\mathbb{P}(K_S \oplus \mathcal{O}_S)$ is not a Calabi–Yau threefold, it is *locally* Calabi–Yau near the zero section. We define

$$N_{g,\iota_*\beta}(\mathbb{P}(K_S \oplus \mathcal{O}_S)) = \int_{[\overline{\mathcal{M}}_{g,0}(\mathbb{P}(K_S\oplus\mathcal{O}_S),\iota_*\beta)]^{vir}} 1 \ ,$$

since $\mathrm{vdim}\, \overline{\mathcal{M}}_{g,0}(\mathbb{P}(K_S \oplus \mathcal{O}_S), \iota_*\beta) = 0$.

Proposition 3.5. *Let S be a $-K$-nef surface, $\iota : S \hookrightarrow \mathbb{P}(K_S \oplus \mathcal{O}_S)$ be the inclusion as the zero section of K_S. Then, for $g \in \mathbb{Z}_{\geq 0}$ and $\beta \in H_2(S, \mathbb{Z})$ such that $\int_\beta c_1(K_S) < 0$, we have*

$$I_{g,\beta}(S) = N_{g,\iota_*\beta}(\mathbb{P}(K_S \oplus \mathcal{O}_S)).$$

Consider the natural $\mathbb{C}^*$-action on $\mathbb{P}(K_S \oplus \mathcal{O}_S)$ as the scalar multiplication in the $\mathbb{P}^1$-fiber direction. There is an induced $\mathbb{C}^*$-action on $\overline{\mathcal{M}}_{g,0}(\mathbb{P}(K_S \oplus \mathcal{O}_S), \iota_*\beta)$. First we show the following

Lemma 3.6. *Let S be a $-K$-nef surface, $\iota : S \hookrightarrow \mathbb{P}(K_S \oplus \mathcal{O}_S)$ be the inclusion as the zero section of K_S. Let $\beta \in H_2(S, \mathbb{Z})$ be a class satisfying $\int_\beta c_1(K_S) < 0$. If a stable map $(f, C) \in \overline{\mathcal{M}}_{g,0}(\mathbb{P}(K_S \oplus \mathcal{O}_S), \iota_*\beta)$, where C is a connected curve of genus g and $f : C \to S$ a morphism such that $[f(C)] = \iota_*\beta$, is fixed by the $\mathbb{C}^*$-action, then the image $f(C)$ is contained in the zero section $\iota(S)$.*

Proof. Denote the $\mathbb{P}^1$-fibration $\mathbb{P}(K_S \oplus \mathcal{O}_S) \to S$ by p, and let $P = [p^{-1}(a)] \in H_2(\mathbb{P}(K_S \oplus \mathcal{O}_S), \mathbb{Z})$ be the class of the fiber $\mathbb{P}^1$ where $a \in S$ is any point. Let $\iota^\infty : S \hookrightarrow \mathbb{P}(K_S \oplus \mathcal{O}_S)$ be the inclusion as the zero section of $\mathcal{O}_S$ (the section at the infinity of the $\mathbb{P}^1$-bundle). Note that for any $\alpha \in H_2(S, \mathbb{Z})$, we have

$$\iota^\infty_*\alpha = \iota_*\alpha - \left(\int_\alpha c_1(K_S)\right) P \ . \tag{3.3}$$

Let $\gamma \in H_2(\mathbb{P}(K_S \oplus \mathcal{O}_S), \mathbb{Z})$. If a stable map $(f, C) \in \overline{\mathcal{M}}_{g,0}(\mathbb{P}(K_S \oplus \mathcal{O}_S), \gamma)$ is fixed by the $\mathbb{C}^*$-action, then the image of an irreducible component C_i of C must be either one of these:
(i) $f(C_i) \subset \iota(S)$,
(ii) $f(C_i) \subset \iota^\infty(S)$,
(iii) $f(C_i) = p^{-1}(a_i)$ $(a_i \in S)$ and $C_i \cong \mathbb{P}^1$.
So assume that irreducible components $C_1, \ldots, C_k$ of C are of type (i) with $[f(C_i)] = \beta_i \in$

$H_2(S,\mathbb{Z})$, $C_{k+1},\ldots,C_r$ are of type (ii) with $[f(C_i)]=\beta_i\in H_2(S,\mathbb{Z})$, and that $C_{r+1},\ldots,C_s$ are of type (iii) with $f:C_i\to p^{-1}(a_i)$ the d_i-fold coverings. Then $[f(C)]=\gamma$ is equivalent to

$$\gamma=\sum_{i=1}^{k}\iota_*\beta_i+\sum_{i=k+1}^{r}\iota_*^{\infty}\beta_i+\sum_{i=r+1}^{s}d_iP=\sum_{i=1}^{r}\iota_*\beta_i+\Big(\sum_{i=r+1}^{s}d_i-\sum_{i=k+1}^{r}\int_{\beta_i}c_1(K_S)\Big)P \quad .$$

Now take $\gamma=\iota_*\beta$ with $\beta\in H_2(S,\mathbb{Z})$ satisfying $\int_\beta c_1(K_S)<0$ and solve the above equation. The assumption that S is $-K$-nef implies that the coefficient of P in the last line is always nonnegative. Therefore it is zero if and only if there is no irreducible components of type (iii) and $\int_{\beta_i}c_1(K_S)=0$ for those of type (ii). Then connectedness of the domain curve C implies either $f(C)\subset\iota(S)$ or $f(C)\subset\iota^{\infty}(S)$. For the latter case, $\int_{[f(C)]}c_1(K_S)=0$ and this contradicts the assumption $\int_\beta c_1(K_S)<0$. Thus $f(C)\subset\iota(S)$. □

Proof. (of Proposition 3.5.) By Lemma 3.6, the $\mathbb{C}^*$-fixed point set in $\overline{\mathcal{M}}_{g,0}(\mathbb{P}(K_S\oplus\mathcal{O}_S),\iota_*\beta)$ is isomorphic to $\overline{\mathcal{M}}_{g,0}(S,\beta)$. We compare perfect obstruction theories on them. Let $E_\bullet=[E_0\to E_1]$ be the perfect obstruction theory on $\overline{\mathcal{M}}_{g,0}(\mathbb{P}(K_S\oplus\mathcal{O}_S),\iota_*\beta)$ restricted to its $\mathbb{C}^*$-fixed point set. By exact sequences (2.3, 2.4), we have

$$\begin{aligned}E_1-E_0=&R^1\pi_*\underline{Hom}(\Omega_\pi,\mathcal{O}_\mathcal{U})-R^0\pi_*\underline{Hom}(\Omega_\pi,\mathcal{O}_\mathcal{U})+R^1\pi_*\mu^*T_X-R^0\pi_*\mu^*T_X\\ =&R^1\pi_*\underline{Hom}(\Omega_\pi,\mathcal{O}_\mathcal{U})-R^0\pi_*\underline{Hom}(\Omega_\pi,\mathcal{O}_\mathcal{U})+R^1\pi_*\mu^*T_S-R^0\pi_*\mu^*T_S\\ &+R^1\pi_*\mu^*K_S-R^0\pi_*\mu^*K_S.\end{aligned}\tag{3.4}$$

Here we have used the normal bundle sequence at the zero-section

$$0\longrightarrow T_S\longrightarrow T_{\mathbb{P}(K_S\oplus\mathcal{O}_S)}|_S\longrightarrow K_S\longrightarrow 0,$$

in the second equality. The first four terms in the second line of (3.4) are the $\mathbb{C}^*$-fixed part of $E_\bullet$ which are identical to the perfect obstruction theory on $\overline{\mathcal{M}}_{g,0}(S,\beta)$. Hence we have

$$[\overline{\mathcal{M}}_{g,0}(\mathbb{P}(K_S\oplus\mathcal{O}_S),\iota_*\beta)^{\mathbb{C}^*}]^{vir}\cong[\overline{\mathcal{M}}_{g,0}(S,\beta)]^{vir}.$$

The two terms in the third line of (3.4) are the $\mathbb{C}^*$-moving part of $E_\bullet$ which give the virtual normal bundle to the $\mathbb{C}^*$-fixed point set. By Lemma 3.4 we have $R^0\pi_*\mu^*K_S=0$. Thus the virtual normal bundle is actually the bundle $R^1\pi_*\mu^*K_S$. Then by Bott's formula (2.14), we have

$$\begin{aligned}N_{g,\iota_*\beta}(\mathbb{P}(K_S\oplus\mathcal{O}_S)) &= \int_{[\overline{\mathcal{M}}_{g,0}(S,\beta)]^{vir}}e_{\mathbb{C}^*}(R^1\pi_*\mu^*K_S)\\ &= \int_{[\overline{\mathcal{M}}_{g,0}(S,\beta)]^{vir}}e(R^1\pi_*\mu^*K_S).\end{aligned}$$

The second equality follows since the rank of $R^1\pi_*\mu^*K_S$ is equal to the virtual dimension of $\overline{\mathcal{M}}_{g,0}(S,\beta)$. □

3.2.3 Deformation invariance

Recall that Gomov–Witten invariants of projective varieties are invariant under deformations (cf. §2.1.3).

Proposition 3.7. *Let S be a $-K$-nef surface and S' be a $-K$-nef surface which is deformation equivalent to S. Let $\beta\in H_2(S,\mathbb{Z})$ be a class satisfying $\int_\beta c_1(K_S)<0$ and $\beta'\in H_2(S',\mathbb{Z})$ be the class corresponding to β under a deformation. Then $I_{g,\beta}(S)=I_{g,\beta'}(S')$ for $g\in\mathbb{Z}_{\geq0}$.*

Proof. Since S and S' are deformation equivalent, $\mathbb{P}(K_S \oplus \mathcal{O}_S)$ and $\mathbb{P}(K_{S'} \oplus \mathcal{O}_{S'})$ are also deformation equivalent. Let $\iota : S \hookrightarrow \mathbb{P}(K_S \oplus \mathcal{O}_S)$ and $\iota' : S' \hookrightarrow \mathbb{P}(K_{S'} \oplus \mathcal{O}_{S'})$ be the inclusions as the zero sections of K_S and $K_{S'}$ respectively. We have

$$I_{g,\beta}(S) = N_{g,\iota_*\beta}(\mathbb{P}(K_S \oplus \mathcal{O}_S)) = N_{g,\iota'_*\beta'}(\mathbb{P}(K_{S'} \oplus \mathcal{O}_{S'})) = I_{g,\beta'}(S').$$

The middle equality follows from the deformation invariance of Gromov–Witten invariants. The first and the third equalities follow from Proposition 3.5. $\square$

4 Gromov–Witten invariants of local toric surfaces

First, we introduce the equivariant Gromov–Witten invariants of local toric surfaces. The Gromov–Witten invariants of local toric surfaces are defined as a certain non-equivariant limit of equivariant ones. Then we give a closed formula for the partition functions of local toric surfaces (Theorem 4.6), which has been stated in Theorem 1.3. Its proof will be given in §5.

4.1 Equivariant Gromov–Witten invariants of local toric surfaces

Let S be a nonsingular complete toric surface (see, e.g. [27, §2.5]). There is a canonical action of $\mathbb{T} = (\mathbb{C}^*)^2$ on S. We regard the canonical bundle K_S of S as a $\mathbb{T}$-equivariant line bundle

$$K_S = \mathcal{O}_S(-\sum_{i=1}^{r} C_i),$$

where $C_1, \cdots, C_r$ are the irreducible toric divisors (cf. [73, §2.1]). By Lemma 3.1, there is a two-term complex $[K_0 \to K_1]$ of vector bundles on $\overline{\mathcal{M}}_{g,0}(S,\beta)$ which is $\mathbb{T}$-equivariant and whose cohomology sheaves are $R^0\pi_*\mu^*K_S$ and $R^1\pi_*\mu^*K_S$. We define

$$e_{\mathbb{T}\times\mathbb{C}^*}(-R^\bullet\pi_*\mu^*K_S) := \frac{e_{\mathbb{T}\times\mathbb{C}^*}(K_1)}{e_{\mathbb{T}\times\mathbb{C}^*}(K_0)} \in A^{\mathrm{vdim}}_{\mathbb{T}\times\mathbb{C}^*}(\overline{\mathcal{M}}_{g,0}(S,\beta)) \otimes Q_{\mathbb{T}\times\mathbb{C}^*}, \tag{4.1}$$

where the second factor in $\mathbb{T}\times\mathbb{C}^*$ acts on K_S by the scalar multiplication fiber-wisely and acts trivially on $\overline{\mathcal{M}}_{g,0}(S,\beta)$.

Definition 4.1. Let us consider the following equivariant integral:

$$N^{\mathbb{T}\times\mathbb{C}^*}_{g,\beta}(K_S) := \int_{[\overline{\mathcal{M}}_{g,0}(S,\beta)]^{vir}_{\mathbb{T}\times\mathbb{C}^*}} e_{\mathbb{T}\times\mathbb{C}^*}(-R^\bullet\pi_*\mu^*K_S) \in \mathbb{Q}(\alpha_1,\alpha_2,\alpha_3), \tag{4.2}$$

Here α_1 and α_2 are the weights of $\mathbb{T}$ and α_3 is that of $\mathbb{C}^*$ (cf. §2.2.2). We call $N^{\mathbb{T}\times\mathbb{C}^*}_{g,\beta}(K_S)$ the *equivariant Gromov–Witten invariant of local toric surface* K_S.

We shall compute $N^{\mathbb{T}\times\mathbb{C}^*}_{g,\beta}(K_S)$ by virtual localization formula (2.13). It turns out that the non-equivariant limit with respect to the second factor

$$\lim_{\alpha_3\to 0} N^{\mathbb{T}\times\mathbb{C}^*}_{g,\beta}(K_S) =: N_{g,\beta}(K_S) \tag{4.3}$$

exists (cf. the last paragraph in §5.4).

Definition 4.2. We call $N_{g,\beta}(K_S)$ the Gromov–Witten invariant of a local toric surface K_S.[7]

[7] This definition was suggested by Hiroshi Iritani. The author is grateful to him for discussions in July 2006 at Hokkaido University.

A priori, $N_{g,\beta}(K_S)$ takes values in $\mathbb{Q}(\alpha_1, \alpha_2)$. However, it turns out that $N_{g,\beta}(K_S) \in \mathbb{Q}$.

Remark 4.3. The total space of K_S is an example of toric Calabi–Yau threefolds, which is called a local toric surface. $N_{g,\beta}(K_S)$ are special cases of Gromov–Witten invariants of toric Calabi–Yau threefolds. See §8.3.

We discuss the relations between $N_{g,\beta}(K_S)$ and $I_{g,\beta}(S)$. In general, they are different. For instance, $I_{g,\beta}(S)$ is a deformation invariant as we mentioned, but $N_{g,\beta}(K_S)$ is not (see Remark 6.2 (i)). However, we have the following

Lemma 4.4. *If S is a $-K$-nef toric surface, then*

$$N_{g,\beta}(K_S) = I_{g,\beta}(S), \tag{4.4}$$

for β satisfying $\int_\beta c_1(K_S) < 0$.

Proof. This follows from (3.2) and Bott's formula (2.14). □

Remark 4.5. The non-equivariant limit with respect to the first factor

$$\lim_{\alpha_1,\alpha_2 \to 0} N_{g,\beta}^{\mathbb{T}\times\mathbb{C}^*}(K_S) =: I_{g,\beta}^{\mathbb{C}^*}(S) \in \mathbb{Q}[\alpha_3, \alpha_3^{-1}] \ ,$$

is equal to so-called the equivariant local Gromov–Witten invariants of S, which have been considered, e.g. in [10]. If we further take the specialization $\alpha_3 = 1$, we recover local Gromov–Witten invariants of S:

$$I_{g,\beta}^{\mathbb{C}^*}(S)|_{\alpha_3=1} = I_{g,\beta}(S).$$

In this sense, the equivariant Gromov–Witten invariants $N_{g,\beta}^{\mathbb{T}\times\mathbb{C}^*}(K_S)$ of K_S are the universal objects which contain both the Gromov–Witten invariants $N_{g,\beta}(K_S)$ of K_S and the local Gromov–Witten invariants $I_{g,\beta}(S)$ of S.

4.2 Topological vertex formula

Let us introduce the following generating function of Gromov–Witten invariants of a local toric surface K_S:

$$F_{K_S}(\lambda, Q) := \sum_{\beta \in H_2(S,\mathbb{Z}), \beta \neq 0} \sum_{g \geq 0} N_{g,\beta}(K_S) \lambda^{2g-2} Q^\beta, \tag{4.5}$$

and its partition function

$$Z_{K_S}(\lambda, Q) := \exp F_{K_S}(\lambda, Q). \tag{4.6}$$

The following formula (4.7) was obtained by Zhou [82] when S is a toric del Pezzo surface.

Theorem 4.6. *Let S be a nonsingular complete toric surface, $C_1, \cdots, C_r$ the irreducible toric divisors on S, and s_i the self-intersection number of C_i. Set $t_i = Q^{[C_i]}$ $(1 \leq i \leq r)$. Then we have*

$$Z_{K_S}(\lambda, Q) = \sum_{(\nu^1,\dots,\nu^r) \in \mathcal{P}^r} \prod_{i=1}^r ((-1)^{s_i} t_i)^{|\nu^i|} e^{\sqrt{-1}\lambda s_i \frac{\kappa(\nu^i)}{2}} W_{\nu^i,\nu^{i+1}}(e^{\sqrt{-1}\lambda}), \tag{4.7}$$

where $W_{\nu^i,\nu^{i+1}}$ has been defined by (2.21).

See §2.3.3 for definitions of combinatorial functions on the set of partitions $\mathcal{P}$. In §5, we shall prove Theorem 4.6 following the arguments in [82]. It also follows from the results for general toric Calabi–Yau threefolds obtained in [59]. The term $W_{\mu,\nu}$ in (4.7) comes from a formula for two-partition Hodge integrals (Theorem 2.6).

4.3 Gopakumar–Vafa invariants

Gopakumar–Vafa invariants can be defined for Gromov–Witten invariants of K_S by exactly the same way as in Definition 2.2.

Definition 4.7. Define the Gopakumar–Vafa invariants $n^g_\beta(K_S) \in \mathbb{Q}$ by the following formula:

$$F_{K_S}(\lambda, Q) = \sum_{\beta \neq 0} \sum_{g=0}^{\infty} \sum_{k=1}^{\infty} \frac{n^g_\beta(K_S)}{k} \left(2 \sin \frac{k\lambda}{2}\right)^{2g-2} Q^{k\beta}. \tag{4.8}$$

By using the topological vertex formula (4.7), Peng [75] and Konishi [48] showed the following

Theorem 4.8 (Konishi, Peng). *Gopakumar–Vafa invariants $n^g_\beta(K_S)$ are integers. Moreover, for each β, $n^g_\beta(K_S)$ is equal to zero for all but finite g.*

We will use this result later.

5 Localization on moduli spaces of stable maps to toric surfaces

The goal of this section is to prove Theorem 4.6 by using virtual localization. Let S be a nonsingular complete toric surface and $\mathbb{T} := (\mathbb{C}^*)^2$. The $\mathbb{T}$-action on S induces that on $\overline{\mathcal{M}}_{g,0}(S, \beta)$ (cf. 2.2.3). We carry out virtual localization on $\overline{\mathcal{M}}_{g,0}(S, \beta)$ as follows. First, in §5.1–5.3, the following four properties are established:

- The $\mathbb{T}$-fixed point set $\overline{\mathcal{M}}_{g,0}(S, \beta)^{\mathbb{T}}$ is a disjoint union of irreducible components.
- Each $\mathbb{T}$-fixed component is isomorphic to a quotient of products of moduli spaces of pointed stable curves by a finite group.
- The virtual fundamental class of a $\mathbb{T}$-fixed component is the ordinary fundamental class of moduli spaces of stable curves.
- The $\mathbb{T}$-equivariant Euler class of the virtual normal bundle to a $\mathbb{T}$-fixed component is expressed in terms of ψ and λ classes on moduli spaces of stable curves.

Here we closely follow [30]. It is straightforward to extend the arguments of *loc.cit.* for any nonsingular projective toric varieties (see [78]). As a result, an equivariant Gromov–Witten invariant of K_S is expressed as a sum over equivalence classes of appropriate graphs (§5.4). Then Theorem 4.6 is proven in §5.5. There we follow [82] in which the case when S is a toric del Pezzo surface has been treated.

5.1 The $\mathbb{T}$-fixed components

We identify the components of the $\mathbb{T}$-fixed point set of $\overline{\mathcal{M}}_{g,0}(S, \beta)$ with a set of (isomorphism classes of) certain decorated graphs.

Definition 5.1. (1) An $\overline{\mathcal{M}}_{g,0}(S, \beta)$-decorated graph G consists of the data (V, E, g, p, d) where:

(i) V is the vertex set,

(ii) $g : V \to \mathbb{Z}_{\geq 0}$ is a genus label,

(iii) $p : V \to S^{\mathbb{T}}$ is a fixed point label, where $S^{\mathbb{T}}$ is the $\mathbb{T}$-fixed point set on S,

(iv) E is the edge set,

 (a) If the edge e connects $v, v' \in V$, then $p(v) \neq p(v')$ and there exists a unique irreducible $\mathbb{T}$-invariant divisor C_e on S which contains $p(v)$ and $p(v')$,

 (b) G is connected,

(v) $d : E \to \mathbb{Z}_{>0}$ is a degree label,

(vi) $g = \sum_{v \in V} g(v) + h^1(G)$, where $h^1(G)$ is the first Betti number of G,

(vii) $\beta = \sum_{e \in E} d(e)[C_e]$, where $[C_e] \in H_2(S, \mathbb{Z})$ is the second homology class of C_e.

(2) An isomorphism of $\overline{\mathcal{M}}_{g,0}(S, \beta)$-decorated graphs

$$f : G = (V, E, g, p, d) \to G' = (V', E', g', p', d')$$

is a pair (f_v, f_e) of bijections

$$f_v : V \to V', \quad f_e : E \to E'$$

which preserve all the structures of decorated graphs. An isomorphism of G to itself is called an automorphism of G.
(3) We denote by $G_{g,\beta}(S)$ the set of isomorphism classes of $\overline{\mathcal{M}}_{g,0}(S, \beta)$-decorated graphs. The automorphism group $\mathrm{Aut}(G)$ of an element $G \in G_{g,\beta}(S)$ is that of a representative of G.

