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An Arrangement Technique of Gray-Code Table for Signal Constellation of Modified QAM and Triangular-Shaped Signal Set

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SUMMARY An arrangement technique of Gray-code table for signal constellations of modified QAM and Triangular-Shaped Signal Set is proposed. This is an optimum mapping technique for both two dimensional rectangular and hexagonal lattice-based constellations which cannot adapt the conventional Gray-code mapping. As our technique achieves most of neighboring symbols differ by one bit, the bit-error-probability (BEP) of the system becomes nearly equal to the symbol-error-probability (SEP). Furthermore, in consideration of this technique, we can have a constellation by which all neighboring symbols differ only by one bit for the rectangular lattice based constellation, but error-performances are worse than those of the original constellations because of a decrease of the corresponding decision regions.

1. Introduction

Digital communication systems employing QAM for transmitting digital signals have found a broad application throughout industry. The growing demands for higher transmission rates over the existing regulated channel bandwidths have dictated that the efficiency of the QAM system should be improved. Therefore, a trend towards a larger alphabet size is occurring as indicated by the intense theoretical and experimental research on 64- and 256-state QAM systems[(1)-(5)]. One of the main advantages for using QAM arises from the relative simplicity of hardware implementation. This is brought about by the orthogonal property of in-phase and quadrature signals. The good performance of QAM in the presence of noise has also made this scheme more attractive for a moderate alphabet size[(6)-(8)]. However, the signal constellation of QAM is not optimum under a peak power limited channels especially with a large alphabet size. This results in a scheme performance deviate from the optimum noise performance. Modified QAM (MQAM) system reforms the signal constellation of QAM, as shown in Fig. 1, preserving the orthogonal property of in-phase and quadrature signals, and the performance is better than that of the conventional QAM constellation under both peak and average signal power constraints[(9)]. Mathematically, these constellations can be categorized as a two-dimensional (2-D) rectangular lattice based constellation because their signals have an orthogonal property. It is well known that the hexagonal lattice based constellation has an optimum sphere packing property[(6)]. Experimental research have showed that the best result can be obtained by applying hexagonal signaling format[(10)]. Therefore, Triangular-Shaped Signal Set (TSSS) achieves better noise performance than the 2-D rectangular lattice based constellation[(11)].

The BEP is an efficient evaluation for digital communication system as it transmits several bits per symbol. Good BEP depends on two main factors, a larger decision region and a minimum Hamming distance (dH) between neighboring symbols. The Gray-code table is useful for the QAM mapping as it has square shaped constellation. But for modified QAM and cross-APK systems, although the signal constellations have orthogonal signals, they are no longer square. Therefore, Gray-code table should be arranged to suit for a non-square shaped constellation, i.e. to achieve minimum Hamming distance between neighboring symbols. Furthermore, for TSSS, since the constellation is based on the hexagonal lattice, some kind of mapping technique is needed to construct this system[(12)].

In this paper, we propose an arrangement technique of Gray-code table for signal constellations of modified QAM and Triangular-Shaped Signal Set. Since these two constellations are based on 2-D rectangular and hexagonal lattice based constellation, our proposed technique has its adaptability to both rectangular and hexagonal lattice based constellations. The main objective of employing our technique is to minimize the BEP, i.e. to minimize the Hamming distance between the neighboring symbols.

In Sect. 2 of this paper, we analyze the characteristics of Gray-code table as well as cyclic Gray-code which we use to construct the Gray-code table. In Sect. 3, we introduce how to arrange the Gray-code table with the characteristic limitations we mention in Sect. 2 for the case of rectangular and hexagonal lattice

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based constellations. Further, we also show that our technique can provide a constellation in which all neighboring symbols differ by one-bit. In Sect. 4, we present the BEP of the considering systems with the optimum mapping yielding from Sect. 3.

2. Characteristics of Gray-Code Table

Two dimensional rectangular lattice based constellation such as QAM, cross-APK and modified QAM can be viewed mathematically as bounded subsets whose elements are of the infinite form $(2p, 2q)+(1, 1)$ where p and q are arbitrary integers. For QAM, as it has square shaped constellation, an optimum mapping can be derived from the same state of Gray-code table in which all neighboring symbols differ by only one bit(13). Figure 1 shows a part of 64-state QAM and modified QAM systems. However, for rectangular lattice based constellation other than QAM, reformation of Gray-code table should be done in order to achieve a better noise performance, i.e., Hamming distances of all neighboring symbols have to be minimized. In this section, we first, analyze cyclic Gray-code which we use to construct Gray-code table and then clarify the characteristics of Gray-code table(14).

2. 1 Cyclic Gray-Code

Cyclic Gray-code is a member of cyclic code whose code word is Gray-code. Figure 2 shows an example for the case of 4-bits. Let X be an arbitrary sequence of information digits with each x_i be an Gray-code shown as follows;

$$X'=(x_0, x_1, \cdots, x_{n-3}, x_{n-2}, x_{n-1})$$

This code has a special property that any cyclic shift of a code word is also another code word. That is,

$$X'=(x_{n-1}, x_0, x_1, \cdots, x_{n-3}, x_{n-2})$$

is also another code word.

There is an important characteristic of cyclic Gray-code that if the Hamming distance of x_i and x_j is unity, then Hamming distances of x_{i+1} and x_{j-1}, x_{i+2} and $x_{j-2}, \cdots, x_{i+l}$ and x_{j-l} are also unity, where $i<j$, $l=(j-i-1)/2$.

Figure 2 shows an example of cyclic Gray-code for the case of 4-bits and its character.