Lemma 5.2. *The $\mathbb{T}$-fixed components of $\overline{\mathcal{M}}_{g,0}(S, \beta)$ are in bijective correspondence with the graph set $G_{g,\beta}(S)$.*

Proof. Let $f : C \to S$ be a $\mathbb{T}$-fixed stable map. Then $f(C)$ must be a $\mathbb{T}$-invariant curve on S and the images of nodes, contracted components, and ramification points must be mapped to $S^{\mathbb{T}}$. It follows that each non-contracted component D must be mapped to an irreducible $\mathbb{T}$-invariant curve on S, which is isomorphic to $\mathbb{P}^1$. Since D is only ramified over two points on $\mathbb{P}^1$, it follows that D is nonsingular and rational. The restriction of f to D is uniquely determined by the degree.

To a $\mathbb{T}$-fixed stable map $f : C \to S$, we associate a graph $G \in G_{g,\beta}(S)$ as follows.

(i) V is the set of connected components of $f^{-1}(S^{\mathbb{T}})$,

(ii) $g(v)$ is the arithmetic genus of the component corresponding to v (defined to be 0 if the component is an isolated point),

(iii) For $v \in V$, $p(v) := f(v)$,

(iv) E is the set of non-contracted irreducible components of C and $C_e := f(e)$ for $e \in E$,

(v) For $e \in E$, $d(e) := \deg(f|_e)$.

This gives a one-to-one correspondence between the $\mathbb{T}$-fixed components of $\overline{\mathcal{M}}_{g,0}(S, \beta)$ and the graph set $G_{g,\beta}(S)$. $\square$

Let $G \in G_{g,\beta}(S)$. The set of $\mathbb{T}$-fixed stable maps with given graph G can be identified with a quotient of a product of moduli space of pointed stable curves by a finite group as follows. Let

$$\overline{\mathcal{M}}_G = \prod_{v \in V} \overline{\mathcal{M}}_{g(v),\mathrm{val}(v)},$$

where $\overline{\mathcal{M}}_{g(v),\mathrm{val}(v)}$ is the moduli space of stable curves of genus $g(v)$ with $\mathrm{val}(v)$ marked points and $\mathrm{val}(v)$ is the valence of v. We consider the automorphism group A_G of $\overline{\mathcal{M}}_G$ which fits in the following exact sequence:

$$1 \longrightarrow \textstyle\prod_{e\in E} \mathbb{Z}/d(e) \longrightarrow A_G \longrightarrow \mathrm{Aut}(G) \longrightarrow 1.$$

Here $\mathrm{Aut}(G)$ acts naturally on $\prod_{e\in E} \mathbb{Z}/d(e)$ and A_G is the semi-direct product $A_G = \prod_{e\in E} \mathbb{Z}/d(e) \ltimes \mathrm{Aut}(G)$.

Lemma 5.3. *Let $\mathcal{Q}_G$ be the $\mathbb{T}$-fixed point component of $\overline{\mathcal{M}}_{g,0}(S,\beta)$ labeled by $G \in G_{g,\beta}(S)$. Then $\mathcal{Q}_G$ can be identified with $\overline{\mathcal{M}}_G/A_G$ as a set.*

5.2 Perfect obstruction theories on the $\mathbb{T}$-fixed components

We study the scheme structure of the $\mathbb{T}$-fixed component $\mathcal{Q}_G$ labeled by $G \in G_{g,\beta}(S)$. It is determined by the perfect obstruction theory on $\overline{\mathcal{M}}_{g,0}(S,\beta)$ restricted to $\mathcal{Q}_G$. Let $E_\bullet = [E_0 \to E_1]$ be the perfect obstruction theory on $\overline{\mathcal{M}}_{g,0}(S,\beta)$. Let $E_{0,G} \to E_{1,G}$ be the restriction of $E_\bullet$ to $\mathcal{Q}_G$. The tangent sheaf $\mathcal{T}^1_G$ and the obstruction sheaf $\mathcal{T}^2_G$ on $\mathcal{Q}_G$ are defined by the following exact sequence:

$$0 \longrightarrow \mathcal{T}^1_G \longrightarrow E_{0,G} \longrightarrow E_{1,G} \longrightarrow \mathcal{T}^2_G \longrightarrow 0. \tag{5.1}$$

Recall that we have the following exact sequence of sheaves on $\mathcal{Q}_G$ (cf. (2.4)):

$$\begin{aligned} 0 &\longrightarrow \mathrm{Ext}^0(\Omega_{\mathrm{C}}, \mathcal{O}_{\mathrm{C}}) \longrightarrow H^0(C, f^*T_S) \longrightarrow \mathcal{T}^1_G \\ &\longrightarrow \mathrm{Ext}^1(\Omega_{\mathrm{C}}, \mathcal{O}_{\mathrm{C}}) \longrightarrow H^1(C, f^*T_S) \longrightarrow \mathcal{T}^2_G \longrightarrow 0. \end{aligned} \tag{5.2}$$

Note that the four term other than $\mathcal{T}^i_G$ in (5.2) are vector bundles on $\mathcal{Q}_G$. Therefore, we have represented them by their fibers at each point $(f, C) \in \mathcal{Q}_G$.

Lemma 5.4. *The $\mathbb{T}$-fixed part $\mathcal{T}^{1,f}_G$ of $\mathcal{T}^1_G$ is isomorphic to that of the fourth term in (5.2). The $\mathbb{T}$-fixed part $\mathcal{T}^{2,f}_G$ of $\mathcal{T}^2_G$ is zero.*

Proof. See [30, the second paragraph in p.503]. (Actually, this follows form proofs of Lemmas 5.8 and 5.10 below.) □

As a result, we have the following

Lemma 5.5. *$\mathcal{Q}_G$ is isomorphic to $\overline{\mathcal{M}}_G/A_G$. Moreover the perfect obstruction theory on $\mathcal{Q}_G$ is trivial in the sense that its virtual fundamental class is a usual fundamental class of $\overline{\mathcal{M}}_G/A_G$.*

Proof. By Lemma 5.4, $\mathcal{T}^{1,f}_G$ is isomorphic to the tangent bundle of $\overline{\mathcal{M}}_G$ and $\mathcal{T}^{2,f}_G = 0$. Then the assertion follows from the exact sequence (5.1). See [30, p.505–506] for detail. □

Then, by Theorem 2.5, we have the following

Proposition 5.6.

$$[\overline{\mathcal{M}}_{g,0}(S,\beta)]^{vir}_{\mathbb{T}} = \iota_* \sum_{G \in G_{g,\beta}(S)} \frac{1}{|A_G|} \frac{[\overline{\mathcal{M}}_G]}{e_{\mathbb{T}}(N^{vir}_G)} \quad \in A^{\mathbb{T}}_{\mathrm{vdim}}(\overline{\mathcal{M}}_{g,0}(S,\beta)) \otimes Q_{\mathbb{T}} \ . \tag{5.3}$$

5.3 Virtual normal bundles of the $\mathbb{T}$-fixed components

We compute the equivariant Euler class of the virtual normal bundle of $\mathcal{Q}_G$. By definition of the virtual normal bundle (cf. Definition 2.4) and the exact sequences (5.1, 5.2), we have

$$\frac{1}{e_{\mathbb{T}}(N_G^{vir})} = \frac{e_{\mathbb{T}}(\mathrm{Ext}^0(\Omega_C, \mathcal{O}_C)^m)}{e_{\mathbb{T}}(\mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)^m)} \cdot \frac{e_{\mathbb{T}}(H^1(C, f^*T_S)^m)}{e_{\mathbb{T}}(H^0(C, f^*T_S)^m)}. \tag{5.4}$$

First, we compute the first factor in (5.4). Let us introduce some notations. Let $G = (V, E, g, p, d) \in G_{g,\beta}(S)$. We set

$$\begin{aligned} V_1 &:= \{v \in V \mid (g(v), \mathrm{val}(v)) = (0,1)\}, \\ V_2 &:= \{v \in V \mid (g(v), \mathrm{val}(v)) = (0,2)\}, \\ V_3 &:= \{v \in V \mid (g(v), \mathrm{val}(v)) \neq (0,1), (0,2)\}. \end{aligned}$$

Definition 5.7. (i) A flag F is a pair (v, e) of a vertex $v \in V$ and an edge $e \in E$ which is incident on v. A flag $F = (v, e)$ is called a flag incident on v We denote the set of flags incident on v by $\mathcal{F}_v$.
(ii) For $v \in V_1$, $\mathcal{F}_v$ consists of a unique element, which is denoted by $F(v)$.
(iii) For $v \in V_2$, there are exactly two elements in $\mathcal{F}_v$ which are denoted by $F_1(v)$ and $F_2(v)$.
(iv) Let $v \in V_2 \cup V_3$. For $F = (v, e) \in \mathcal{F}_v$, we define $\psi_F := c_1(L_F)$ (the ψ-class associated to F), where L_F is the line bundle over $\overline{\mathcal{M}}_G$ whose fiber is the cotangent line to the component e of the domain curve at the node v.
(v) For a $\mathbb{T}$-invariant divisor C and a $\mathbb{T}$-fixed point $p \in S^{\mathbb{T}}$ on S, we denote the $\mathbb{T}$-weight of $\mathcal{O}_S(C)$ at p by $\omega_p^C = \omega_p^C(\alpha_1, \alpha_2) \in R_{\mathbb{T}}$. Here α_1, α_2 are the weights of the standard representations of factors of $\mathbb{T}$ (cf. §2.2.2).
(vi) For a flag $F = (v, e)$, we define $\omega_F := \dfrac{\omega_{p(v)}^{C_e}}{d(e)} \in R_{\mathbb{T}}$.

We have the following

Lemma 5.8. *(i)*

$$e_{\mathbb{T}}(\mathrm{Ext}^0(\Omega_C, \mathcal{O}_C)^m) = \prod_{v \in V_1} \omega_{F(v)}.$$

(ii)

$$e_{\mathbb{T}}(\mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)^m) = \prod_{v \in V_3} \prod_{F \in \mathcal{F}_v} (\omega_F - \psi_F) \times \prod_{v \in V_2} (\omega_{F_1(v)} + \omega_{F_2(v)}).$$

Proof. See [30, p.503–504] or [78, §7.1–7.2]. □

Next, we compute the second factor in (5.4). Let us introduce notations. Let Σ be the fan of S and Σ_n be the set of n-cones of Σ. Note that Σ_2 is in one-to-one correspondence with $S^{\mathbb{T}}$, and Σ_1 is in one-to-one correspondence with the set of irreducible $\mathbb{T}$-invariant divisors on S. Let $\Sigma_2 = \{\sigma_1, \dots, \sigma_r\}$, $\Sigma_1 = \{\rho_1 \dots, \rho_r\}$, where numberings are given as in Figure 1. Let p_i be the $\mathbb{T}$-fixed point corresponding to σ_i and C_i be the irreducible $\mathbb{T}$-invariant divisor corresponding to ρ_i. For $i = 1, \dots, r$, we set

$$\begin{aligned} V_j^{(i)} &:= \{v \in V_j \mid p(v) = p_i\}, \quad j = 1,2,3, \\ V^{(i)} &:= V_1^{(i)} \cup V_2^{(i)} \cup V_3^{(i)}, \\ E^{(i)} &:= \{e \in E \mid C_e = C_i\}. \end{aligned}$$

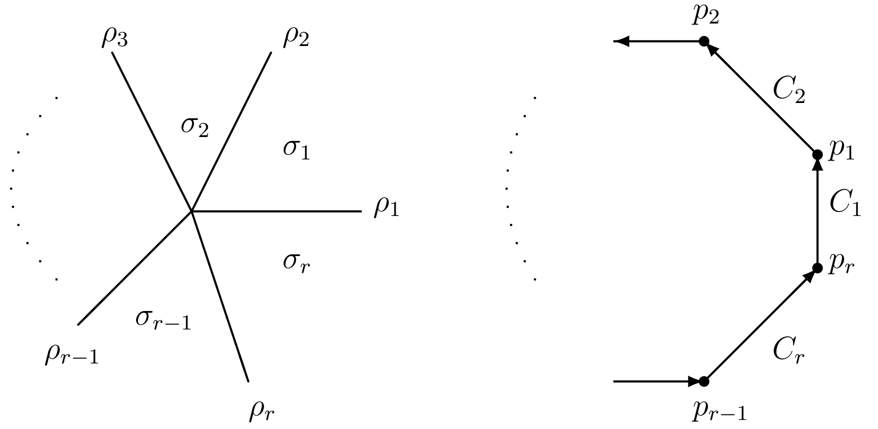


Figure 1: Σ (left) and its dual graph (right).

Definition 5.9. (i) We set

$$\omega_i^{\mathrm{in}} := \omega_{p_i}^{C_i}, \quad \omega_i^{\mathrm{out}} := \omega_{p_i}^{C_{i+1}},$$

where we have regarded i as an element of $\mathbb{Z}/r$.
(ii) Let $s_i := C_i^2$ be the self intersection number of C_i. For $d \in \mathbb{Z}_{\geq 1}$, we define

$$P_{(i)}(d) := \frac{d^{2d}}{(d!)^2} \frac{1}{(\omega_{i-1}^{\mathrm{in}})^d (\omega_i^{\mathrm{out}})^d} \times \begin{cases} \prod_{a=1}^{-s_i d-1} \left(\frac{a}{d}\, \omega_i^{\mathrm{out}} + \omega_i^{\mathrm{in}}\right) & \text{if } s_i < 0, \\ \prod_{a=0}^{s_i d} \left(-\frac{a}{d}\, \omega_i^{\mathrm{out}} + \omega_i^{\mathrm{in}}\right)^{-1} & \text{if } s_i \geq 0. \end{cases} \tag{5.5}$$

Then we have the following

Lemma 5.10.

$$\frac{e_{\mathbb{T}}(H^1(C, f^*T_S)^m)}{e_{\mathbb{T}}(H^0(C, f^*T_S)^m)} = \prod_{i=1}^{r} \left\{ \prod_{v \in V_2^{(i)}} \omega_i^{\mathrm{in}} \cdot \omega_i^{\mathrm{out}} \times \prod_{v \in V_3^{(i)}} (\omega_i^{\mathrm{in}} \cdot \omega_i^{\mathrm{out}})^{\mathrm{val}(v)-1} \cdot \Lambda_{g(v)}^{\vee}(\omega_i^{\mathrm{in}}) \cdot \Lambda_{g(v)}^{\vee}(\omega_i^{\mathrm{out}}) \right\}$$
$$\times \prod_{i=1}^{r} \prod_{e \in E^{(i)}} P_{(i)}\left(d(e)\right) \ , \tag{5.6}$$

where $\Lambda_{g(v)}^{\vee}(u)$ has been defined in §2.3.

Proof. The proof is identical to that of Lemma 5.13 below. See [30, p.504–505] or [78, §7.3.]. □

By eq.(5.4), Lemmas 5.8 and 5.10, we have the following

Proposition 5.11.

$$\frac{1}{e_{\mathbb{T}}(N_G^{vir})} = \prod_{i=1}^{r} \Big\{ \prod_{v \in V_1^{(i)}} \omega_{F(v)} \times \prod_{v \in V_2^{(i)}} \frac{\omega_i^{\mathrm{in}} \cdot \omega_i^{\mathrm{out}}}{\omega_{F_1(v)} + \omega_{F_2(v)}}$$
$$\times \prod_{v \in V_3^{(i)}} \frac{(\omega_i^{\mathrm{in}} \cdot \omega_i^{\mathrm{out}})^{\mathrm{val(v)}-1} \cdot \Lambda_{g(v)}^{\vee}(\omega_i^{\mathrm{in}}) \cdot \Lambda_{g(v)}^{\vee}(\omega_i^{\mathrm{out}})}{\prod_{F \in \mathcal{F}_v} (\omega_F - \psi_F)} \Big\} \tag{5.7}$$
$$\times \prod_{i=1}^{r} \prod_{e \in E^{(i)}} P_{(i)}\left(d(e)\right) \ .$$

5.4 Equivariant Gromov–Witten invariants of K_S

We compute the $\mathbb{T}\times\mathbb{C}^*$-equivariant Euler class $e_{\mathbb{T}\times\mathbb{C}^*}(-R^\bullet\pi_*\mu^*K_S)$ on $\overline{\mathcal{M}}_G$. First, we introduce some notations.

Definition 5.12. (i) Let s_i^{can} be the degree of the canonical bundle $\mathcal{O}_S(-\sum_{j=1}^r C_j)$ of S pulled back to C_i. By adjunction formula, it is given by

$$s_i^{\mathrm{can}} = -s_i - 2.$$

(ii) Let ω_i^{can} be the $\mathbb{T}$-weight of the canonical bundle $\mathcal{O}_S(-\sum_{j=1}^r C_j)$ at p_i. It is given by

$$\omega_i^{\mathrm{can}} = -\omega_i^{\mathrm{in}} - \omega_i^{\mathrm{out}}.$$

(iii) For $d\in\mathbb{Z}_{\geq 1}$, we define

$$Q_{(i)}(d) := \begin{cases} \prod_{a=1}^{-ds_i^{\mathrm{can}}}\left(\frac{a}{d}\,\omega_i^{\mathrm{out}} + \omega_i^{\mathrm{can}} + \alpha_3\right) & \text{if } s_i^{\mathrm{can}} < 0, \\ \prod_{a=0}^{ds_i^{\mathrm{can}}}\left(-\frac{a}{d}\,\omega_i^{\mathrm{out}} + \omega_i^{\mathrm{out}} + \alpha_3\right)^{-1} & \text{if } s_i^{\mathrm{can}} \geq 0. \end{cases} \tag{5.8}$$

Lemma 5.13. *We have*

$$\begin{aligned} \iota_G^*\, e_{\mathbb{T}\times\mathbb{C}^*}(-R^\bullet\pi_*\mu^*K_S) = &\prod_{i=1}^r\Big\{\prod_{v\in V_3^{(i)}}\Lambda_{g(v)}^\vee(\omega_i^{\mathrm{can}})\cdot(\omega_i^{\mathrm{can}}+\alpha_3)^{\mathrm{val}(v)-1} \times \prod_{v\in V_2^{(i)}}(\omega_i^{\mathrm{can}}+\alpha_3)\Big\} \\ &\times\prod_{i=1}^r\prod_{e\in E^{(i)}} Q_{(i)}\left(d(e)\right)\ . \end{aligned} \tag{5.9}$$

Proof. Let $(f,C)\in\overline{\mathcal{M}}_G$. We have the following normalization sequence

$$0\to\mathcal{O}_C\to\bigoplus_{e\in E}\mathcal{O}_{C_e}\oplus\bigoplus_{v\in V_3}\mathcal{O}_{C_v}\to\bigoplus_{v\in V_2}\mathcal{O}_{p(v)}\oplus\bigoplus_{v\in V_3}\mathcal{O}_{p(v)}^{\oplus|\mathcal{F}_v|}\to 0,$$

where in the second term, we have denote by C_v the f-contracted component corresponding to $v\in V_3$. By twisting f^*K_S and taking cohomology, we have

$$\begin{aligned} 0&\to H^0(C,f^*K_S)\to\bigoplus_{e\in E}H^0(C_e,f^*K_S)\oplus\bigoplus_{v\in V_3}H^0(C_v,f^*K_S) \\ &\to\bigoplus_{v\in V_2}(K_S)_{p(v)}\oplus\bigoplus_{v\in V_3}(K_S)_{p(v)}^{\oplus|\mathcal{F}_v|}\to H^1(C,f^*K_S) \\ &\to\bigoplus_{e\in E}H^1(C_e,f^*K_S)\oplus\bigoplus_{v\in V_3}H^1(C_v,f^*K_S)\to 0. \end{aligned}$$

Therefore, as an element in K-group, we have

$$\begin{aligned} H^1(C,f^*K_S)-H^0(C,f^*K_S) = &\sum_{e\in E}H^1(C_e,f^*K_S)+\sum_{v\in V_3}H^1(C_v,f^*K_S) \\ &+\sum_{v\in V_2}(K_S)_{p(v)}+\sum_{v\in V_3}\mathrm{val}(v)(K_S)_{p(v)} \\ &-\sum_{e\in E}H^0(C_e,f^*K_S)-\sum_{v\in V_3}(K_S)_{p(v)}. \end{aligned} \tag{5.10}$$

For the second term in the RHS of (5.10), we have

$$H^1(C_v, f^*K_S) \cong H^1(C_v, \mathcal{O}_{C_v}) \otimes (K_S)_{p(v)} \cong H^0(C_v, \omega_{C_v})^\vee \otimes (K_S)_{p(v)}.$$

Other terms in the RHS of (5.10) are trivial vector bundles over $\overline{\mathcal{M}}_G$. Then (5.9) follows immediately. □

Here we collect some useful formulas:

$$\begin{aligned} \omega_i^{\mathrm{in}} &= -\omega_{i-1}^{\mathrm{out}}, && (5.11)\\ \omega_i^{\mathrm{out}} &= -\omega_{i-1}^{\mathrm{in}}, && (5.12)\\ \omega_i^{\mathrm{in}} - s_i\omega_i^{\mathrm{out}} &= \omega_{i-1}^{\mathrm{out}}. && (5.13)\end{aligned}$$

Lemma 5.14. *Let $Q'_{(i)}(d) := Q_{(i)}(d)|_{\alpha_3=0}$. Then we have*

$$\begin{aligned} P_{(i)}(d)\cdot Q'_{(i)}(d) = &\frac{d^{2d}}{(d!)^2}\frac{(-1)^{ds_i+1}}{(\omega_{i-1}^{\mathrm{out}})^d(\omega_i^{\mathrm{out}})^d}\\ &\times\prod_{a=1}^{d-1}\left(\frac{a}{d}\,\omega_i^{\mathrm{out}}+\omega_i^{\mathrm{in}}\right)\cdot\prod_{a=1}^{d-1}\left(\frac{a}{d}\,\omega_{i-1}^{\mathrm{in}}+\omega_{i-1}^{\mathrm{out}}\right)\ . \end{aligned} \tag{5.14}$$

Proof. We have three cases:

$$\begin{cases} \text{(i) } s_i\geq 0,\ s_i^{\mathrm{can}}<0 \iff s_i\geq 0,\\ \text{(ii) } s_i<0,\ s_i^{\mathrm{can}}<0 \iff s_i=-1,\\ \text{(iii) } s_i<0,\ s_i^{\mathrm{can}}\geq 0 \iff s_i\leq -2. \end{cases}$$

For the case (i), we have

$$Q'_{(i)}(d) = \underbrace{\prod_{a=1}^{d-1}\left(\frac{a}{d}\,\omega_i^{\mathrm{out}}+\omega_i^{\mathrm{can}}\right)}_{(a)}\times\underbrace{\prod_{a=d}^{s_id+d}\left(\frac{a}{d}\,\omega_i^{\mathrm{out}}+\omega_i^{\mathrm{can}}\right)}_{(b)}\times\underbrace{\prod_{a=s_id+1}^{s_id+2d-1}\left(\frac{a}{d}\,\omega_i^{\mathrm{out}}+\omega_i^{\mathrm{can}}\right)}_{(c)}\ .$$

We have

$$P_{(i)}(d)\times(b) = (-1)^{s_id+1}, \tag{5.15}$$

$$(a) = \prod_{a=1}^{d-1}\left(\frac{a-d}{d}\omega_i^{\mathrm{out}}-\omega_i^{\mathrm{in}}\right) = \prod_{a=1}^{d-1}\left(-\frac{a}{d}\omega_i^{\mathrm{out}}-\omega_i^{\mathrm{in}}\right) = (-1)^{d-1}\left(\frac{a}{d}\omega_i^{\mathrm{out}}+\omega_i^{\mathrm{in}}\right), \tag{5.16}$$

$$(c) = \prod_{a=1}^{d-1}\{(s_i+\frac{a}{d})\omega_i^{\mathrm{out}}-\omega_i^{\mathrm{in}}\} = \prod_{a=1}^{d-1}\{\frac{a}{d}\omega_i^{\mathrm{out}}+(s_i\omega_i^{\mathrm{out}}-\omega_i^{\mathrm{in}})\} = (-1)^{d-1}\prod_{a=1}^{d-1}\left(\frac{a}{d}\,\omega_{i-1}^{\mathrm{in}}+\omega_{i-1}^{\mathrm{out}}\right)\ . \tag{5.17}$$

In the third equality in (5.17) we have used (5.12, 5.13). By equations (5.15, 5.16, 5.17), we get the claim for the case (i). Proofs for other cases are similar. □

Let us further introduce notations.

Definition 5.15. (i) For μ^1, $\mu^2 \in \mathcal{P}$, we define

$$
\begin{aligned}
\Omega(\omega^1,\omega^2,\alpha\ ;\ \mu^1,\mu^2) &:= (\sqrt{-1})^{l(\mu^1)+l(\mu^2)} \\
&\times \prod_{i=1}^{l(\mu^1)} \frac{(\mu_i^1)^{\mu_i^1}}{\mu_i^1!} \prod_{a=1}^{\mu_i^1-1} \left(\frac{a}{\mu_i^1}\ \omega^2+\omega^1 \right) \times \prod_{i=1}^{l(\mu^2)} \frac{(\mu_i^2)^{\mu_i^2}}{\mu_i^2!} \prod_{a=1}^{\mu_i^2-1} \left(\frac{a}{\mu_i^2}\ \omega^1+\omega^2 \right) \\
&\times \left\{ \omega^1\omega^2\left(-\omega^1-\omega^2-\alpha\right) \right\}^{l(\mu^1)+l(\mu^2)-1} \times \frac{1}{(\omega^2)^{|\mu^1|}} \cdot \frac{1}{(\omega^1)^{|\mu^2|}}\ ,
\end{aligned} \tag{5.18}
$$

where ω^i and α are parameters.
(ii) For $\delta \in \mathbb{Z}$ and a partition μ, we define

$$
R_\delta(\omega^1,\omega^2,\alpha\ ;\ \mu) := \prod_{a=1}^{l(\mu)} \frac{R_\delta^0(\omega^1,\omega^2,\alpha\ ;\ \mu_i)}{R_\delta^0(\omega^1,\omega^2,0\ ;\ \mu_i)}, \tag{5.19}
$$

where

$$
R_\delta^0(\omega^1,\omega^2,\alpha\ ;\ d) := \begin{cases} \prod_{a=1}^{-d\delta} \left\{ \frac{a}{d}\ \omega^1+(-\omega^1-\omega^2+\alpha) \right\} & \text{if } \delta<0, \\ \prod_{a=0}^{d\delta} \left\{ -\frac{a}{d}\ \omega^1+(-\omega^1-\omega^2+\alpha) \right\}^{-1} & \text{if } \delta \geq 0, \end{cases}
$$

for $d \in \mathbb{Z}_{\geq 1}$.

Definition 5.16. Let $G=(V,E,g,p,d) \in G_{g,\beta}(S)$. For $v \in V^{(i)}$, we define partitions μ_v^{in} and μ_v^{out} as follows:

$$
\begin{aligned}
\mu_v^{\text{in}} &:= \left\{ d(e) \mid (v,e) \in \mathcal{F}_v,\ C_e=C_i \right\}, && (5.20)\\
\mu_v^{\text{out}} &:= \left\{ d(e) \mid (v,e) \in \mathcal{F}_v,\ C_e=C_{i+1} \right\}. && (5.21)
\end{aligned}
$$

For $i=1,\ldots,r$, we define partitions $\nu^{(i)}$ by

$$
\nu^{(i)} := \left\{ d(e) \mid e \in E^{(i)} \right\}\ . \tag{5.22}
$$

We have

$$
l(\mu_v^{\text{in}})+l(\mu_v^{\text{out}}) = \text{val}(v)\ , \tag{5.23}
$$

$$
\bigsqcup_{v \in V^{(i)}} \mu_v^{\text{in}} = \bigsqcup_{v \in V^{(i-1)}} \mu_v^{\text{out}} = \nu^{(i)}\ . \tag{5.24}
$$

Theorem 5.17. *We have*

$$
\begin{aligned}
N_{g,\beta}^{\mathbb{T}\times\mathbb{C}^*}(K_S) &= \sum_{G \in G_{g,\beta}(S)} \frac{1}{|\text{Aut(G)}|} \cdot \frac{1}{\prod_{i=1}^r \prod_{j=1}^{l(\nu^{(i)})} \nu_j^{(i)}} \\
&\times \prod_{i=1}^r R_{-2-s_i}(\omega_i^{\text{in}},\omega_i^{\text{out}},\alpha_3\ ;\ \nu^{(i)}) \cdot (-1)^{|\nu^{(i)}|s_i} \\
&\times \prod_{i=1}^r \Big\{ \prod_{v \in V^{(i)}} \Omega(\omega_i^{\text{in}},\omega_i^{\text{out}},\alpha_3\ ;\ \mu_v^{\text{in}},\mu_v^{\text{out}}) \\
&\quad \cdot \int_{\overline{\mathcal{M}}_{g(v),\ l(\mu_v^{\text{in}})+l(\mu_v^{\text{out}})}} \frac{\Lambda_{g(v)}^\vee(\omega_i^{\text{in}})\ \Lambda_{g(v)}^\vee(\omega_i^{\text{out}})\ \Lambda_{g(v)}^\vee(-\omega_i^{\text{in}}-\omega_i^{\text{out}}+\alpha_3)}{\prod_{j=1}^{l(\mu_v^{\text{in}})} \left(\frac{\omega_i^{\text{out}}}{(\mu_v^{\text{in}})_j}-\psi_j \right) \prod_{j=1}^{l(\mu_v^{\text{out}})} \left(\frac{\omega_i^{\text{in}}}{(\mu_v^{\text{out}})_j}-\psi_{j+l(\mu_v^{\text{out}})} \right)} \Big\}
\end{aligned} \tag{5.25}
$$

Proof. Note that the first line of the RHS comes from $|A_G|$, the second line is the contributions from edges and the last two lines are the contributions from vertices. We deal with the the signs coming form Lemma 5.14, as follows.

$$\begin{aligned}\prod_{i=1}^{r}\prod_{e\in E^{(i)}}(-1)^{d(e)s_i+1} &= \prod_{i=1}^{r}\prod_{e\in E^{(i)}}(-1)^{d(e)s_i}\times\prod_{i=1}^{r}\prod_{e\in E^{(i)}}(-1)\\ &= \prod_{i=1}^{r}(-1)^{|\nu^{(i)}|s_i}\times\underbrace{\prod_{i=1}^{r}\prod_{v\in V^{(i)}}\left(\sqrt{-1}\right)^{l(\mu_v^{\mathrm{in}})+l(\mu_v^{\mathrm{out}})}}_{(*)}\end{aligned} \tag{5.26}$$

We insert the term $(*)$ into Ω. The rest of the verification is straightforward. □

From (5.25), we can see that the non-equivariant limit

$$\lim_{\alpha_3\to 0} N_{g,\beta}^{\mathbb{T}\times\mathbb{C}^*}(K_S) = N_{g,\beta}(K_S)$$

exists.

5.5 Gromov–Witten invariants of K_S

In this section, we complete the proof of Theorem 4.6. Let us introduce some notations on graphs.

Definition 5.18. (i) We set $G_\beta(S) := \bigsqcup_{g\geq 0} G_{g,\beta}(S)$. Let $G_\beta^\bullet(S)$ be the disconnected version of $G_\beta(S)$ (i.e. graphs may be disconnected).
(ii) For $\vec{\eta} = (\eta_1, \ldots, \eta_r) \in \mathcal{P}^r$, we set

$$G_{\beta,\vec{\eta}}(S) := \left\{G \in G_\beta(S) \mid \nu^{(i)} = \eta^i\right\}, \quad G_{\beta,\vec{\eta}}^\bullet(S) := \left\{G \in G_\beta^\bullet(S) \mid \nu^{(i)} = \eta^i\right\}.$$

(iii) For $G \in G_\beta(S)$ or $\mathrm{G}_{\beta,\vec{\eta}}^\bullet(\mathrm{S})$, we define

$$\chi(G) := -2\left(\sum_{v\in V} g(v) + |E| - |V|\right).$$

Definition 5.19. (i) Given a decorated graph $G = (V, E, g, p, d) \in G_\beta(S)$, the decorated graph $\mathbb{G} = (V, E, p, d)$ obtained from G by ignoring the genus label g is called *the type of* G. We denote by $\mathbb{G}_\beta(S)$ the set of types of decorated graphs in $G_\beta(S)$. Its disconnected version is denoted by $\mathbb{G}_\beta^\bullet(S)$. We have natural maps

$$\psi : G_\beta(S) \to \mathbb{G}_\beta(S), \quad G_\beta^\bullet(S) \to \mathbb{G}_\beta^\bullet(S).$$

(ii) For $\vec{\eta} \in \mathcal{P}^r$ we define $\mathbb{G}_{\beta,\vec{\eta}}(S)$ and $\mathbb{G}_{\beta,\vec{\eta}}^\bullet(S)$ as in Definition 5.18 (ii).

By Theorem 5.17, we have

$$N_{g,\beta}(K_S) = \sum_{G\in G_{g,\beta}(S)} F(G), \tag{5.27}$$

here $F(G)$ is defined by

$$\begin{aligned} F(G) :=& \frac{1}{|\mathrm{Aut(G)}|}\cdot\frac{1}{\prod_{i=1}^{r}\prod_{j=1}^{l(\nu^{(i)})}\nu_j^{(i)}}\\ &\times\prod_{i=1}^{r}(-1)^{|\nu^{(i)}|s_i}\times\prod_{i=1}^{r}\prod_{v\in V^{(i)}} z_{\mu_v^{\mathrm{in}}}\cdot z_{\mu_v^{\mathrm{out}}}\cdot G_{g(v),\mu_v^{\mathrm{in}},\mu_v^{\mathrm{out}}}\left(\frac{\omega_i^{\mathrm{in}}}{\omega_i^{\mathrm{out}}}\right), \end{aligned} \tag{5.28}$$

where $G_{g,\mu^+,\mu^-}(\tau)$ has been defined by (2.18).

Lemma 5.20. *For $\mathbb{G} \in \mathbb{G}_{\beta,\vec{\eta}}$, we have*

$$\sum_{G \in \psi^{-1}(\mathbb{G})} F(G)\, \lambda^{-\chi(G)} = H(\mathbb{G}), \tag{5.29}$$

here $H(\mathbb{G})$ is given by

$$\begin{aligned} H(\mathbb{G}) :=& \frac{1}{|\mathrm{Aut}(\mathbb{G})|} \cdot \frac{1}{\prod_{i=1}^{r} \prod_{j=1}^{l(\eta^i)} \eta_j^i} \\ &\times \prod_{i=1}^{r} (-1)^{|\eta^i| s_i} \times \prod_{i=1}^{r} \prod_{v \in V^{(i)}} z_{\mu_v^{\mathrm{in}}} \cdot z_{\mu_v^{\mathrm{out}}} \cdot G_{\mu_v^{\mathrm{in}},\, \mu_v^{\mathrm{out}}} \left(\lambda; \frac{\omega_i^{\mathrm{in}}}{\omega_i^{\mathrm{out}}} \right) \ , \end{aligned} \tag{5.30}$$

where $G_{\mu^+,\mu^-}(\lambda;\tau)$ has been define by (2.19).

Proof. By expanding the RHS of (5.29), we have a sum over all decorated graphs[8] which have the type $\mathbb{G}$. For each $G \in \psi^{-1}(\mathbb{G})$, there are exactly $|\mathrm{Aut}(\mathbb{G})|/|\mathrm{Aut}(G)|$ representatives of G. This completes the proof. □

Let

$$F_{K_S}(\lambda)_\beta := \sum_{g \geq 0} N_{g,\beta}(K_S) \lambda^{2g-2}.$$

Then by Lemma 5.20, we have

$$F_{K_S}(\lambda)_\beta = \sum_{\vec{\eta} \in \mathcal{P}^r} \sum_{\mathbb{G} \in \mathbb{G}_{\beta,\vec{\eta}}} H(\mathbb{G}) \ .$$

We have

$$\begin{aligned} F_{K_S}(\lambda, Q) &= \sum_{\substack{\beta \in H_2(S,\mathbb{Z}) \\ \beta \neq 0}} F_{K_S}(\lambda)_\beta Q^\beta \\ &= \sum_{\substack{\beta \in H_2(S,\mathbb{Z}) \\ \beta \neq 0}} \Big(\sum_{\vec{\eta} \in \mathcal{P}^r} \sum_{\mathbb{G} \in \mathbb{G}_{\beta,\vec{\eta}}} H(\mathbb{G}) \Big) Q^\beta \\ &= \sum_{\substack{\vec{\eta} \in \mathcal{P}^r \\ \eta \neq \vec{\emptyset}}} \sum_{\mathbb{G} \in \mathbb{G}_{\beta,\vec{\eta}}} H(\mathbb{G}) Q^\beta \ , \end{aligned} \tag{5.31}$$

where $\vec{\emptyset} = (\emptyset, \ldots, \emptyset)$. If we set $t_i := Q^{[C_i]}$ $(1 \leq i \leq r)$, we have

$$F_{K_S}(\lambda, t_1, \ldots, t_r) = \sum_{\substack{\vec{\eta} \in \mathcal{P}^r \\ \eta \neq \vec{\emptyset}}} \sum_{\mathbb{G} \in \mathbb{G}_{\beta,\vec{\eta}}} H(\mathbb{G}) \prod_{i=1}^{r} t_i^{|\eta^i|} \ . \tag{5.32}$$

Now we take exponentials of the both sides of (5.32). As is well-known, we obtain a sum over disconnected graphs (see e.g. [48, Appendix A]):

$$Z_{K_S}(\lambda, t_1, \ldots, t_r) = 1 + \sum_{\substack{\vec{\eta} \in \mathcal{P}^r \\ \eta \neq \vec{\emptyset}}} \sum_{\mathbb{G} \in \mathbb{G}^{\bullet}_{\beta,\vec{\eta}}} H(\mathbb{G}) \prod_{i=1}^{r} t_i^{|\eta^i|} \ . \tag{5.33}$$

[8] Note that this is *not* a sum over isomorphism classes of decorated graphs.

Lemma 5.21.

$$\sum_{\mathbb{G}\in\mathbb{G}^\bullet_{\beta,\vec{\eta}}(S)} H(\mathbb{G}) = \frac{1}{\prod_{i=1}^r |\mathrm{Aut}(\eta^i)| \prod_{j=1}^{l(\eta^i)} \eta^i_j} \times \prod_{i=1}^r (-1)^{|\eta^i| s_i} \times \prod_{i=1}^r z_{\eta^i} \cdot z_{\eta^{i+1}} \cdot G^\bullet_{\eta^i,\,\eta^{i+1}} \left(\lambda; \frac{\omega_i^{\mathrm{in}}}{\omega_i^{\mathrm{out}}}\right) . \tag{5.34}$$

Proof. This can be shown by expanding the RHS using the definition (2.20) of $G^\bullet$ and (5.24). □

Proof. (of Themrem 4.6.) By Lemma 5.21, we have

$$Z_{K_S}(\lambda, t_1, \ldots, t_r) = 1 + \sum_{\substack{\vec{\eta}\in\mathcal{P}^r \\ \eta\neq\vec{\emptyset}}} \prod_{i=1}^r z_{\eta^i} \times \prod_{i=1}^r \big((-1)^{s_i} t_i\big)^{|\eta^i|} \times \prod_{i=1}^r G^\bullet_{\eta^i,\,\eta^{i+1}} \left(\lambda; \frac{\omega_i^{\mathrm{in}}}{\omega_i^{\mathrm{out}}}\right) . \tag{5.35}$$

Now we apply Theorem 2.6. We have

$$\begin{aligned} Z_{K_S} =& 1 + \sum_{\substack{\vec{\eta}\in\mathcal{P}^r \\ \eta\neq\vec{\emptyset}}} \prod_{i=1}^r z_{\eta^i} \times \prod_{i=1}^r \big((-1)^{s_i} t_i\big)^{|\eta^i|} \\ & \times \prod_{i=1}^r \sum_{\substack{\mu^i,\,\mu^i_+ \\ |\mu^i|=|\eta^i| \\ |\mu^i_+|=|\eta^{i+1}|}} \frac{\chi_{\mu^i}(C_{\eta^i})}{z_{\eta^i}} \frac{\chi_{\mu^i_+}(C_{\eta^{i+1}})}{z_{\eta^{i+1}}} e^{\sqrt{-1}\left(\kappa(\mu^i)\frac{\omega_i^{\mathrm{in}}}{\omega_i^{\mathrm{out}}} + \kappa(\mu^i_+)\frac{\omega_i^{\mathrm{out}}}{\omega_i^{\mathrm{in}}}\right)\lambda/2} W_{\mu^i,\,\mu^i_+}(e^{\sqrt{-1}\lambda}) . \end{aligned} \tag{5.36}$$

By taking sum over η^i and using orthogonality of characters (2.15) for each i, we can see that only terms which satisfy

$$\mu^i_+ = \mu^{i+1} \quad \text{for all} \quad i \in \mathbb{Z}/r$$

survive. Hence we have

$$\begin{aligned} Z_{K_S} &= \sum_{(\mu^1,\ldots,\mu^r)\in\mathcal{P}^r} \prod_{i=1}^r \big((-1)^{s_i} t_i\big)^{|\mu^i|} \cdot e^{\sqrt{-1}\left(\kappa(\mu^i)\frac{\omega_i^{\mathrm{in}}}{\omega_i^{\mathrm{out}}} + \kappa(\mu^{i+1})\frac{\omega_i^{\mathrm{out}}}{\omega_i^{\mathrm{in}}}\right)\lambda/2} \cdot W_{\mu^i,\,\mu^{i+1}}(e^{\sqrt{-1}\lambda}) \\ &= \sum_{(\mu^1,\ldots,\mu^r)\in\mathcal{P}^r} \prod_{i=1}^r \big((-1)^{s_i} t_i\big)^{|\mu^i|} \cdot e^{\sqrt{-1}\kappa(\mu^i)\left(\frac{\omega_i^{\mathrm{in}}}{\omega_i^{\mathrm{out}}} + \frac{\omega_{i-1}^{\mathrm{out}}}{\omega_{i-1}^{\mathrm{in}}}\right)\lambda/2} \cdot W_{\mu^i,\,\mu^{i+1}}(e^{\sqrt{-1}\lambda}) \\ &= \sum_{(\mu^1,\ldots,\mu^r)\in\mathcal{P}^r} \prod_{i=1}^r \big((-1)^{s_i} t_i\big)^{|\mu^i|} \cdot e^{\sqrt{-1}\kappa(\mu^i) s_i \lambda/2} \cdot W_{\mu^i,\,\mu^{i+1}}(e^{\sqrt{-1}\lambda}) . \end{aligned}$$

In the last equality, we have used (5.12, 5.13). □

6 Local Gromov–Witten invariants of cubic surfaces

In this section, we give a proof of Theorem 1.4 (Theorem 6.8). In §6.1, an example which illustrate the argument is given. We summarize necessary facts on cubic surfaces S_3 in §6.2. In §6.3, the $-K$-nef toric surface S_3^0 is introduced. (A proof of the deformation equivalence of S_3 and S_3^0 is given in §6.6.) Formulas for the generating functions of local Gromov–Witten invariants of S_3^0 and S_3 are obtained in §6.4. Numerical results are presented in §6.4.3. In §6.5, local Gopakumar–Vafa invariants of S_3 are studied.

6.1 Example of $-K$-nef toric degeneration

We illustrate our arguments in §6 with an example. Consider the Hirzebruch surface $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$ $(n \geq 0)$. We denote the homology class of the canonical section of $\mathbb{F}_n \to \mathbb{P}^1$ with self-intersection number $-n$ by B_n and that of the fiber by F. It is well-known that $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ is deformation equivalent to $\mathbb{F}_2$. Note that $\mathbb{F}_2$ is a $-K$-nef toric surface but not a del Pezzo surface. We call $\mathbb{F}_2$ is a $-K$-nef toric degeneration of $\mathbb{F}_0$.

There is a one parameter deformation $\pi : \mathcal{X} \to T$ of $\mathbb{F}_2$ such that $X_0 = \pi^{-1}(t_0) \cong \mathbb{F}_2$ and $X_t = \pi^{-1}(t) \cong \mathbb{F}_0$ if $t \neq t_0$, where t_0 is some fixed base point. See e.g. [68, Ch.1, Theorem 4.2] for an explicit description of the deformation space. The relative Picard group of $\pi : \mathcal{X} \to T$ is a trivial rank 2 local system, where the class F on X_t deforms to the class F on X_0, and the class B_0 on X_t deforms to $B_2 + F$ on X_0. By Lemma 4.4 and Proposition 3.7, we have

$$N_{g,aB_0+bF}(K_{\mathbb{F}_0}) = I_{g,aB_0+bF}(\mathbb{F}_0) = I_{g,aB_2+(a+b)F}(\mathbb{F}_2) = N_{g,aB_2+(a+b)F}(K_{\mathbb{F}_2}), \qquad (6.1)$$

for $a \geq 0$, $b \geq 0$, $(a,b) \neq (0,0)$. Note that

- $N_{g,aB_0+bF}(K_{\mathbb{F}_0}) = 0$ unless $a \geq 0$, $b \geq 0$.
- $N_{g,cB_2+dF}(K_{\mathbb{F}_2}) = 0$ unless $c \geq 0$, $d \geq 0$.
- Therefore, the difference between them only appears at a multiple of $[B_2]$ which does not corresponds to an effective class on $\mathbb{F}_0$.

The first two statements follow from the virtual localization formula (2.13). Now we translate these into an identity between partition functions $Z_{K_{\mathbb{F}_0}}(\lambda, Q_{B_0}, Q_F)$ and $Z_{K_{\mathbb{F}_2}}(\lambda, Q_{B_2}, Q_F)$, where $Q_{B_0} = Q^{[B_0]}$, $Q_{B_2} = Q^{[B_2]}$ and $Q_F = Q^{[F]}$.

Proposition 6.1. *As an identity between formal power series in two variables Q_1, Q_2, we have*

$$Z_{K_{\mathbb{F}_0}}(\lambda, Q_1Q_2, Q_2) = \frac{Z_{K_{\mathbb{F}_2}}(\lambda, Q_1, Q_2)}{Z_{K_{\mathbb{F}_2}}(\lambda, Q_1, 0)}. \qquad (6.2)$$

Proof. The denominator of the RHS of (6.2) subtracts the contributions from multiples of $[B_2]$. The substitution of the variable in the LHS is determined by the identification $Q_{B_2} = Q_{B_0}Q_F^{-1}$. □

Note that the identity (6.2) is a nontrivial combinatorial identity by Theorem 4.6. Our strategy is to apply the same arguments to a cubic surface S_3 (which is *non-toric*) and its $-K$-nef toric degeneration S_3^0. This time, we shall obtain a closed formula for the partition function of local Gromov–Witten invariants of S_3 from that of S_3^0.

Remark 6.2. (i) In general, $\mathbb{F}_n$ and $\mathbb{F}_m$ are deformation equivalent if and only if $n \equiv m \mod 2$. However, there is no relation such as (6.1) between Gromov–Witten invariants of $K_{\mathbb{F}_n}$ and $K_{\mathbb{F}_m}$ if one of which is not a $-K$-nef surface. Therefore Gromov–Witten invariants of local toric surfaces are not deformation invariant. Hence $-K$-nef condition is essential for our argument. We cannot apply our method to S_1 and S_2, del Pezzo surfaces of degrees 1 and 2, since they do not admits $-K$-nef toric degenerations.
(ii) In [16], another approach to local Gromov–Witten invariants of non-toric del Pezzo surfaces based on the ruled vertex formula [17]. Although it is still conjectural, it may cover all del Pezzo surfaces including S_1 and S_2.
(iii) The $-K$-nef toric degeneration S_3^0 of S_3 has been used in [79] to study quantum cohomology of S_3.

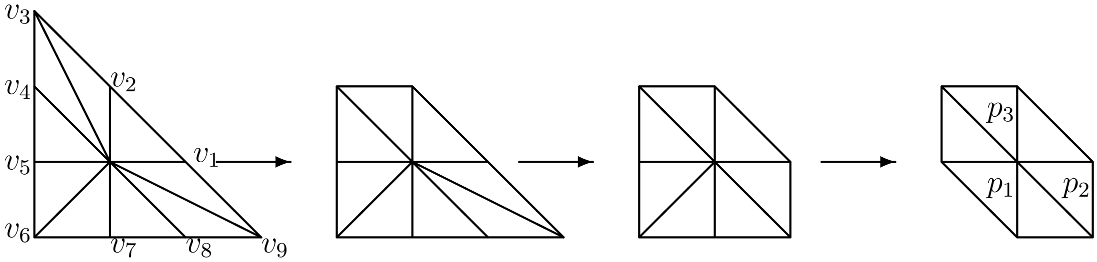


Figure 2: $S_3^0 \to S_4^0 \to S_5^0 \to S_6$.

6.2 Cubic surfaces S_3

Here we summarize basic facts on cubic surfaces, see e.g. [31, Ch. V.4] for details.

Let $S_3 \subset \mathbb{P}^3$ be a nonsingular cubic surface. S_3 is realized as a blowing up $\pi : S_3 \to \mathbb{P}^2$ at six points in a general position. Let $e_1, \cdots, e_6$ be the classes of the exceptional curves of π and l be the class of a line in $\mathbb{P}^2$ pulled back by π. Then $l, e_1, \cdots, e_6$ is a basis of $\mathrm{Pic}(S_3)$. Their intersections are

$$l^2 = 1, \quad e_i^2 = -1, \quad l.e_i = 0, \quad e_i.e_j = 0 \text{ if } i \neq j.$$

Let h be the class of the hyperplane section of $\mathbb{P}^3$. Then we have

$$h = -K_{S_3} = 3l - \sum_{i=1}^{6} e_i \ .$$

It is a classical fact that S_3 contains exactly twenty-seven lines which are given as follows:

$$e_i \ (i = 1, \cdots, 6), \quad l - e_i - e_j \ (1 \leq i < j \leq 6), \quad 2l - \sum_{i \neq j} e_i \ \ (j = 1, \cdots, 6).$$

Each one of these is an exceptional curve of the first kind. These twenty-seven lines are the minimal generators of the Mori cone (the cone generated by effective 1-cycles modulo numerical equivalence) (cf. [67, (0.6)]).

It is well-known that the Weyl group W_{E_6} of type E_6 acts on $\mathrm{Pic}(S_3)$ as symmetries of configurations of twenty seven lines. Its generators are given as follows.

$$\begin{aligned} s_i :\ & e_i \leftrightarrow e_{i+1} \ \ (1 \leq i \leq 5), \\ s_6 :\ & e_1 \mapsto l - e_2 - e_3, \quad e_2 \mapsto l - e_1 - e_3, \\ & e_3 \mapsto l - e_1 - e_2, \quad l \mapsto 2l - e_1 - e_2 - e_3. \end{aligned}$$

It is known [19, §4] that W_{E_6} coincides with the group of automorphisms of $\mathrm{Pic}(S_3)$ which preserve the intersection form, the canonical class, and the semigroup of effective classes. Hereafter we identify $\mathrm{Pic}(S_3)$ with $H^2(S_3, \mathbb{Z}) \cong H_2(S_3, \mathbb{Z})$. We have the following

Lemma 6.3. *$I_{g,\beta}(S_3) = I_{g,w(\beta)}(S_3)$ for $w \in W_{E_6}$.*

Proof. See [34, §2.4]. □

6.3 $-K$-nef toric degenerations of S_3, S_4, and S_5

Let S_3^0, S_4^0, and S_5^0 be the $-K$-nef toric surfaces whose fans are given in Figure 2. Here the nine one-dimensional cones of S_3^0 are generated by

$$\begin{gathered} v_1 = (1, 0), \quad v_2 = (0, 1), \quad v_3 = (-1, 2), \quad v_4 = (-1, 1), \quad v_5 = (-1, 0), \\ v_6 = (-1, -1), \quad v_7 = (0, -1), \quad v_8 = (1, -1), \quad v_9 = (2, -1). \end{gathered}$$

Let the fan of the toric del Pezzo surface S_6 be given in Figure 2 and let p_1, p_2, p_3 be the torus fixed points of S_6 corresponding to the two-dimensional cones generated by $(v_5, v_7), (v_8, v_1), (v_2, v_4)$. S_3^0 (resp. S_4^0, S_5^0) is obtained by blowing up S_6 at p_1, p_2, p_3 (resp. p_1, p_2 and p_1). S_k^0 contains (-2)-curves and its anticanonincal divisor is nef but not ample.

Proposition 6.4. *S_k^0 $(k = 3, 4, 5)$ is deformation equivalent to S_k.*

A proof will be given in §6.6 (see Proposition 6.13).

Now let us explain the geometry of the $-K$-nef toric surface S_3^0. The torus-invariant divisors C_i $(1 \le i \le 9)$ corresponding to v_i have the intersections:

$$C_i.C_{i+1} = 1, \quad C_i.C_j = 0 \ (j \neq i, i \pm 1), \quad C_i^2 = \begin{cases} -1 & (i = 3, 6, 9) \ , \\ -2 & (i = 1, 2, 4, 5, 7, 8) \ , \end{cases} \tag{6.3}$$

and the canonical divisor $K_{S_3^0}$ is rationally equivalent to $-C_1 - \cdots - C_9$. The Mori cone of S_3^0 is generated by $C_1, \ldots, C_9$ [73, Proposition 2.26].

Note that $\mathrm{Pic}(S_3^0) \cong \mathrm{Pic}(S_3)$ and an isomorphism is given by the following.

$$\begin{aligned} &C_1 \mapsto e_2 - e_5, && C_2 \mapsto l - e_2 - e_3 - e_6, && C_3 \mapsto e_6, \\ &C_4 \mapsto e_3 - e_6, && C_5 \mapsto l - e_1 - e_3 - e_4, && C_6 \mapsto e_4, \\ &C_7 \mapsto e_1 - e_4, && C_8 \mapsto l - e_1 - e_2 - e_5, && C_9 \mapsto e_5. \end{aligned} \tag{6.4}$$

This is explained as follows. First, in S_6, we regard the torus-invariant divisors C'_1, C'_4, C'_7 corresponding to v_1, v_4, v_7 as the exceptional curves of blowing up of $\mathbb{P}^2$ and identify them with e_2, e_3, e_1. The torus-invariant divisors C'_2, C'_5, C'_8 corresponding to v_2, v_5, v_8 are identified with the proper transforms $l - e_2 - e_3, l - e_1 - e_3, l - e_1 - e_2$ of lines in $\mathbb{P}^2$. Then in S_3^0, C_3, C_6, C_9 are exceptional curves of the blowup at p_3, p_1, p_2 and we identify them with e_6, e_4, e_5. For $i = 1, 2, 4, 5, 7, 8$, C_i is the proper transform of C'_i. (This identification can be seen from the construction of a deformation in the proof of Proposition 6.13.)

From here on, we identify $\mathrm{Pic}(\mathrm{S}_3^0)$ with $H^2(S_3^0, \mathbb{Z}) \cong H_2(S_3^0, \mathbb{Z})$.

Corollary 6.5. *For $g \in \mathbb{Z}_{\ge 0}$ and $\beta \in H_2(S_3, \mathbb{Z})$ such that $K_{S_3}.\beta < 0$,*

$$I_{g,\beta}(S_3) = I_{g,\beta'}(S_3^0) \ ,$$

where $\beta' \in H_2(S_3^0, \mathbb{Z})$ is the class corresponding to β by eq. (6.4).

Proof. This follows from Propositions 3.7 and 6.4. □

Remark 6.6. The statements similar to Theorem 6.5 hold for S_4, S_5: local Gromov–Witten invariants of S_4 and S_5 are the same as those of S_4^0 and S_5^0 and the latter is obtained from those of S_3^0 (see Corollary 9.7). Their generating functions also have expressions analogous to the formula for S_3 (which will be stated in Theorem 6.8).

6.4 Generating function of local Gromov–Witten invariants

6.4.1 Formula for S_3^0

First we consider the generating function of local Gromov–Witten invariants of S_3^0 with $\beta \in H_2(S_3^0, \mathbb{Z})$ such that $K_{S_3^0}.\beta < 0$. Take a basis $c_1, \ldots, c_7$ of $H_2(S_3^0, \mathbb{Z})$ and let $X_1, \ldots, X_7$ be associated formal variables. For $\beta = a_1c_1 + \cdots + a_7c_7 \in H_2(S_3^0, \mathbb{Z})$, denote $X_1^{a_1} \dots X_7^{a_7}$ by X^β. We define the generating function as

$$F^{\mathrm{loc}}_{S_3^0}(\lambda, X) := \sum_{\substack{\beta \in H_2(S_3^0,\mathbb{Z}),\\ K_{S_3^0}.\beta<0}} \sum_{g\geq 0} I_{g,\beta}(S_3^0)\lambda^{2g-2}X^\beta,$$

and its partition function

$$Z^{\mathrm{loc}}_{S_3^0}(\lambda, X) := \exp F^{\mathrm{loc}}_{S_3^0}(\lambda, X)$$

Second we consider the partition function of Gromov–Witten invariants of the local toric surface $Z_{K_{S_3^0}}$. By Theorem 4.6, we have the following formula:

$$Z_{K_{S_3^0}}(\lambda, t_1, \cdots, t_9) = \prod_{i=1}^{9} \sum_{\nu^i} ((-1)^{s_i} t_i)^{|\nu^i|} e^{\sqrt{-1}\lambda s_i \frac{\kappa(\nu^i)}{2}} W_{\nu^i,\nu^{i+1}}(e^{\sqrt{-1}\lambda}), \tag{6.5}$$

where $t_i = X^{[C_i]}$ $(1 \leq i \leq 9)$ and $s_i = C_i^2$ (See (6.3)), and each ν^i $(1 \leq i \leq 9)$ runs over the set of partitions ($\nu^{10} = \nu^1$ is assumed). Define $Z_{(-2,0)}(t)$ by

$$Z_{(-2,0)}(\lambda, t) := \exp\left[-\sum_{j\geq 1} \frac{1}{j} \left(2 \sin \frac{j\lambda}{2}\right)^{-2} t^j \right].$$

Lemma 6.7. *Under $t_i = X^{[C_i]}$ $(1 \leq i \leq 9)$, we have*

$$Z^{\mathrm{loc}}_{S_3^0}(\lambda, X) = \frac{Z_{K_{S_3^0}}(\lambda, t_1, \cdots, t_9)}{\prod_{i=1,4,7} Z_{(-2,0)}(t_i) Z_{(-2,0)}(t_{i+1}) Z_{(-2,0)}(t_i t_{i+1})}.$$

Proof. By Lemma 4.4, we have $I_{g,\beta}(S_3^0) = N_{g,\beta}(K_{S_3^0})$ for β such that $K_{S_3^0}.\beta < 0$. Note also that $N_{g,\beta}(K_{S_3^0}) = 0$ if there is no effective divisors of the form $\sum_{1\leq i\leq 9} a_i[C_i]$ $(a_i \in \mathbb{Z}_{\geq 0})$ which are rationally equivalent to β because $\overline{\mathcal{M}}_{g,0}(S_3^0, \beta)^{\mathbb{T}}$ is empty.

We subtract the contributions coming from classes β which does not satisfy $K_{S_3^0}.\beta < 0$ from $Z_{K_{S_3^0}}$. By the above remark, such effective classes are of the forms $a[C_1] + b[C_2]$, $a[C_4] + b[C_5]$ or $a[C_7] + b[C_8]$ $(a, b \in \mathbb{Z}_{\geq 0})$. Therefore

$$\exp\left[\sum_{\substack{\beta \in H_2(S_3^0,\mathbb{Z})\\ K_{S_3^0}.\beta\geq 0}} \sum_{g\geq 0} N_{g,\beta}(K_{S_3^0})\lambda^{2g-2}X^\beta \right] = \prod_{i=1,4,7} \exp\left[\sum_{a,b\in\mathbb{Z}_{\geq 0}} \sum_{g\geq 0} N_{g,a[C_i]+b[C_{i+1}]}(K_{S_3^0})\lambda^{2g-2} t_i^a t_{i+1}^b \right]. \tag{6.6}$$

The $i = 1$ factor is easily obtained by setting $t_3 = t_4 = \cdots = t_9 = 0$ in (6.5). It is equal to

$$Z_{K_{S_3^0}}(\lambda, t_1, \cdots, t_9)|_{t_3=t_4=\cdots=t_9=0} = Z_{(-2,0)}(\lambda, t_1) Z_{(-2,0)}(\lambda, t_2) Z_{(-2,0)}(\lambda, t_1t_2).$$

The $i = 4, 7$ factors are similar. Dividing (6.5) by (6.6), we obtain

$$\exp\left[\sum_{\substack{\beta \in H_2(S_3^0,\mathbb{Z}),\\ K_{S_3^0}.\beta<0}} \sum_{g\geq 0} N_{g,\beta}(K_{S_3^0})\lambda^{2g-2}X^\beta \right] = \frac{Z_{K_{S_3^0}}(\lambda, t_1, \cdots, t_9)}{\prod_{i=1,4,7} Z_{(-2,0)}(\lambda, t_i) Z_{(-2,0)}(\lambda, t_{i+1}) Z_{(-2,0)}(\lambda, t_i t_{i+1})}.$$

Thus we complete our proof. $\square$

6.4.2 Formula for S_3

Next we study the generating function of local Gromov–Witten invariants of S_3. Let $Q = (Q_1, \ldots, Q_6, Q_7)$ be a set of formal variables and denote $Q_1^{a_1} Q_2^{a_2} \ldots Q_7^{a_7}$ by Q^β for $\beta = a_1 e_1 + \cdots + a_6 e_6 + a_7 l \in H_2(S_3, \mathbb{Z})$. Define

$$F_d^{\mathrm{loc}}(\lambda, Q) := \sum_{\substack{\beta \in H_2(S_3,\mathbb{Z}),\\ -K_{S_3}.\beta = d}} \sum_{g \in \mathbb{Z}_{\geq 0}} I_{g,\beta}(S_3) \lambda^{2g-2} Q^\beta \ , \qquad (d \in \mathbb{Z}_{\geq 1}),$$

$$F_{S_3}^{\mathrm{loc}}(\lambda, Q) := \sum_{d \geq 1} F_d^{\mathrm{loc}}(\lambda, Q),$$

$$Z_{S_3}^{\mathrm{loc}}(\lambda, Q) := \exp F_{S_3}^{\mathrm{loc}}(\lambda, Q).$$

The following is the main result of the part II.

Theorem 6.8. *With the following identification of the parameters*

$$\begin{aligned} &t_1 = Q^{e_2 - e_5}, \quad t_2 = Q^{l - e_2 - e_3 - e_6}, \quad t_3 = Q^{e_6}, \quad t_4 = Q^{e_3 - e_6}, \quad t_5 = Q^{l - e_1 - e_3 - e_4}, \\ &t_6 = Q^{e_4}, \quad t_7 = Q^{e_1 - e_4}, \quad t_8 = Q^{l - e_1 - e_2 - e_5}, \quad t_9 = Q^{e_5}, \end{aligned} \tag{6.7}$$

we have

$$Z_{S_3}^{\mathrm{loc}}(\lambda, Q) = \frac{Z_{K_{S_3^0}}(\lambda, t_1, \cdots, t_9)}{\prod_{i=1,4,7} Z_{(-2,0)}(\lambda, t_i) Z_{(-2,0)}(\lambda, t_{i+1}) Z_{(-2,0)}(\lambda, t_i t_{i+1})}. \tag{6.8}$$

Proof. The identification (6.7) is determined by (6.4). Under this identifications of parameters, we have

$$Z_{S_3}^{\mathrm{loc}}(\lambda, Q) = Z_{S_3^0}^{\mathrm{loc}}(\lambda, t_1, \cdots, t_9),$$

by Corollary 6.5. Then (6.8) follows from Lemma 6.7. □

Remark 6.9. In [18], Diaconescu and Florea obtained a formula for F_{S_3} which is different from ours (eq.(3.14) for $k = 5$ in *loc.cit.*). It would be an interesting problem to show that these two formulas are equivalent.

6.4.3 Numerical results up to degree 6

Define the orbit sum $m(\beta)$ for $\beta \in H_2(S_3, \mathbb{Z})$ by

$$m(\beta) := \frac{1}{\#\{w \in W_{E_6} \mid w(\beta) = \beta\}} \sum_{w \in W_{E_6}} Q^{w(\beta)} \ .$$

By Lemma 6.3, $F_d^{\mathrm{loc}}(\lambda, Q)$ can be expanded in terms of these. Here we list $F_d^{\mathrm{loc}}(\lambda, Q)$ up to $d = 6$. Let $\mathsf{b}[k] := (2 \sin \frac{k\lambda}{2})^2$.

$$F_1^{\mathrm{loc}} = \frac{1}{\mathsf{b}[1]} m(e_6) \ , \qquad F_2^{\mathrm{loc}} = \frac{1}{2 \cdot \mathsf{b}[2]} m(2e_6) + \frac{-2}{\mathsf{b}[1]} m(-e_1 + l) \ ,$$

$$F_3^{\mathrm{loc}} = \frac{1}{3 \cdot \mathsf{b}[3]} m(3e_6) + \frac{3}{\mathsf{b}[1]} m(l) + \Big(-4 + \frac{27}{\mathsf{b}[1]} \Big) m(-e_1 - e_2 - e_3 - e_4 - e_5 - e_6 + 3l) \ ,$$

$$F_4^{\mathrm{loc}} = \frac{1}{4\cdot \mathsf{b}[4]}\, m(4e_6) + \frac{-2}{2\cdot \mathsf{b}[2]}\, m(-2e_1+2l) + \frac{-4}{\mathsf{b}[1]}\, m(-e_1-e_2+2l)$$
$$+ \Big(5 + \frac{-32}{\mathsf{b}[1]}\Big)\, m(-e_1-e_2-e_3-e_4-e_5+3l)$$

$$F_5^{\mathrm{loc}} = \frac{1}{5\cdot \mathsf{b}[5]}\, m(5e_6) + \frac{5}{\mathsf{b}[1]}\, m(-e_1+2l) + \Big(-6 + \frac{35}{\mathsf{b}[1]}\Big)\, m(-e_1-e_2-e_3-e_4+3l)$$
$$+ \Big(7\cdot \mathsf{b}[1] - 68 + \frac{205}{\mathsf{b}[1]}\Big)\, m(-2e_1-e_2-e_3-e_4-e_5-e_6+4l)\ ,$$

$$F_6^{\mathrm{loc}} = \frac{1}{6\cdot \mathsf{b}[6]}\, m(6e_6) - \frac{2}{3\cdot \mathsf{b}[3]}\, m(-3e_1+3l) + \Big(\frac{3}{2\cdot \mathsf{b}[2]} - \frac{6}{\mathsf{b}[1]}\Big)\, m(2l)$$
$$+ \Big(7 - \frac{36}{\mathsf{b}[1]}\Big)\, m(-e_1-e_2-e_3+3l)$$
$$+ \Big(-8\cdot \mathsf{b}[1] + 72 - \frac{198}{\mathsf{b}[1]}\Big)\, m(-2e_1-e_2-e_3-e_4-e_5+4l)$$
$$+ \Big(9\cdot \mathsf{b}[1]^2 - 108\cdot \mathsf{b}[1] + 498 - \frac{936}{\mathsf{b}[1]}\Big)\, m(-e_1-e_2-e_3-e_4-e_5-e_6+4l)$$
$$+ \Bigg(\frac{1}{2}\Big(-4 + \frac{27}{\mathsf{b}[2]}\Big) - 10\cdot \mathsf{b}[1]^3 + 141\cdot \mathsf{b}[2]^2 - 846\cdot \mathsf{b}[1] + 2636 - \frac{3780}{\mathsf{b}[1]}\Bigg)$$
$$\times\, m(-2e_1-2e_2-2e_3-2e_4-2e_5-2e_6+6l)\ .$$

6.5 Gopakumar–Vafa invariants

The local Gipakumar–Vafa invariants $n^g_\beta(S_3)$ ($g \in \mathbb{Z}_{\geq 0},\ \beta \in H_2(S_3,\mathbb{Z})$) are defined by the following expansion:

$$F^{\mathrm{loc}}_{S_3}(\lambda, Q) = \sum_{\beta\in H_2(S_3,\mathbb{Z})} \sum_{g\in\mathbb{Z}_{\geq 0}} \sum_{k\geq 1} \frac{n^g_\beta(S_3)}{k}\Big(2\sin\frac{k\lambda}{2}\Big)^{2g-2} Q^{k\beta}.$$

Proposition 6.10. *Gopakumar–Vafa invariants $n^g_\beta(S_3)$ of S_3 are integers. Moreover, for each β, $n^g_\beta(S_3)$ is equal to zero for all but finite g.*

Proof. This follows from Theorem 6.8 and the same statement for the toric surface S_3^0 (cf. Theorem 4.8). □

From the results in §6.4.3, we can obtain $n^g_\beta(S_3)$ for all β satisfying $d = -K_{S_3}\cdot\beta \leq 6$. They are listed in Table 1, in which we set $\beta = a_1e_1 + \cdots + a_6e_6 + a_7l$.

Remark 6.11. The results in Table 1 are in agreement with previous results in [66, Table 3], [54, Table 1, $n=6$], [11, Table 7, $X_3(1,1,1,1)$] obtained by the B-model calculation based on local mirror symmetry. Also compare with [43, Table 7].

d	$\beta=(a_1,\dots,a_6,a_7)$	$\lvert\mathcal{O}(\beta)\rvert$	$g(\beta)$	g	0	1	2	3	4	5
1	$(0,0,0,0,0,1,0)$	27	0		1					
2	$(-1,0,0,0,0,0,1)$	27	0		-2					
3	$(0,0,0,0,0,0,1)$	72	0		3					
	$(-1,-1,-1,-1,-1,-1,3)$	1	1		27	-4				
4	$(-1,-1,0,0,0,0,2)$	216	0		-4					
	$(-1,-1,-1,-1,-1,0,3)$	27	1		-32	5				
5	$(-1,0,0,0,0,0,2)$	432	0		5					
	$(-1,-1,-1,-1,0,0,3)$	216	1		35	-6				
	$(-2,-1,-1,-1,-1-1,4)$	27	2		205	-68	7			
6	$(-2,-1,0,0,0,0,3)$	432	0		-6					
	$(0,0,0,0,0,0,2)$	72	0		-6					
	$(-1,-1,-1,0,0,0,3)$	720	1		-36	7				
	$(-2,-1,-1,-1,-1,0,4)$	270	2		-198	72	-8			
	$(-1,-1,-1,-1,-1,-1,4)$	72	3		-936	498	-108	9		
	$(-2,-2,-2,-2,-2,-2,6)$	1	4		-3780	2636	-846	141	-10	

Table 1: Gopakumar–Vafa invariants $n^g_\beta(S_3)$ up to $d \le 6$. Here $\mathcal{O}(\beta)$ is the W_{E_6}-orbit of β and $g(\beta)$ is the arithmetic genus of a curve which belongs to β which is given by $\beta.(\beta+K_{S_3})/2+1$. One can observe that $n^g_\beta(S_3)$ are zero if $g > g(\beta)$.

6.6 Appendix: $-K$-nef toric surfaces

6.6.1 Classification of $-K$-nef toric surfaces

The following classification is due to Batyrev [5] (see also [11, Table 1]).

Lemma 6.12. *There are exactly sixteen $-K$-nef toric surfaces, whose fans are shown in Figure 3.*

We will refer the $-K$-nef toric surfaces using the numbers shown in frames in Figure 3.

Proof. The minimal $-K$-nef toric surfaces are $\mathbb{P}^2$, $\mathbb{P}^1\times\mathbb{P}^1$, and the Hirzebruch surface $\mathbb{F}_2$, which are No. 1, No. 2, and No. 4 respectively. $-K$-nef toric surfaces are obtained from them by blowing up at a torus-fixed point successively. By the $-K$-nef condition, we must blow-up at a torus-fixed point which is not on a torus-fixed (-2)-curve. All possible patterns of blowing-ups are listed in Figure 3. Note that No. 13, 15, and 16 can no longer be blown-up to $-K$-nef toric surfaces, since all of their torus-fixed points are on a torus-fixed (-2)-curve. This completes the classification. □

6.6.2 Deformation of $-K$-nef toric surfaces

Proposition 6.13. *A $-K$-nef toric surface has a smooth versal deformation family of dimension $h^1(\Theta)$, whose general member is a del Pezzo surface of degree c_1^2.*

$h^1(\Theta)$ and c_1^2 are given in Table 2.

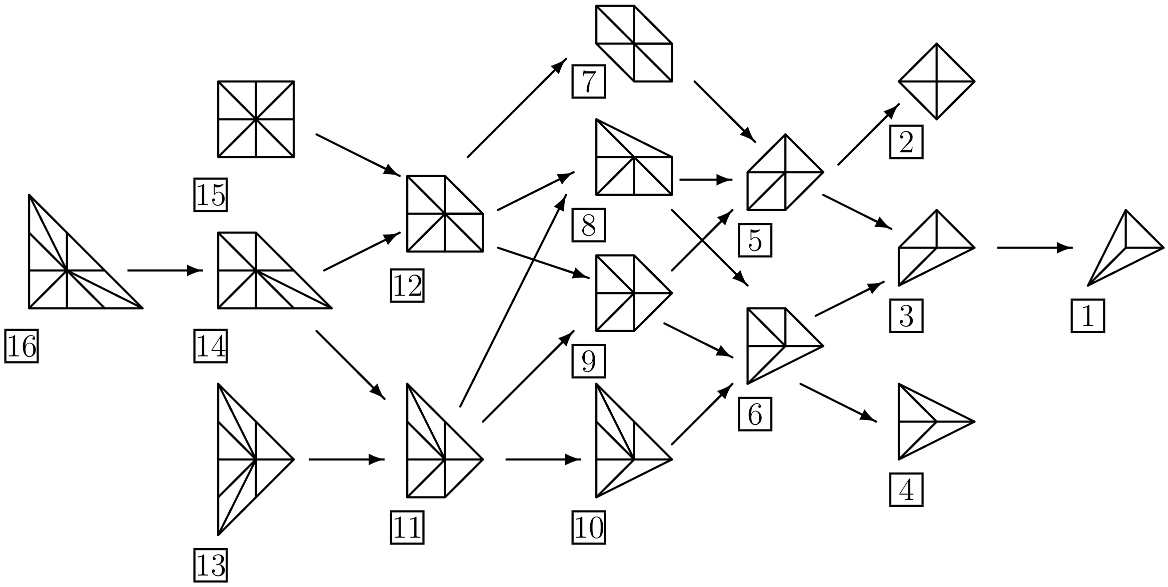


Figure 3: Classification of $-K$-nef toric surfaces. The arrows indicate blow-downs. The numbers in frames are reference numbers. Note that S_3^0, S_4^0, and S_5^0 introduced in §6.3 are No. 16, 14, and 12, respectively.

Proof. Note that $h^2(\Theta) = 0$ for any smooth compact toric surface (Corollary 6.15). This implies smoothness of a versal deformation family [46].

Versal deformation families of $-K$-nef toric surfaces are constructed inductively as follows. Let $\pi : \tilde{S} \to S$ be one of the blowing-ups in Figure 3. Let $P \in S$ be the center of the blowing-up π which is the intersection of two torus-fixed curves C_1 and C_2 (see Figure 4). By comparing Table 2 with Figure 3, we have

$$h^1(\tilde{S}, \Theta) = \begin{cases} h^1(S, \Theta) & \text{if } C_1^2 > -1,\ C_2^2 > -1\ , \\ h^1(S, \Theta) + 1 & \text{if } C_1^2 = -1,\ C_2^2 > -1\ , \\ h^1(S, \Theta) + 2 & \text{if } C_1^2 = C_2^2 = -1\ . \end{cases} \tag{6.9}$$

Since smooth rational curves on complex surfaces with self-intersection ≥ -1 is stable under small deformations [47, Example in p.86] (see also [4, IV. 3.1]), a complete deformation family of $\tilde{S}$ can be found as a simultaneous blowing-up of a complete deformation family of S. Furthermore, by eq. (6.9), we can find a versal deformation family of $\tilde{S}$ as follows. First, we consider a versal deformation family $\mathcal{S}$ of S on which C_1 and C_2 deform holomorphically. If both of C_1 and C_2 have self-intersection > -1, simultaneous blowing up of $\mathcal{S}$ at P gives a versal deformation family of $\tilde{S}$ which is of dimension $h^1(S, \Theta)$. If $C_1^2 = -1$ and $C_2^2 > -1$, we move the center P in the C_2 direction (see Figure 4) and blow $\mathcal{S}$ up simultaneously to get a versal deformation family of $\tilde{S}$ which is of dimension $h^1(S, \Theta) + 1$. If $C_1^2 = C_2^2 = -1$, we move the center P in the whole direction and blow $\mathcal{S}$ up simultaneously to get a versal deformation family of $\tilde{S}$ which is of dimension $h^1(S, \Theta) + 2$.
Thus we can find versal deformation families of $-K$-nef toric surfaces inductively. It is easy to see that their general members are del Pezzo surfaces. □

6.6.3 Unobstructedness

Let X be a smooth compact toric surface, $D := D_1 + \cdots + D_r$ be the sum of all torus invariant divisors $D_1, \cdots, D_r$, and $\Theta(-\log D)$ be the sheaf of germs of holomorphic vector fields with

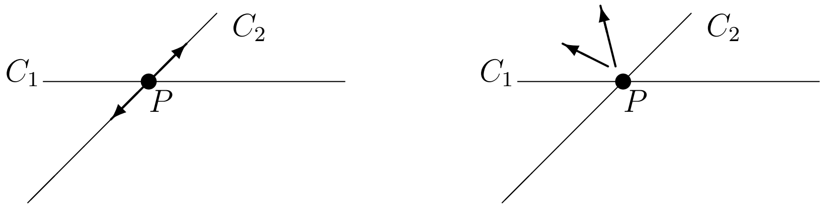


Figure 4: The center P of a blowing-up (C_1 and C_2 are torus-fixed curves) and its moving. The left is the case with $C_1^2 = -1$, $C_2^2 \geq 0$ and the right is the case with $C_1^2 = C_2^2 = -1$.

	I	II	III		IV		V				VI		VII			$VIII$
No.	1	3	2	4	5	6	7	8	9	10	11	12	13	14	15	16
c_1^2	9	8	8		7		6				5		4			3
c_2	3	4	4		5		6				7		8			9
$-\frac{7c_1^2-5c_2}{6}$	-8	-6	-6		-4		-2				0		2			4
$h^0(\Theta)$	8	6	6	7	4	5	2	4	3	5	3	2	3	2	2	2
$h^2(\Theta)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$h^1(\Theta)$	0	0	0	1	0	1	0	2	1	3	3	2	5	4	4	6

Table 2: Eight deformation types and $h^1(\Theta)\left(= -\frac{7c_1^2-5c_2}{6} + h^0(\Theta) + h^2(\Theta)\right)$.

logarithmic zeros along D.

Lemma 6.14. $H^2(X, \Theta(-\log D)) = 0$.

Proof. Since $\Theta(-\log D) = \mathcal{O} \otimes_{\mathbb{Z}} N$ (cf. [73, Proposition 3.1]), where N is the 2-dimensional lattice such that the fan of X sits in $N \otimes \mathbb{R}$. $H^2(X, \Theta(-\log D)) = H^2(X, \mathcal{O} \otimes_{\mathbb{Z}} N) = H^2(X, \mathcal{O} \oplus \mathcal{O}) = 0$, since $H^2(X, \mathcal{O}) = 0$ (cf. [73, Corollary 2.8]). □

Corollary 6.15. $H^2(X, \Theta) = 0$.

Proof. From the exact sequence (cf. [73, Theorem 3.12])

$$0 \longrightarrow \Theta(-\log D) \longrightarrow \Theta \longrightarrow \oplus_{i=1}^r \mathcal{O}(D_i)|_{D_i} \longrightarrow 0\ ,$$

and Lemma 6.14, we have $H^2(X, \Theta) = 0$. □

Part III

Topological vertex and its flop invarinace

In this part, we prove Theorem 1.6. In §7.1, the topological vertex is introduced. In §7.2, Theorem 1.5 is proved (Theorem 7.9). Some combinatorial formulas are collected in §7.3. In §8, definitions of toric Calabi–Yau threefolds and their Gromov–Witten invariants are given. Then, a method to write down their partition functions is explained. In §9.1, we study the transformations of partition functions under a flop and prove Theorem 1.6 (Corollary 9.4). As an application, we consider the canonical bundles K_S and $K_{\hat{S}}$ of a complete smooth toric surface S and its blow-up $\hat{S}$ (§9.2), and show that Gromov–Witten invariants of K_S are obtained from those of $K_{\hat{S}}$ (Corollary 9.8). In §9.3, an example is studied and the relationship with Nekrasov's partition function is discussed.

7 Topological vertex

7.1 Topological vertex

We use the following definition of the topological vertex given in [72]. See §2.3.1 for notations on combinatorial quantities.

Definition 7.1.

$$C_{\lambda_1,\lambda_2,\lambda_3}(q) \stackrel{\text{def.}}{=} q^{\frac{1}{2}\kappa(\lambda_3)} s_{\lambda_2}(q^{\rho}) \sum_{\mu\in\mathcal{P}} s_{\lambda_1/\mu}(q^{\lambda_2^t+\rho}) s_{\lambda_3^t/\mu}(q^{\lambda_2+\rho}) \ , \tag{7.1}$$

where $s_{\mu/\nu}(q^{\mu+\rho})$ (resp. $s_\mu(q^\rho)$) is the skew-Schur function with the specialization of variables:

$$s_{\mu/\nu}(x_i = q^{\mu_i - i + \frac{1}{2}}) \quad (\text{resp. } s_\mu(x_i = q^{-i+\frac{1}{2}})) \ .$$

Remark 7.2. (i) We have

$$W_{\mu,\nu}(q) = q^{\frac{1}{2}\kappa(\nu)} C_{\emptyset,\mu,\nu^t}(q), \tag{7.2}$$

where $\emptyset$ is the empty partition. (See (2.21) for definition of $W_{\mu,\nu}(q)$.)
(ii) In [72], it has been shown that $C_{\lambda_1,\lambda_2,\lambda_3}(q)$ is invariant under the cyclic permutation of $\lambda_1, \lambda_2, \lambda_3$. This is not manifest from the expression (7.1).

7.2 Topological vertex under flops

Take four partitions $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. These will be fixed throughout the rest of §7.2. We define

$$Z_0(q, Q_0) \stackrel{\text{def.}}{=} \sum_{\mu\in\mathcal{P}} (-Q_0)^{|\mu|} C_{\lambda_1,\lambda_2,\mu^t}(q) C_{\lambda_3,\lambda_4,\mu}(q) \ , \tag{7.3}$$

$$Z_0^+(q, Q_0^+) \stackrel{\text{def.}}{=} \sum_{\mu\in\mathcal{P}} (-Q_0^+)^{|\mu|} C_{\lambda_1,\mu^t,\lambda_4}(q) C_{\lambda_3,\mu,\lambda_2}(q) \ . \tag{7.4}$$

We also set

$$Z_{(-1,-1)}(q, Q) = \prod_{k=1}^{\infty} (1 - Qq^k)^k, \tag{7.5}$$

and

$$Z_0'(q, Q_0) \stackrel{\text{def.}}{=} \frac{Z_0(q, Q_0)}{Z_{(-1,-1)}(q, Q_0)} \ , \qquad Z_0^{+\prime}(q, Q_0^+) \stackrel{\text{def.}}{=} \frac{Z_0^+(q, Q_0^+)}{Z_{(-1,-1)}(q, Q_0^+)} \ .$$

The goal of this section is to show an identity relating $Z_0'(q, Q_0)$ and $Z_0^{+\prime}(q, Q_0^+)$ under the identification $Q_0^+ = Q_0^{-1}$ (Theorem 7.9). Formulas necessary for proofs can be found in §7.3.

Remark 7.3. Let us mention the geometrical meaning of the above formal power series. $Z_{(-1,-1)}(q, Q_0)$ is the partition function of the toric Calabi–Yau threefold $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ (see §8.3, also [24, Theorem 3] and [80]). $Z_0(q, Q_0)$ and $Z_0^+(q, Q_0^+)$ appear as local contributions in the partition functions of toric Calabi–Yau threefolds related by a flop such that both a flopping curve and a flopped curve are rational and have normal bundles isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ (see Figure 8).

7.2.1 Individual calculations

First, we compute $Z_0'(q,Q_0)$ and $Z_0^{+\prime}(q,Q_0^+)$ respectively. Following [36], introduce

$$
\begin{aligned}
f_\mu(q) &= \frac{q}{q-1}\sum_{i\geq 1}(q^{\mu_i-i}-q^{-i}) \ , \\
f_{\mu,\nu}(q) &= (q-2+q^{-1})f_\mu(q)f_\nu(q)+f_\mu(q)+f_\nu(q) \ ,
\end{aligned}
$$

and let $C_k(\mu,\nu)$ be the expansion coefficients in the Laurent polynomial $f_{\mu,\nu}(q)$:

$$
f_{\mu,\nu}(q)=\sum_{k\in\mathbb{Z}}C_k(\mu,\nu)q^k \ .
$$

Proposition 7.4. *We have*

$$
\begin{aligned}
Z_0'(q,Q_0) &= q^{\frac{1}{2}\kappa(\lambda_2)+\frac{1}{2}\kappa(\lambda_4)}s_{\lambda_1}(q^\rho)s_{\lambda_3}(q^\rho)\prod_{k\in\mathbb{Z}}(1-Q_0q^k)^{C_k(\lambda_1^t,\lambda_3^t)} \\
&\times\sum_\tau(-Q_0)^{|\tau|}s_{\lambda_2^t/\tau^t}(q^{\lambda_1+\rho},Q_0q^{-\lambda_3-\rho})s_{\lambda_4^t/\tau}(q^{\lambda_3+\rho},Q_0q^{-\lambda_1-\rho}) \ ,
\end{aligned}
\tag{7.6}
$$

$$
\begin{aligned}
Z_0^{+\prime}(q,Q_0^+) &= s_{\lambda_1}(q^\rho)s_{\lambda_3}(q^\rho)\prod_{k\in\mathbb{Z}}(1-Q_0^+q^k)^{C_k(\lambda_1,\lambda_3)} \\
&\times\sum_\tau(-Q_0^+)^{|\tau|}s_{\lambda_2/\tau}(q^{\lambda_3^t+\rho},Q_0^+q^{-\lambda_1^t-\rho})s_{\lambda_4/\tau^t}(q^{\lambda_1^t+\rho},Q_0^+q^{-\lambda_3^t-\rho}) \ .
\end{aligned}
\tag{7.7}
$$

Proof. By definition (7.1) of the topological vertex, we have

$$
\begin{aligned}
Z_0(q,Q_0) = &\sum_\mu(-Q_0)^{|\mu|}q^{\frac{1}{2}\kappa(\lambda_2)}s_{\lambda_1}(q^\rho)\sum_T s_{\mu^t/T}(q^{\lambda_1^t+\rho})s_{\lambda_2^t/T}(q^{\lambda_1+\rho}) \\
&\qquad q^{\frac{1}{2}\kappa(\lambda_4)}s_{\lambda_3}(q^\rho)\sum_{T'}s_{\mu/T'}(q^{\lambda_3^t+\rho})s_{\lambda_4^t/T'}(q^{\lambda_3+\rho}) \\
= &\ q^{\frac{1}{2}\kappa(\lambda_2)+\frac{1}{2}\kappa(\lambda_4)}s_{\lambda_1}(q^\rho)s_{\lambda_3}(q^\rho)\sum_{T,T'}(-Q_0)^{|T|}s_{\lambda_2^t/T}(q^{\lambda_1+\rho})s_{\lambda_4^t/T'}(q^{\lambda_3+\rho}) \\
&\qquad \sum_\mu s_{\mu^t/T}(-Q_0q^{\lambda_1^t+\rho})s_{\mu/T'}(q^{\lambda_3^t+\rho}) \ .
\end{aligned}
$$

We perform the sum with respect to μ by using (7.13):

$$
\begin{aligned}
Z_0(q,Q_0) = &\ q^{\frac{1}{2}\kappa(\lambda_2)+\frac{1}{2}\kappa(\lambda_4)}s_{\lambda_1}(q^\rho)s_{\lambda_3}(q^\rho) \\
&\prod_{i,j\geq 1}(1-Q_0q^{h_{\lambda_1^t,\lambda_3^t}(i,j)})\sum_\tau s_{T^t/\tau}(q^{\lambda_3^t+\rho})s_{(T')^t/\tau^t}(-Q_0q^{\lambda_1^t+\rho}) \\
&\sum_{T,T'}(-Q_0)^{|T|}s_{\lambda_2^t/T}(q^{\lambda_1+\rho})s_{\lambda_4^t/T'}(q^{\lambda_3+\rho}) \\
= &\ q^{\frac{1}{2}\kappa(\lambda_2)+\frac{1}{2}\kappa(\lambda_4)}s_{\lambda_1}(q^\rho)s_{\lambda_3}(q^\rho)\prod_{i,j\geq 1}(1-Q_0q^{h_{\lambda_1^t,\lambda_3^t}(i,j)}) \\
&\sum_\tau(-Q_0)^{|\tau|}\sum_T s_{\lambda_2^t/T}(q^{\lambda_1+\rho})s_{T/\tau^t}(Q_0q^{-\lambda_3-\rho})\sum_{T'}s_{\lambda_4^t/T'}(q^{\lambda_3+\rho})s_{T'/\tau}(Q_0q^{-\lambda_1-\rho}) \ .
\end{aligned}
$$

Here for $\mu, \nu \in \mathcal{P}$,

$$h_{\mu,\nu}(i,j) \stackrel{\text{def.}}{=} \mu_i - i + \nu_j - j + 1 \ .$$

In passing to the second line, we have used (7.16). By using (7.14), we have

$$Z_0(q,Q_0) = q^{\frac{1}{2}\kappa(\lambda_2)+\frac{1}{2}\kappa(\lambda_4)} s_{\lambda_1}(q^\rho) s_{\lambda_3}(q^\rho) \prod_{i,j\geq 1} (1 - Q_0 q^{h_{\lambda_1^t,\lambda_3^t}(i,j)}) \\ \sum_\tau (-Q_0)^{|\tau|} s_{\lambda_2^t/\tau^t}(q^{\lambda_1+\rho}, Q_0 q^{-\lambda_3-\rho}) s_{\lambda_4^t/\tau}(q^{\lambda_3+\rho}, Q_0 q^{-\lambda_1-\rho}) \ .$$

Applying Lemma 7.10, we obtain (7.6). One can also compute $Z_0^+(q,Q_0^+)$ in a similar way. □

Remark 7.5. This type of computation also appears in a proof of the large N factorization formula of the S-matrix of $\mathrm{U}(N)$ Chern–Simons theory. See [41].

The next corollary is a consequence of Proposition 7.4.

Corollary 7.6. *$Z_0^{+\prime}(q,Q_0^+)$ is a polynomial in Q_0^+ of degree at most $|\lambda_1|+|\lambda_2|+|\lambda_3|+|\lambda_4|$. Moreover, if $\lambda_3=\lambda_4=\emptyset$, $Z_0^{+\prime}(q,Q_0^+)$ is a polynomial in Q_0^+ of degree $|\lambda_1|+|\lambda_2|$.*

Similar statement also holds for $Z_0'(q,Q_0)$.

Proof. The first statement follows if we apply (7.10) and (7.15) to the expression (7.7). To prove the second statement, we show that the top term does not vanish. By (7.10), we have

$$\prod (1 - Q_0^+ q^k)^{C_k(\lambda_1^t,\emptyset)} = (-1)^{|\lambda|} q^{-\frac{1}{2}\kappa_{\lambda_1}} (Q_0^+)^{|\lambda_1|} + (\text{terms of lower degree in } Q_0^+).$$

Substituting this into (7.7) with λ_3,λ_4 set to $\emptyset$, and using (7.15), we obtain the claim. □

7.2.2 Comparison

Next, we compare $Z_0'(q,Q_0)$ with $Z_0^{+\prime}(q,Q_0^+)$ under the identification $Q_0^+ = Q_0^{-1}$. First we have the following

Lemma 7.7. *Under the identification $Q_0^+ = Q_0^{-1}$, we have*

$$\sum_\tau (-Q_0^+)^{|\tau|} s_{\lambda_2/\tau}(q^{\lambda_3^t+\rho}, Q_0^+ q^{-\lambda_1^t-\rho}) s_{\lambda_4/\tau^t}(q^{\lambda_1^t+\rho}, Q_0^+ q^{-\lambda_3^t-\rho}) \\ = (-Q_0)^{-|\lambda_2|-|\lambda_4|} \sum_\tau (-Q_0)^{|\tau|} s_{\lambda_2^t/\tau^t}(q^{\lambda_1+\rho}, Q_0 q^{-\lambda_3-\rho}) s_{\lambda_4^t/\tau}(q^{\lambda_3+\rho}, Q_0 q^{-\lambda_1-\rho}) \ . \quad (7.8)$$

Proof. Under $Q_0^+ = Q_0^{-1}$, we have

$$\begin{aligned}
(\mathrm{LHS}) &= \sum_\tau (-Q_0^{-1})^{|\tau|} s_{\lambda_2/\tau}(q^{\lambda_3^t+\rho}, Q_0^{-1} q^{-\lambda_1^t-\rho}) s_{\lambda_4/\tau^t}(q^{\lambda_1^t+\rho}, Q_0^{-1} q^{-\lambda_3^t-\rho}) \\
&= \sum_\tau (-Q_0^{-1})^{|\tau|} (-1)^{|\lambda_2|+|\lambda_4|} s_{\lambda_2^t/\tau^t}(q^{-\lambda_3-\rho}, Q_0^{-1} q^{\lambda_1+\rho}) s_{\lambda_4^t/\tau}(q^{-\lambda_1-\rho}, Q_0^{-1} q^{\lambda_3+\rho}) \ , \\
&= \sum_\tau (-Q_0)^{-|\lambda_2|+|\tau|-|\lambda_4|+|\tau|} (-Q_0^{-1})^{|\tau|} s_{\lambda_2^t/\tau^t}(q^{\lambda_1+\rho}, Q_0 q^{-\lambda_3^t-\rho}) s_{\lambda_4^t/\tau}(q^{\lambda_3+\rho}, Q_0 q^{-\lambda_1^t-\rho}) \\
&= (\mathrm{RHS}) \ .
\end{aligned}$$

Note that we have used the property (7.16) in the second line and the homogeneity (7.15) of skew Schur functions in the third line. □

Lemma 7.8. *The following identity holds:*

$$\prod_k (1-Q_0^{-1}q^k)^{C_k(\lambda_1,\lambda_3)} = (-Q_0)^{-|\lambda_1|-|\lambda_3|} q^{\frac{1}{2}\kappa(\lambda_1)+\frac{1}{2}\kappa(\lambda_3)} \prod_k (1-Q_0 q^k)^{C_k(\lambda_1^t,\lambda_3^t)}.$$

Proof. By (7.11), we have

$$\begin{aligned}
\prod_k (1-Q_0^{-1}q^k)^{C_k(\lambda_1,\lambda_3)} &= \prod_k (1-Q_0^{-1}q^{-k})^{C_k(\lambda_1^t,\lambda_3^t)} \\
&= Q_0^{-\frac{1}{2}(|\lambda_1|+|\lambda_3|)} q^{-\frac{1}{4}(\kappa(\lambda_1^t)+\kappa(\lambda_3^t))} \prod_k \left(Q_0^{\frac{1}{2}} q^{\frac{k}{2}} - Q_0^{-\frac{1}{2}} q^{-\frac{k}{2}}\right)^{C_k(\lambda_1^t,\lambda_3^t)} .
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\prod_k (1-Q_0 q^k)^{C_k(\lambda_1^t,\lambda_3^t)} &= Q_0^{\frac{1}{2}(|\lambda_1|+|\lambda_3|)} q^{\frac{1}{4}(\kappa(\lambda_1^t)+\kappa(\lambda_3^t))} \prod_k \left(Q_0^{-\frac{1}{2}} q^{-\frac{k}{2}} - Q_0^{\frac{1}{2}} q^{\frac{k}{2}}\right)^{C_k(\lambda_1^t,\lambda_3^t)} \\
&= (-1)^{|\lambda_1|+|\lambda_3|} Q_0^{\frac{1}{2}(|\lambda_1|+|\lambda_3|)} q^{\frac{1}{4}(\kappa(\lambda_1^t)+\kappa(\lambda_3^t))} \prod_k \left(Q_0^{\frac{1}{2}} q^{\frac{k}{2}} - Q_0^{-\frac{1}{2}} q^{-\frac{k}{2}}\right)^{C_k(\lambda_1^t,\lambda_3^t)} .
\end{aligned}$$

By comparing the above two equations and by using a symmetry (7.9) of a κ-factor, we get the claim. □

The following is the main result in this section [9]

Theorem 7.9. *Under the identification $Q_0^+ = Q_0^{-1}$, we have*

$$Z_0^{+\prime}(q, Q_0^+) = (-Q_0)^{-(|\lambda_1|+|\lambda_2|+|\lambda_3|+|\lambda_4|)} q^{\frac{1}{2}(\kappa(\lambda_1)-\kappa(\lambda_2)+\kappa(\lambda_3)-\kappa(\lambda_4))} Z_0'(q, Q_0) .$$

Proof. This follows from Proposition 7.4 and Lemmas 7.7 and 7.8. □

7.3 Appendix: Combinatorial formulas

We collect some combinatorial formulas which are used in §7. Our basic references are [62, 22, 72, 83]. For $\mu \in \mathcal{P}$, $\kappa(\mu)$ has the following important property:

$$\kappa(\mu^t) = -\kappa(\mu) , \tag{7.9}$$

where μ^t denotes the conjugate partition (the partition obtained by the transposition of the Young diagram of μ).

It is known that $C_k(\mu,\nu)$ are nonzero integers for finitely many values of k ([83, Theorem 5.1]), and have the following properties ([22, §3.1], [83, §5.3]):

$$\sum_k C_k(\mu,\nu) = |\mu| + |\nu| , \qquad \sum_k k C_k(\mu,\nu) = \frac{1}{2}(\kappa(\mu)+\kappa(\nu)) , \tag{7.10}$$

$$C_k(\mu,\nu) = C_{-k}(\mu^t,\nu^t) . \tag{7.11}$$

The following lemma is proved in [22, Lemma in §C], [83, Proposition 6.1]:

[9] The case of $\lambda_1 = \lambda_4 = \emptyset$ was proved by Hiroaki Kanno in his unpublished note in January 2004. The author is grateful to him for kindly providing the note.

Lemma 7.10. *For $\mu, \nu \in \mathcal{P}$, the following identity holds:*

$$\prod_{i,j\geq 1}\left(1-Qq^{h_{\mu,\nu}(i,j)}\right) = Z_{(-1,-1)}(q,Q)\prod_{k}\left(1-Qq^{k}\right)^{C_k(\mu,\nu)} .$$

Here are some properties of skew-Schur function. The following formulas are useful in performing the summations over partitions ([62, p.93,(5.10)]). Let $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ be sets of variables and $(x, y) = (x_1, x_2, \dots, y_1, y_2, \dots)$.

$$\sum_{\lambda\in\mathcal{P}} s_{\lambda/\lambda_1}(x)s_{\lambda/\lambda_2}(y) = \prod_{i,j\geq 1}(1-x_iy_j)^{-1}\sum_{\mu\in\mathcal{P}} s_{\lambda_2/\mu}(x)s_{\lambda_1/\mu}(y) \ , \tag{7.12}$$

$$\sum_{\lambda\in\mathcal{P}} s_{\lambda/\lambda_1}(x)s_{\lambda^t/\lambda_2}(y) = \prod_{i,j\geq 1}(1+x_iy_j)\sum_{\mu\in\mathcal{P}} s_{\lambda_2^t/\mu}(x)s_{\lambda_1^t/\mu^t}(y) \ , \tag{7.13}$$

$$\sum_{\xi\in\mathcal{P}} s_{\mu/\xi}(x)s_{\xi/\nu}(y) = s_{\mu/\nu}(x,y) \ . \tag{7.14}$$

Other properties are ([83, Proposition 4.1]):

$$s_{\mu/\nu}(Qx) = Q^{|\mu|-|\nu|}s_{\mu/\nu}(x) \ , \tag{7.15}$$

where $Qx = (Qx_1, Qx_2, \dots)$.

$$s_{\lambda/\mu}(q^{\nu+\rho}) = (-1)^{|\lambda|-|\mu|}s_{\lambda^t/\mu^t}(q^{-\nu^t-\rho}) \ . \tag{7.16}$$

8 Toric Calabi–Yau threefolds and Partition functions

8.1 Toric Calabi–Yau threefolds

In this section, we give definitions of toric Calabi–Yau threefolds and their partition functions. Basic references for toric varieties are [27, 73]. We follow [49] for a definition of toric Calabi–Yau threefolds and their partition functions.

8.1.1 Toric Calabi–Yau threefolds

Definition 8.1. A toric Calabi–Yau threefold is a three-dimensional smooth toric variety X associated with a finite fan Σ satisfying following conditions:

(i) the primitive generator $\vec{\omega}$ of every 1-cone satisfies $\vec{\omega}\cdot\vec{u} = 1$ where $\vec{u} = (0, 0, 1)$;

(ii) all maximal cones are three dimensional;

(iii) $|\Sigma| \cap \{z = 1\}$ is connected, where $|\Sigma| = \bigcup_{\sigma\in\Sigma}\sigma \subset \mathbb{R}^3$ is the support of Σ and z is the third coordinate of $\mathbb{R}^3$.

The condition (i) is equivalent to the condition that K_X is trivial (the Calabi–Yau condition) and the condition (ii) implies that $\pi_1(X) = 0$, $A_2(X) \cong \mathrm{Pic}(X) \cong H^2(X, \mathbb{Z})$ and they are torsion free, where $A_2(X)$ is the group of all Weil divisors modulo rational equivalence. (cf.[27, §3.4]). The condition (iii) will be used to define a certain connected graph in §8.2.1. Note that

Σ cannot be complete by the condition (i). Hence a toric Calabi–Yau threefold X is never compact.

Let us give an example of a toric Calabi–Yau threefold. Let Δ be the two dimensional nonsingular complete fan, which defines a toric surface S. Let $\{\rho_1, \ldots, \rho_r\}$ be the set of 1-cones of Δ and v_i be the primitive lattice vector generating ρ_i. For Δ, we associate a three dimensional fan Σ whose three-dimensional cone is generated by $(v_i, 1) \in \mathbb{R}^3$. Then the toric Calabi–Yau threefold associated to Σ is the total space of the canonical bundle K_S of S (*a local toric surface*). Another example is the total space of a rank 2 vector bundle over $\mathbb{P}^1$ with degree -2 (*a local toric curve*). More complicated examples can be found in §9.3.

8.1.2 Intersection theory

We briefly describe necessary facts on intersection theory on a toric Calabi–Yau threefold X. Recall that the subset $\Sigma_n \subset \Sigma$ of n-cones is in one-to-one correspondence with the set of $(3-n)$-dimensional torus invariant subvarieties in X. Let $\Sigma_1 = \{\rho_1, \ldots, \rho_r\}$ be the set of 1-cones. Denote by $D_{\rho_i} \subset X$ $(1 \leq i \leq r)$ the torus invariant Weil divisor corresponding to ρ_i. Let Σ_2' be the set of 2-cones which lie in the interior of $|\Sigma|$:

$$\Sigma_2' = \{\tau \in \Sigma_2 | \, \tau \subset |\Sigma| \setminus \partial|\Sigma|\}.$$

It is in one-to-one correspondence with the set of torus invariant (hence rational) curves in X. Let us write $\Sigma_2' = \{\tau_1, \ldots, \tau_p\}$ and let $C_{\tau_i} \subset X$ denote the rational curve corresponding to τ_i. We consider the following additive groups

$$TZ^1(X) := \bigoplus_{\rho \in \Sigma_1} \mathbb{Z} D_\rho, \quad TZ_1^c(X) := \bigoplus_{\tau \in \Sigma_2'} \mathbb{Z} C_\tau.$$

There is a $\mathbb{Z}$-bilinear paring

$$TZ^1(X) \times TZ_1^c(X) \to \mathbb{Z}, \tag{8.1}$$

induced by intersection numbers between D_{ρ_i}'s and C_{τ_i}'s. Numerical equivalence relations on $TZ^1(X)$ and $TZ_1^c(X)$ are defined as follows. For $D \in TZ^1(X)$, $D \equiv 0$ if $D.C = 0$ for any $C \in TZ_1^c(X)$. For $C \in TZ_1^c(X)$, $C \equiv 0$ if $D.C = 0$ for any $D \in TZ^1(X)$. We define

$$TN^1(X) := TZ^1(X)/\equiv, \quad TN_1^c(X) := TZ_1^c(X)/\equiv.$$

Note that $TN^1(X)_{\mathbb{R}} := TN^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$ and $TN_1^c(X)_{\mathbb{R}} := TN_1^c(X) \otimes_{\mathbb{Z}} \mathbb{R}$ are dual to each other by the paring (8.1).

Let us explain how to obtain the numerical equivalence relations. By the condition (ii) in Definition 8.1, $TN^1(X)$ is isomorphic to $A_2(X)$. Therefore numerical equivalence relations are given by

$$\sum_{j=1}^{r} a_{ij} D_{\rho_j} = 0, \quad (i = 1, 2, 3),$$

where $A = (a_{ij})$ is the $3 \times r$ matrix

$$A = \big(\vec{\omega}_1, \ldots, \vec{\omega}_r\big),$$

and $\vec{\omega}_i$ $(1 \leq i \leq r)$ are the primitive lattice vector generating ρ_i. Let us introduce the following injective map

$$l_X : TN_1^c(X) \to \{l \in \mathbb{Z}^r \,|\, A.l = \vec{0}\}, \quad Z \mapsto (D_{\rho_1}.Z, \ldots, D_{\rho_r}.Z). \tag{8.2}$$

Then numerical equivalence relations on $TN_1^c(X)$ can be read from linear relations between the lattice vectors $l_X([C_{\tau_1}]), \ldots, l_X([C_{\tau_p}])$.

Remark 8.2. Intersection numbers between D_{ρ_i}'s and C_{τ_i}'s can be computed as follows. First, if ρ_j and τ_i spans a 3-cone, $D_{\rho_j}.C_{\tau_i} = 1$ and if ρ_j and τ_i do not span a cone in the fan, $D_{\rho_j}.C_{\tau_i} = 0$ (cf.[27, §5.1]). If two 1-cones, say ρ_1, ρ_2, are contained in τ_i, then $D_{\rho_1}.C_{\tau_i}$ and $D_{\rho_2}.C_{\tau_i}$ are obtained by solving the equation $A.l_X([C_{\tau_i}]) = \vec{0}$. In particular, they satisfy the relation $D_{\rho_1}.C_{\tau_i} + D_{\rho_2}.C_{\tau_i} = -2$ by the condition (i) in Definition 8.1.

Let $\tau \in \Sigma_2'$ and $\rho_1, \rho_2 \in \Sigma_1$ be as above. By looking at the gluing of local coordinate systems around C_τ, we see that its normal bundle is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(D_{\rho_1}.C_\tau) \oplus \mathcal{O}_{\mathbb{P}^1}(D_{\rho_2}.C_\tau)$. We call C_τ a $(-1,-1)$-*curve* if its normal bundle is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

8.2 Partition functions

Let X be a toric Calabi–Yau threefold and Σ be its fan. We explain how to write down the partition function of X.

8.2.1 Toric graphs

First, consider the following directed graph Γ_X (called *a toric graph*) with labels on edges of a certain type. The vertex set is

$$V(\Gamma_X) = V_3(\Gamma_X) \cup V_1(\Gamma_X),$$

where

$$V_3(\Gamma_X) = \{v_\sigma | \sigma \in \Sigma_3(X)\}, \quad V_1(\Gamma_X) = \{v_\tau | \tau \in \Sigma_2(X) \setminus \Sigma_2'(X)\}.$$

The edge set is

$$E(\Gamma_X) = E_3(\Gamma_X) \cup E_1(\Gamma_X),$$

where

$$E_3(\Gamma_X) = \{e_\tau | \tau \in \Sigma_2'(X)\}, \quad E_1(\Gamma_X) = \{e_\tau | \tau \in \Sigma_2(X) \setminus \Sigma_2'(X)\}.$$

An edge $e_\tau \in E_3(\Gamma_X)$ joins $v_\sigma, v_{\sigma'} \in V_3(\Gamma)$ if and only if $\tau = \sigma \cap \sigma'$ (see Figure 5) and an edge $e_\tau \in E_1(\Gamma)$ joins $v_\sigma \in V_3(\Gamma_X)$ and $v_\tau \in V_1(\Gamma_X)$ if and only if σ is a unique 3-cone such that τ is a face of σ. This defines a finite planner graph. Note that $V_3(\Gamma_X) \neq \emptyset$ by the condition (ii) in Definition 8.1. A vertex in $V_3(\Gamma_X)$ is trivalent and a vertex in $V_1(\Gamma_X)$ is univalent. A graph Γ_X is connected by the condition (iii) in Definition 8.1.

The direction of edges can be taken arbitrarily. The label $n : E_3(\Gamma_X) \to \mathbb{Z}$, called the *framing*, is given as follows:

$$n(e_\tau) = \frac{D_{\rho_1}.C_\tau - D_{\rho_2}.C_\tau}{2},$$

where $\tau \in \Sigma_2'$ and $\rho_1, \rho_2 \in \Sigma_1$ are as shown in Figure 5. Note that C_τ is a $(-1,-1)$-curve if and only if $n(e_\tau) = 0$.

Let us consider the case of a local toric surface $X = K_S$. Its toric graph is a cyclic graph formed by $V_3(\Gamma_X)$ and $E_3(\Gamma_X)$ together with exterior vertices $V_1(\Gamma_X)$ and edges $E_1(\Gamma_X)$ (see Figure 6). Its inner edges $E_3(\Gamma_X)$ come from torus invariant curves C in S and their framings are given by $C^2 - 1$, where C^2 is the self-intersection number of C in S, if we take the clockwise orientations on $E_3(\Gamma_X)$ as in Figure 6.

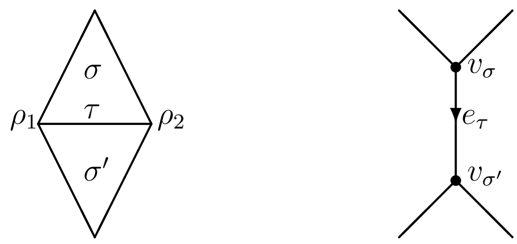


Figure 5: Fan (section at $z = 1$) and toric graph

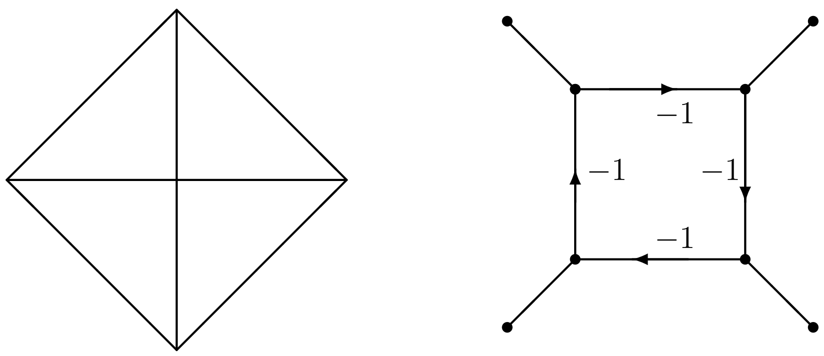


Figure 6: Fan (section at $z = 1$) (left) and toric graph (right) of $K_{\mathbb{P}^1\times\mathbb{P}^1}$. Numbers attached to $E_3(\Gamma_X)$ are framings with respect to the orientations given in the figure.

8.2.2 Partition functions

Secondly, we write down the partition function from Γ_X. Let

$$\mathcal{P}(\Gamma_X) = \{\vec{\lambda} : E_3(\Gamma) \to \mathcal{P}\}.$$

Take the set of formal variables $\vec{Q} = (Q_e)_{e\in E_3(\Gamma_X)}$ associated to $E_3(\Gamma_X)$. Then the partition function of X is a formal power series in $\vec{Q}$ given by

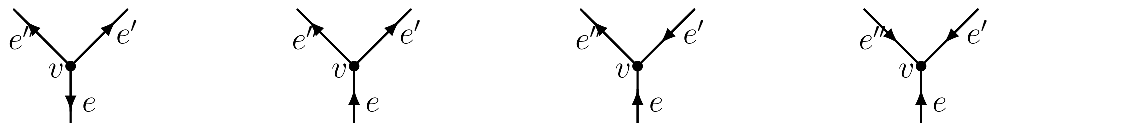
$$Z_X(q, \vec{Q}) = \sum_{\vec{\lambda}\in\mathcal{P}(\Gamma_X)} \prod_{e\in E_3(\Gamma_X)} (-1)^{|\vec{\lambda}(e)|(n_e+1)} q^{\frac{\kappa(\vec{\lambda}(e))}{2} n(e)} Q_e^{|\vec{\lambda}(e)|} \prod_{v\in V_3(\Gamma_X)} C_{\vec{\lambda}_v}(q)\ . \tag{8.3}$$

Here $C_{\vec{\lambda}_v}(q)$ is the topological vertex defined in (7.1) and $\vec{\lambda}_v$ ($v \in V_3(\Gamma_X)$, $\vec{\lambda} \in \mathcal{P}(\Gamma_X)$) is as in Figure 7 (for $e \in E(\Gamma_X) \setminus E_3(\Gamma_X)$, set $\vec{\lambda}(e)$ to $\emptyset$). We remark that the partition function does not depend on the directions of edges since the framing changes the sign if one gives the opposite direction to an edge $e \in E_3(\Gamma_X)$ and it is compensated by (7.9) and the summation.

Remark 8.3. By Remark 7.2 (i) and the last paragraph in §8.2.1, it is immediate to see that the formula (4.7) can be recovered from the formula (8.3) for local toric surfaces.

8.3 Gromov–Witten invariants of toric Calabi–Yau threefolds

Now we explain how to obtain Gromov–Witten invariants of a toric Calabi–Yau threefold X from the partition function $Z_X(q, \vec{Q})$. First, we give a precise definition of Gromov–Witten invariants for toric Calabi–Yau threefolds.



$(\vec{\lambda}(e), \vec{\lambda}(e'), \vec{\lambda}(e''))$ $(\vec{\lambda}(e)^t, \vec{\lambda}(e'), \vec{\lambda}(e''))$ $(\vec{\lambda}(e)^t, \vec{\lambda}(e')^t, \vec{\lambda}(e''))$ $(\vec{\lambda}(e)^t, \vec{\lambda}(e')^t, \vec{\lambda}(e'')^t)$

Figure 7: $\vec{\lambda}_v$

8.3.1 Gromov–Witten invariants of toric Calabi–Yau threefolds

Heuristically, Gromov-Witten invariants of X is defined as

$$\int_{[\overline{\mathcal{M}}_{g,0}(X,\beta)]^{\mathrm{vir}}} 1 \ . \tag{8.4}$$

However the integral is in general not well-defined, since X is non-compact. A rigorous definition of Gromov–Witten invariants for toric Calabi–Yau threefolds has been given in [59], which we now explain. The basic idea is to *define* the integral (8.4) as an equivariant integral by using the virtual localization formula (2.13). Recall that the 3-dimensional algebraic torus $(\mathbb{C}^*)^3$ acts on X. There is a distinguished 2-dimensional subtorus $\mathbb{T} \subset (\mathbb{C}^*)^3$ defined as follows. Let $X^{(\mathbb{C}^*)^3}$ be the fixed point set of the $(\mathbb{C}^*)^3$-action, which is nonempty by the condition (ii) in Definition 8.1. Let $p \in X^{(\mathbb{C}^*)^3}$ be a fixed point. Then $(\mathbb{C}^*)^3$ acts on T_pX and $\wedge^3 T_pX$, where T_pX is the tangent space at p. The action of $(\mathbb{C}^*)^3$ on the complex line $\wedge^3 T_pX$ gives an irreducible character

$$\alpha : (\mathbb{C}^*)^3 \to \mathbb{C}^*.$$

By conditions (i) and (iii) in Definition 8.1, the character α is independent of the choice of p. We define $\mathbb{T} := \mathrm{Ker}\ \alpha \cong (\mathbb{C}^*)^2$. Then $\mathbb{T}$ acts on X preserving the holomorphic 3-form on X (the *equivariantly Calabi–Yau* condition). The following is a generalization of Gromov–Witten invariants of local toric surfaces (cf. §4).

Definition 8.4. We define the Gromov–Witten invariant $N_{g,\beta}(X)$ of X with genus g, degree $\beta \neq 0$ as follows :

$$N_{g,\beta}(X) := \int_{[\overline{\mathcal{M}}_{g,0}(X,\beta)^{\mathbb{T}}]^{vir}} \frac{1}{e_{\mathbb{T}}(N^{vir})}, \tag{8.5}$$

where the RHS is a $\mathbb{T}$-equivariant integral and $e_{\mathbb{T}}(N^{vir})$ is the $\mathbb{T}$-equivariant Euler class of the virtual normal bundle N^{vir} of $\overline{\mathcal{M}}_{g,0}(X,\beta)^{\mathbb{T}}$ (cf. §2.2.3).

A priori, $N_{g,\beta}(X)$ takes value in $\mathbb{Q}(\alpha_1, \alpha_2)$, where α_1, α_2 are the weights of the standard representations of $\mathbb{T}$. However, it has been shown in [59, Theorem 5.9] that $N_{g,\beta}(X) \in \mathbb{Q}$, which is a nontrivial consequence from the equivariantly Calabi–Yau condition.

For a toric Calabi–Yau threefold X, the Gopakumar–Vafa invariants $n^g_\beta(X)$ are defined in terms of Gromov–Witten invariants as in Definition 4.7. It is worth mentioning that Konishi [49] has shown that the same statement as in Theorem 4.8 holds for any toric Calabi–Yau threefolds.

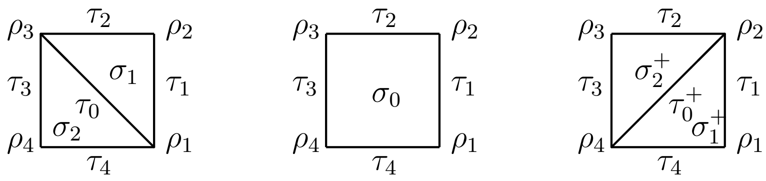


Figure 8: Fans (sections at $z = 1$): Σ (left), Σ_0 (middle) and Σ^+ (right). The generators $\vec{\omega}_1, \dots, \vec{\omega}_4$ of $\rho_1, \dots, \rho_4$ satisfy the relation $\vec{\omega}_1 + \vec{\omega}_3 = \vec{\omega}_2 + \vec{\omega}_4$.

8.3.2 Partition functions and Gromov–Witten invariants

The claim of [2, 59] is that the Gromov–Witten invariant $N_{g,\beta}(X)$ of X with the genus g and the degree β is obtained form $Z_X(q, \vec{Q})$ as follows:

$$\sum_{g\geq 0} N_{g,\beta}(X)\lambda^{2g-2} = \sum_{\substack{\vec{d}=(d_e)_{e\in E_3(\Gamma_X)},\\ \vec{d}[\vec{C}]=[\beta]}} F_{\vec{d}}(e^{\sqrt{-1}\lambda}), \tag{8.6}$$

where $[\vec{C}] = ([C_e])_{e\in E_3(\Gamma_X)}$, $C_e \subset X$ is the rational curve corresponding to e, $\vec{d}[\vec{C}] := \sum_{e\in E_3(\Gamma_X)} d_e[C_e]$, and $F_{\vec{d}}(q)$ is given by

$$\log Z_X(q, \vec{Q}) = \sum_{\vec{d}=(d_e)_{e\in E_3(\Gamma_X)}} F_{\vec{d}}(q)\vec{Q}^{\vec{d}},$$

where $\vec{Q}^{\vec{d}} = \prod_{e\in E_3(\Gamma_X)} Q_e^{d_e}$.

The formula (8.6) can be explained as follows. Since $\mathbb{T}$ and $(\mathbb{C}^*)^3$ have the same 0- and 1-dimensional orbits, their induced actions on $\overline{\mathcal{M}}_{g,0}(X, \beta)$ has the same fixed point set. One can carry out virtual localization as in the case with local toric surfaces in §5. This time, one encounters *three-partition special triple Hodge integrals* (cf. §2.3), which give rise to the topological vertex $C_{\vec{\lambda}_v}(q)$. This was first observed in [16], and almost established in [59] (see the remark below).

Remark 8.5. Precisely speaking, the partition function obtained in [59] has the expression almost same as (8.3) except that $C_{\vec{\lambda}_v}(q)$ is replaced by $\tilde{\mathcal{W}}_{\vec{\lambda}_v}(q)$. Here $\tilde{\mathcal{W}}_{\lambda_1,\lambda_2,\lambda_3}(q)$ is a rational function in $q^{\frac{1}{2}}$ similar to $C_{\lambda_1,\lambda_2,\lambda_3}(q)$ but has a slightly different expression. It is conjectured that $\tilde{\mathcal{W}}_{\lambda_1,\lambda_2,\lambda_3}(q) = C_{\lambda_1,\lambda_2,\lambda_3}(q)$ [59, Conjecture 8.3]. Here we use $C_{\lambda_1,\lambda_2,\lambda_3}(q)$ assuming that the conjecture is true. (The conjecture is true if at least one λ_i is empty.)

9 Transformations of partition functions

9.1 Flop invariance

In this section, we study the transformation of the partition function of toric Calabi–Yau threefolds under a flop.

9.1.1 Flop

Let X be a toric Calabi–Yau threefold and let Σ be its fan. Assume that X contains at least one $(-1,-1)$-curve C_0. Denote the corresponding 2-cone by τ_0. Near τ_0, the fan looks like the

	X		X^+	
2-cone	$\tau_0, \tau_1, \dots, \tau_4$	τ	$\tau_0^+, \tau_1, \dots, \tau_4$	τ
curve	$C_0, C_1, \dots, C_4$	C_τ	$C_0^+, C_1^+, \dots, C_4^+$	C_τ^+
edge	$e_0, e_1, \dots, e_4$	e_τ or just e	$e_0^+, e_1^+, \dots, e_4^+$	e_τ or just e
variable	$Q_0, Q_1, \dots, Q_4$	Q_e	$Q_0^+, Q_1^+, \dots, Q_4^+$	Q_e

Table 3:

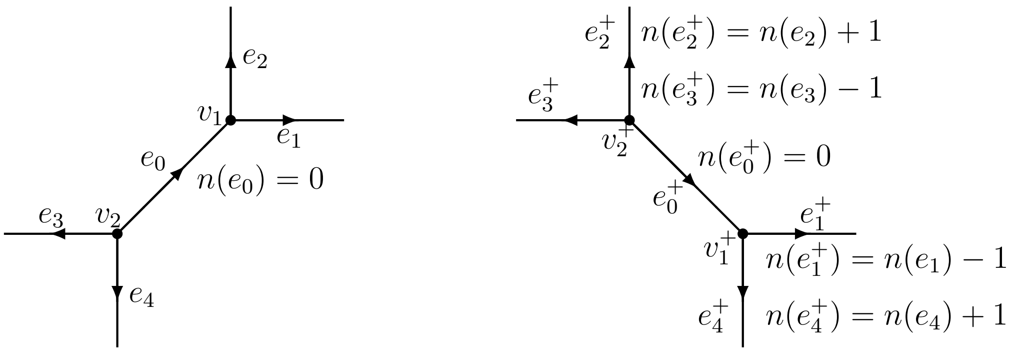


Figure 9: Toric graphs Γ_X (left) and Γ_{X^+} (right).

left diagram in Figure 8. We set

$$\Sigma_0 = (\Sigma \setminus \{\tau_0, \sigma_1, \sigma_2\}) \cup \{\sigma_0\}, \quad \Sigma^+ = (\Sigma \setminus \{\tau_0, \sigma_1, \sigma_2\}) \cup \{\tau_0^+, \sigma_1^+, \sigma_2^+\}$$

where $\tau_0, \sigma_1, \sigma_2, \sigma_0, \tau_0^+ \sigma_1^+, \sigma_2^+$ are cones shown in Figure 8. Let X_0 be the singular toric variety associated with the fan Σ_0 and X^+ be the toric Calabi–Yau threefold associated with the fan Σ^+. We denote by C_0^+ the $(-1,-1)$-curve on X corresponding to τ_0^+. Then associated to the evident maps $\Sigma \to \Sigma_0$ and $\Sigma^+ \to \Sigma_0$, there are the following birational maps:

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & X^+ \\ f \searrow & & \swarrow f^+ \\ & X_0 & \end{array}$$

The maps f, f^+ are small contractions whose exceptional sets are C_0, C_0^+ respectively. The birational map $\phi = (f^+)^{-1} \circ f$ is called the flop with respect to C_0. Note that ϕ is an isomorphism in codimension 1. Therefore, there is a canonical isomorphism $TN^1(X) \cong TN^1(X^+)$ induced by ϕ. In turn, this induces an isomorphism $\phi_* : TN_1^c(X)_{\mathbb{R}} \to TN_1^c(X^+)_{\mathbb{R}}$ via intersection paring.

From here on, we proceeds assuming that $\tau_1, \dots, \tau_4 \in \Sigma_2'$.[10] We use the notations shown in Table 3.

Lemma 9.1. *Under the flop $\phi : X \dashrightarrow X^+$, the curve classes transform as follows.*

$$\phi_*[C_0] = -[C_0^+], \quad \phi_*[C_i] = [C_i^+] + [C_0^+], \quad \phi_*[C_\tau] = [C_\tau^+] \quad \text{for } \tau \in \Sigma_2'(X) \setminus \{\tau_0, \dots, \tau_4\}.$$

Proof. This follows from computations of intersection numbers between curves and divisors. The first statement follows from $l_X([C_0]) = -l_{X^+}([C_0^+])$. Proofs of the others are similar. □

Let Γ_X be a toric graph of X. Near the edge e_0, the graph looks like the left diagram in Figure 9. Under the flop ϕ, the toric diagram (and the framings) changes as follows.

[10] For other cases, results on partition functions can be recovered by setting to zero the formal variables associated to any of $\tau_1, \dots, \tau_4$ which are not in Σ_2'.

Lemma 9.2. *A graph obtained from Γ_X by replacing the left diagram in Figure 9 with the right is a toric graph of X^+.*

9.1.2 Transformation of partition function

We associate the same formal variables $\vec{Q} = (Q_e)$ to edges in $E_3(\Gamma_X) \setminus \{e_0, \ldots, e_4\}$ and those in $E_3(\Gamma_{X^+}) \setminus \{e_0^+, \ldots, e_4^+\}$ and write the partition functions of X and X^+ as $Z_X(q, \vec{Q}, Q_0, Q_1, Q_2, Q_3, Q_4)$ and $Z_{X^+}(q, \vec{Q}, Q_0^+, Q_1^+, Q_2^+, Q_3^+, Q_4^+)$ respectively. It is immediate to check that

$$\begin{aligned} Z_X(q, \vec{0}, Q_0, 0, 0, 0, 0) &= Z_{(-1,-1)}(q, Q_0), \\ Z_{X^+}(q, \vec{0}, Q_0^+, 0, 0, 0, 0) &= Z_{(-1,-1)}(q, Q_0^+). \end{aligned} \tag{9.1}$$

We set

$$\begin{aligned} Z'_X(q, \vec{Q}, Q_0, Q_1, Q_2, Q_3, Q_4) &\stackrel{\text{def.}}{=} \frac{Z_X(q, \vec{Q}, Q_0, Q_1, Q_2, Q_3, Q_4)}{Z_X(q, \vec{0}, Q_0, 0, 0, 0, 0)}, \\ Z'_{X^+}(q, \vec{Q}, Q_0^+, Q_1^+, Q_2^+, Q_3^+, Q_4^+) &\stackrel{\text{def.}}{=} \frac{Z_{X^+}(q, \vec{Q}, Q_0^+, Q_1^+, Q_2^+, Q_3^+, Q_4^+)}{Z_{X^+}(q, \vec{0}, Q_0^+, 0, 0, 0, 0)}. \end{aligned}$$

Now we will compare these. To do so, we should identify the formal variables so that the identification is compatible with Lemma 9.1:

$$Q_0 = (Q_0^+)^{-1}, \qquad Q_i = Q_0^+ Q_i^+.$$

Theorem 9.3. *(i) The coefficients of $\vec{Q}^{\vec{d}} Q_0^{d_0} Q_1^{d_1} \ldots Q_4^{d_4}$ in $Z'_X(q, \vec{Q}, Q_0, Q_1, Q_2, Q_3, Q_4)$ is zero if $d_0 > d_1 + d_2 + d_3 + d_4$. A similar result holds for X^+.*

(ii) Under the identification $Q_0 = (Q_0^+)^{-1}$, $Q_i = Q_0^+ Q_i^+$, we have

$$Z'_X(q, \vec{Q}, Q_0, Q_1, Q_2, Q_3, Q_4) = Z'_{X^+}(q, \vec{Q}, Q_0^+, Q_1^+, Q_2^+, Q_3^+, Q_4^+).$$

(This is an equality between two formal power series in $\vec{Q}, Q_0^+, \ldots, Q_4^+$.)

Proof. The statement (i) follows from the first statement of Corollary 7.6.

To see (ii), let

$$\mathcal{P}'(\Gamma_X) = \{\vec{\nu} : E_3(\Gamma_X) \setminus \{e_0\} \to \mathcal{P}\}$$

and define $\vec{\nu}_v \in \mathcal{P}^3$ for $\vec{\nu} \in \mathcal{P}'(\Gamma_X)$ and $v \in V_3(\Gamma_X) \setminus \{v_1, v_2\}$ in the same way as $\vec{\lambda}_v$ (Figure 7). After (8.3), $Z_X(q, \vec{Q}, Q_0, Q_1, Q_2, Q_3, Q_4)$ is written as follows:

$$\begin{aligned} &Z_X(q, \vec{Q}, Q_0, Q_1, Q_2, Q_3, Q_4) \\ &= \sum_{\vec{\nu} \in \mathcal{P}'(\Gamma_X)} \prod_{e \in E_3(\Gamma_X) \setminus \{e_0, \ldots, e_4\}} (-1)^{(n(e)+1)|\vec{\nu}(e)|} Q_e^{|\vec{\nu}(e)|} \prod_{v \in V_3(\Gamma_X) \setminus \{v_1, v_2\}} C_{\vec{\nu}_v}(q) \\ &\quad \times \underbrace{\prod_{i=1}^{4} (-1)^{(n(e_i)+1)|\vec{\nu}(e_i)|} Q_i^{|\vec{\nu}(e_i)|} \sum_{\mu \in \mathcal{P}} C_{\vec{\nu}(e_1), \vec{\nu}(e_2), \mu^t}(q) C_{\vec{\nu}(e_3), \vec{\nu}(e_4), \mu}(q) (-Q_0)^{|\mu|}}_{(a)}. \end{aligned}$$

Similarly, $Z_{X^+}(q,\vec{Q},Q_0^+,Q_1^+,Q_2^+,Q_3^+,Q_4^+)$ is written as follows:

$$\begin{aligned}
&Z_{X^+}(q,\vec{Q},Q_0^+,Q_1^+,Q_2^+,Q_3^+,Q_4^+)\\
&=\sum_{\vec{\nu}\in\mathcal{P}'(\Gamma_{X^+})}\prod_{e\in E_3(\Gamma_{X^+})\setminus\{e_0^+,\dots,e_4^+\}}(-1)^{(n(e)+1)|\vec{\nu}(e)|}Q_e^{|\vec{\nu}(e)|}\prod_{v\in V_3(\Gamma_{X^+})\setminus\{v_1^+,v_2^+\}}C_{\vec{\nu}_v}(q)\\
&\times\underbrace{\prod_{i=1}^{4}(-1)^{(n(e_i^+)+1)|\vec{\nu}(e_i^+)|}(Q_i^+)^{|\vec{\nu}(e_i^+)|}\sum_{\mu\in\mathcal{P}}C_{\vec{\nu}(e_1^+),\mu^t,\vec{\nu}(e_4^+)}(q)C_{\vec{\nu}(e_3^+),\mu,\vec{\nu}(e_2^+)}(q)(-Q_0^+)^{|\mu|}}_{(b)}.
\end{aligned}$$

Here

$$\mathcal{P}'(\Gamma_{X^+})=\{\vec{\nu}:E_3(\Gamma_{X^+})\setminus\{e_0^+\}\to\mathcal{P}\}$$

and for $\vec{\nu}\in\mathcal{P}'(\Gamma_{X^+})$ and $v\in V_3(\Gamma_{X^+})\setminus\{v_1^+,v_2^+\}$, $\vec{\nu}_v\in\mathcal{P}^3$ is defined in the same way.

Since Γ_X and Γ_{X^+} are identical outside the diagrams described in Figure 9, $E_3(\Gamma_X)\setminus\{e_0,\dots,e_4\}=E_3(\Gamma_{X^+})\setminus\{e_0^+,\dots,e_4^+\}$, $V_3(\Gamma_X)\setminus\{v_1,v_2\}=V_3(\Gamma_{X^+})\setminus\{v_1^+,v_2^+\}$ and we have a natural bijection $p:\mathcal{P}'(\Gamma_X)\to\mathcal{P}'(\Gamma_{X^+})$ such that $p(\vec{\nu})=\vec{\nu}^+$ iff $\vec{\nu}(e)=\vec{\nu}^+(e)$ for all $e\in E_3(\Gamma_X)\setminus\{e_0,\dots,e_4\}$ and $\vec{\nu}(e_i)=\vec{\nu}^+(e_i^+)$ for $1\le i\le 4$. Under this identification, we could see that the two partition functions have the same expressions except for the factors (a) and (b). Taking into account the change in framings, we have

$$\left.\frac{(a)}{Z_{(-1,-1)}(q,Q_0)}\right|_{Q_0=(Q_0^+)^{-1},Q_i=Q_0^+Q_i^+}=\frac{(b)}{Z_{(-1,-1)}(q,Q_0^+)}$$

by Theorem 7.9. □

We finish this subsection by restating Theorem 9.3 in terms of Gromov–Witten invariants.

Corollary 9.4. *For $\beta\in H_2(X,\mathbb{Z})$ such that β is not a multiple of $[C_0]$,*

$$N_{g,\phi_*(\beta)}(X^+)=N_{g,\beta}(X).$$

Moreover,

$$N_{g,d[C_0]}(X)=N_{g,d[C_0^+]}(X^+)=N_{g,d[\mathbb{P}^1]}(\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)).$$

The same results also hold for the Gopakumar–Vafa invariants.

Proof. Theorem 9.3 implies that $\log Z_X(q,\vec{Q},Q_0,\dots,Q_4)$ and $\log Z_{X^+}(q,\vec{Q},Q_0^+,\dots,Q_4^+)$ are written in the following form:

$$\log Z'_X(q,\vec{Q},Q_0,\dots,Q_4)=\sum_{\vec{d}}\sum_{\substack{d_0,\dots,d_4\ge 0,\\ d_1+\dots+d_4\ge d_0}}F_{\vec{d},d_0,d_1,d_2,d_3,d_4}(q)\vec{Q}^{\vec{d}}Q_0^{d_0}\dots Q_4^{d_4},$$

$$\log Z'_{X^+}(q,\vec{Q},Q_0^+,\dots,Q_4^+)=\sum_{\vec{d}}\sum_{\substack{d_0,\dots,d_4\ge 0,\\ d_1+\dots+d_4\ge d_0}}F_{\vec{d},d_0,d_1,d_2,d_3,d_4}(q)\vec{Q}^{\vec{d}}(Q_0^+)^{d_1+\dots+d_4-d_0}(Q_1^+)^{d_1}\dots(Q_4^+)^{d_4}.$$

Comparing with (8.6), we obtain the first statement. The second statement follows from (9.1). □

Remark 9.5. In [38, §4.1], Iqbal and Kashani–Poor studied the special case such that the 2-cones $\tau_2,\tau_4\notin\Sigma'_2$ and the curves C_1,C_3 have normal bundles $\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}}(-1)$ or $\mathcal{O}_{\mathbb{P}^1}(-2)\oplus\mathcal{O}_{\mathbb{P}^1}(0)$. They obtained the result of Lemma 9.1 and proved the second statement of Theorem 9.3 in that case.

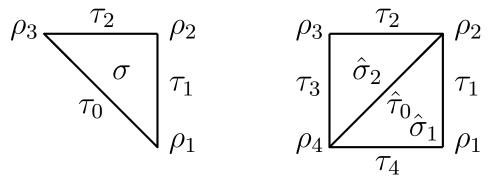


Figure 10: Fans (sections at $z = 1$) : Σ (left) and $\hat{\Sigma}$ (right). The generators $\vec{\omega}_1, \ldots, \vec{\omega}_4$ of $\rho_1, \ldots, \rho_4$ satisfy the relation $\vec{\omega}_1 + \vec{\omega}_3 = \vec{\omega}_2 + \vec{\omega}_4$.

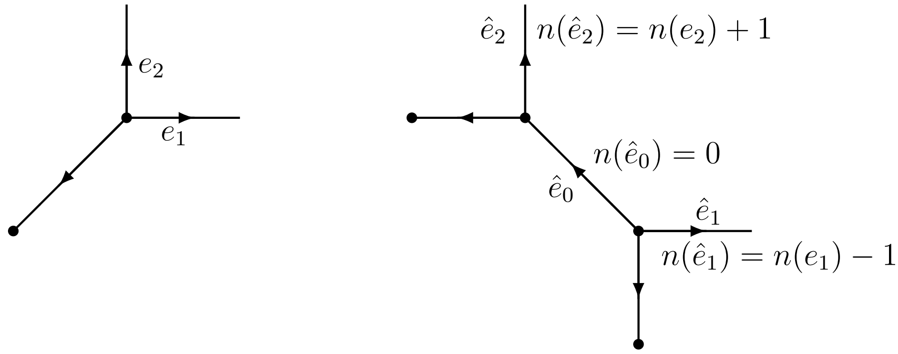


Figure 11: Toric graphs Γ_X (left) and $\Gamma_{\hat{X}}$ (right).

9.2 Application to toric surface and its blowup

9.2.1 Small modification of toric Calabi–Yau threefolds

Let X be a toric Calabi–Yau threefold and Σ be its fan. Let $\sigma \in \Sigma_3$ be a 3-cone such that one of its three 2-dimensional faces τ_0 lies on the boundary of the support of the fan $|\Sigma|$. Let $\hat{\Sigma}$ be the following fan (see Figure 10):

$$\hat{\Sigma} = (\Sigma \setminus \{\tau_0, \sigma\}) \cup \{\rho_4, \hat{\tau}_0, \tau_3, \tau_4, \hat{\sigma}_1, \hat{\sigma}_2\},$$

and let $\hat{X}$ be the toric Calabi–Yau threefold associated with the fan Σ. We call $\hat{X}$ a toric Calabi–Yau threefold obtained from X by a small modification. We compare the partition function of $\hat{X}$ and that of X. For the rational curves, edges and formal variables, we use the following notations in Table 4. Note that the rational curve $\hat{C}_0$ corresponding to $\hat{\tau}_0$ is a $(-1, -1)$-curve.

	X		$\hat{X}$	
2-cone	τ_1, τ_2	τ	$\hat{\tau}_0, \tau_1, \tau_2$	τ
curve	C_1, C_2	C_τ	$\hat{C}_0, \hat{C}_1, \hat{C}_2$	$\hat{C}_\tau$
edge	e_1, e_2	e_τ or just e	$\hat{e}_0, \hat{e}_1, \hat{e}_2$	e_τ or just e
variable	Q_1, Q_2	Q_e	$\hat{Q}_0, \hat{Q}_1, \hat{Q}_2$	Q_e

Table 4:

Let Γ_X be a toric graph of X. Near the edge corresponding to the 2-cone τ_0, the graph looks like the left diagram in Figure 11. Figure 12 shows an example of small modifications of toric Calabi–Yau threefolds. After the small modification, the toric diagram (and the framings) changes as follows.

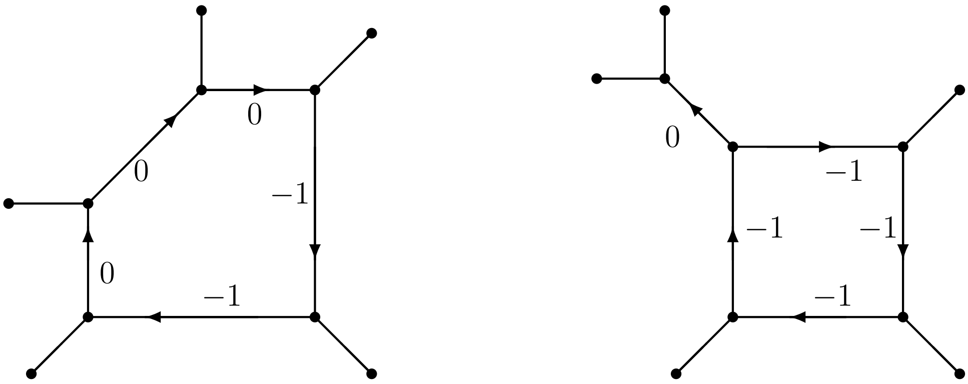


Figure 12: Toric graphs for a small modification of local $\mathbb{P}^1 \times \mathbb{P}^1$ (left) and its flop (right). Compare with Figure 6.

Lemma 9.6. *A graph obtained from Γ_X by replacing the left diagram in Figure 11 with the right is a toric graph of $\hat{X}$.*

We study the transformation of the partition function under a small modification. We associate the same formal variables $\vec{Q} = (Q_e)$ to edges in $E_3(\Gamma_X) \setminus \{e_1, e_2\}$ and those in $E_3(\Gamma_{\hat{X}}) \setminus \{\hat{e}_0, \hat{e}_1, \hat{e}_2\}$ and write the partition functions of X and $\hat{X}$ as $Z_X(q, \vec{Q}, Q_1, Q_2)$ and $Z_{\hat{X}}(q, \vec{Q}, \hat{Q}_0, \hat{Q}_1, \hat{Q}_2)$. It is immediate to check that

$$Z_{\hat{X}}(q, \vec{0}, \hat{Q}_0, 0, 0) = Z_{(-1,-1)}(q, \hat{Q}_0). \tag{9.2}$$

We define

$$Z'_{\hat{X}}(q, \vec{Q}, \hat{Q}_0, \hat{Q}_1, \hat{Q}_2) = \frac{Z_{\hat{X}}(q, \vec{Q}, \hat{Q}_0, \hat{Q}_1, \hat{Q}_2)}{Z_{\hat{X}}(q, \vec{0}, \hat{Q}_0, 0, 0)}.$$

The main result of this section is

Theorem 9.7. *(i) Coefficients of $\vec{Q}^{\vec{d}} \hat{Q}_0^{d_0} \hat{Q}_1^{d_1} \hat{Q}_2^{d_2}$ in $Z'_{\hat{X}}(q, \vec{Q}, \hat{Q}_0, \hat{Q}_1, \hat{Q}_2)$ vanish unless $d_1 + d_2 \geq d_0$.*

(ii)

$$Z_X(q, \vec{Q}, Q_1, Q_2) = Z'_{\hat{X}}(q, \vec{Q}, Q_0^{-1}, Q_1 Q_0, Q_2 Q_0)|_{Q_0 \to 0}\ .$$

Proof. The statement (i) follows from Corollary 7.6. To prove (ii), we consider the toric Calabi–Yau threefold $\hat{X}^+$ obtained from $\hat{X}$ by the flop of $\hat{C}_0$. Let Q_0, $\hat{Q}_1^+$ and $\hat{Q}_2^+$ be the formal variables correspond to the flopped curves $\hat{C}_0^+$, $\hat{C}_1^+$ and $\hat{C}_2^+$ respectively. Let $\hat{\Sigma}^+$ be the fan of $\hat{X}^+$. A natural inclusion $\Sigma \hookrightarrow \hat{\Sigma}^+$ induces that of toric varieties $X \hookrightarrow \hat{X}^+$. Under this map, we identify $\hat{Q}_1^+$ and $\hat{Q}_2^+$ with Q_1 and Q_2 respectively. Then by Theorem 7.9, we have

$$Z'_{\hat{X}^+}(q, \vec{Q}, Q_0, Q_1, Q_2) = Z'_{\hat{X}}(q, \vec{Q}, Q_0^{-1}, Q_1 Q_0, Q_2 Q_0).$$

Then (ii) follows, since the $Q_0 \to 0$ limit of the LHS is equal to $Z_X(q, \vec{Q}, Q_1, Q_2)$. □

9.2.2 Toric surface and its blowup

Let S be a complete smooth toric surface and $\hat{S}$ its blowup at a torus fixed point. The exceptional curve of $p : \hat{S} \to S$ is denoted by E. Let $X = K_S$ and $\hat{X} = K_{\hat{S}}$. Then $\hat{X}$ is a small modification of X and E is a $(-1,-1)$-curve on $\hat{X}$ added by the small modification. Applying Theorem 9.7 to this case, we obtain the following

Corollary 9.8. *(i) For $\beta \in H_2(\hat{S}, \mathbb{Z})$ such that β is not an multiple of E and satisfying $\beta.E < 0$,*

$$N_{g,\beta}(K_{\hat{S}}) = 0 \ .$$

(ii) For $\beta \in H_2(\hat{S}, \mathbb{Z})$ such that $\beta.E = 0$,

$$N_{g,\beta}(K_{\hat{S}}) = N_{g,p_*\beta}(K_S) \ .$$

(iii) For a multiple of $[E]$,

$$N_{g,d[E]}(K_{\hat{S}}) = N_{g,d}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)).$$

Proof. The statements (i) and (ii) follow from Theorem 9.7 (i) and (ii) respectively. The statement (iii) follows from (9.2). □

Especially, Gromov–Witten invariants of K_S are obtained from those of $K_{\hat{S}}$. The same statements also hold for the Gopakumar–Vafa invariants. See e.g. [11, Table 1, 10] for some related numerical data.

9.3 Example and geometric engineering

In this section, we first give an example of §9.1. Then we will discuss its relation with Nekrasov's partition function [70] along the same lines with [36, 37, 22, 23, 83].

Let X and X^+ be the toric Calabi–Yau threefolds associated with the left and right toric graphs in Figure 13, respectively. X contains two copies of $\mathbb{P}^1 \times \mathbb{P}^1$ disjoint to each other and X^+ is obtained by a flop of a unique $(-1,-1)$-curve in X. In this example, formal variables should be assigned as in Figure 13: the five variables for X are independent and the nine variables for X^+ have the four relations $Q_{F_i} = Q^+_{F_i} Q^+_0$ $(i = 1, 2)$ and $Q_{B_i} = Q^+_{B_i} Q^+_0$ $(i = 1, 2)$. The variables of X and X^+ should be identified by $Q^+_0 = Q_0^{-1}$ and Q_{F_i}, Q_{B_i} of $X^+ = Q_{F_i}, Q_{B_i}$ of X.

Let us compute the partition function Z_X of X (we omit the variables). By Proposition 7.4, we have

$$Z_X = \sum_{\mu_1^1,\mu_2^1,\mu_1^2,\mu_2^2} \prod_{k=1}^{2} (Q_{B_k})^{|\mu_1^k|+|\mu_2^k|} s^2_{\mu_1^k}(q^\rho) s^2_{\mu_2^k}(q^\rho) \prod_{i,j\geq 1} \left(1 - Q_{F_k} q^{h_{\mu_1^k,(\mu_2^k)^t}(i,j)}\right)^{-2}$$
$$\sum_\lambda \ (-Q_0)^{|\lambda|} \ s_{\lambda^t}(q^{\mu_2^1+\rho}, Q_{F_1} q^{\mu_1^1+\rho}) \ s_\lambda(q^{(\mu_2^2)^t+\rho}, Q_{F_2} q^{(\mu_1^2)^t+\rho}) \ .$$

We can perform the sum in the last factor by (7.13):

$$\sum_\lambda \quad (-Q_0)^{|\lambda|} s_{\lambda^t}(q^{\mu_2^1+\rho}, Q_{F_1} q^{\mu_1^1+\rho}) s_\lambda(q^{(\mu_2^2)^t+\rho}, Q_{F_2} q^{(\mu_1^2)^t+\rho})$$
$$= \prod_{i,j\geq 1} \quad \left(1 - Q_0 q^{h_{\mu_2^1,(\mu_2^2)^t}(i,j)}\right) \left(1 - Q_0 Q_{F_1} q^{h_{\mu_1^1,(\mu_2^2)^t}(i,j)}\right)$$
$$\left(1 - Q_0 Q_{F_2} q^{h_{\mu_2^1,(\mu_1^2)^t}(i,j)}\right) \left(1 - Q_0 Q_{F_1} Q_{F_2} q^{h_{\mu_1^1,(\mu_1^2)^t}(i,j)}\right) \quad .$$

Therefore Theorem 9.3 implies that the partition function Z_{X^+} is obtained from Z_X by replacing

$$\prod_{i,j\geq 1}\left(1-Q_0 q^{h_{\mu_2^1,(\mu_2^2)^t}(i,j)}\right) \to \prod_k \left(1-(Q_0^+)^{-1}q^k\right)^{C_k(\mu_2^1,(\mu_2^2)^t)} \prod_{k\geq 1}\left(1-Q_0^+ q^k\right)^k ,$$

and replacing Q_0 in other factors by $(Q_0^+)^{-1}$.

From the discussions in [42, §2.1], it seems natural to expect that the partition function of X reproduces Nekrasov's partition function for a gauge theory with a product gauge group and with a matter. We want to clarify this statement. Let us set

$$Z_X^{\text{inst}} = \frac{Z_X}{Z_X|_{Q_{B_1}=Q_{B_2}=0}} .$$

Then, by the same method with [22, 83], we can show the following

Proposition 9.9. *Let*

$$q=e^{-2R\hbar} , \quad Q_{F_1}=e^{-4Ra_1} , \quad Q_{F_2}=e^{-4Ra_2} , \quad Q_0=e^{2R(a_1+a_2-m)} .$$

Then we have

$$\begin{aligned} Z_X^{\text{inst}} = & \sum_{\mu_1^1,\mu_2^1,\mu_1^2,\mu_2^2} \prod_{k=1}^2 \left(\frac{Q_{B_k}}{2^4 Q_{F_k}}\right)^{|\mu_1^k|+|\mu_2^k|} \prod_{l,n=1}^2 \prod_{i,j\geq 1} \frac{\sinh R\left(a_{ln}^{(k)}+\hbar\left(\mu_{1,i}^k-\mu_{2,j}^k+j-i\right)\right)}{\sinh R\left(a_{ln}^{(k)}+\hbar\left(j-i\right)\right)} \\ & \times q^{\frac{1}{2}\left(\kappa(\mu_1^1)+\kappa(\mu_2^1)-\kappa(\mu_1^2)-\kappa(\mu_2^2)\right)} (2^2 Q_0)^{|\mu_1^1|+|\mu_2^1|+|\mu_1^2|+|\mu_2^2|} (Q_{F_1}^{\frac{1}{2}})^{2|\mu_1^1|+|\mu_1^2|+|\mu_2^2|} (Q_{F_2}^{\frac{1}{2}})^{|\mu_1^1|+|\mu_2^1|+2|\mu_1^2|} \\ & \prod_{l,n=1}^2 \prod_{i,j\geq 1} \frac{\sinh R\left(a_{ln}^{(1,2)}+m+\hbar\left(j-i\right)\right)}{\sinh R\left(a_{ln}^{(1,2)}+m+\hbar\left(\mu_{l,i}^1-\mu_{n,j}^2+j-i\right)\right)} , \end{aligned}$$

where

$$a_{11}^{(k)}=a_{22}^{(k)}=0 , \quad a_{12}^{(k)}=-a_{21}^{(k)}=2a_k ,$$

and

$$a_{11}^{(1,2)}=a_1+a_2 , \quad a_{21}^{(1,2)}=-a_1+a_2 , \quad a_{12}^{(1,2)}=a_1-a_2 , \quad a_{22}^{(1,2)}=-a_1-a_2 .$$

By Proposition 9.9, it is easy to see that the $R\to 0$ limit of

$$Z_X^{\text{inst}}|_{q=e^{-2R\hbar},\ Q_{B_k}=2^2\Lambda_k,\ Q_{F_k}=e^{-4Ra_k},\ Q_0=e^{2R(a_1+a_2-m)}}$$

is equal to the instanton part of Nekrasov's partition function of 4-dimensional $\mathrm{SU}(2)\times\mathrm{SU}(2)$ gauge theory with a matter in the bifundamental representation $(\mathbf{2},\bar{\mathbf{2}})$ (cf. [70, (66)]). It is straightforward to generalize this example to those which engineer more general quiver gauge theories of type A.

Remark 9.10. It is immediate to see that $Z_{X^+}^{\text{inst}} = Z_{X^+}/(Z_{X^+}|_{Q_{B_1}=Q_{B_2}=0})$ also coincides with the same Nekrasov's partition function with a similar variable identification in the limit $R\to 0$. More generally, Theorem 9.3 may imply that if toric Calabi–Yau threefolds X and X^+ are related by flops with respect to $(-1,-1)$-curves and if the partition function of X reproduces Nekrasov's partition function for a gauge theory, then the partition function of X^+ also reproduces it [11].

[11] Showing this statement, which was suggested by Yuji Tachikawa, was the author's original motivation for the problem. The author is grateful to him for discussions in August 2005 at Santa Barbara.

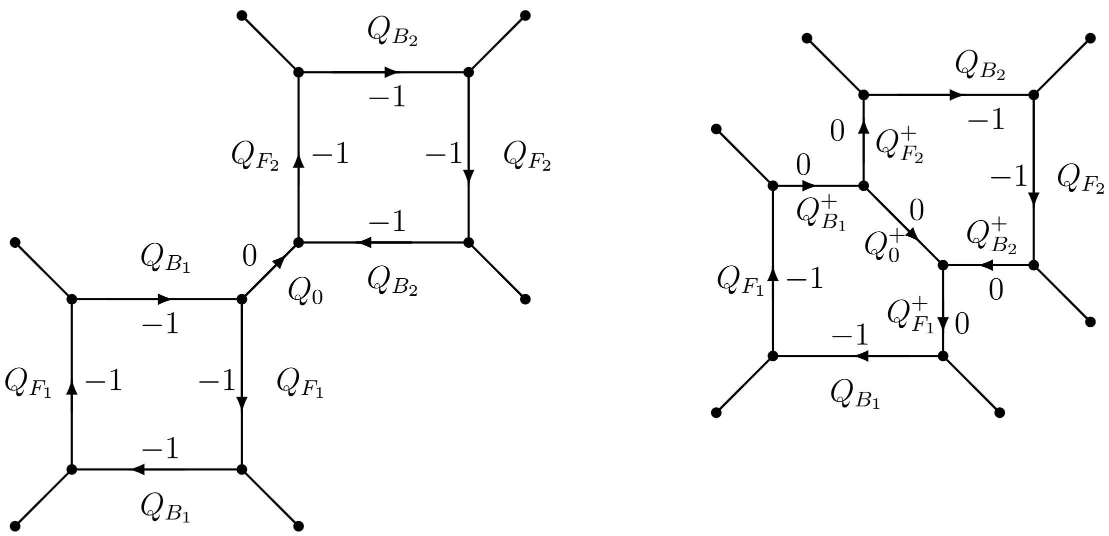


Figure 13: Toric Calabi–Yau threefold which contains two disjoint $\mathbb{P}^1 \times \mathbb{P}^1$ connected by a $(-1,-1)$-curve (left) and its flop (right).

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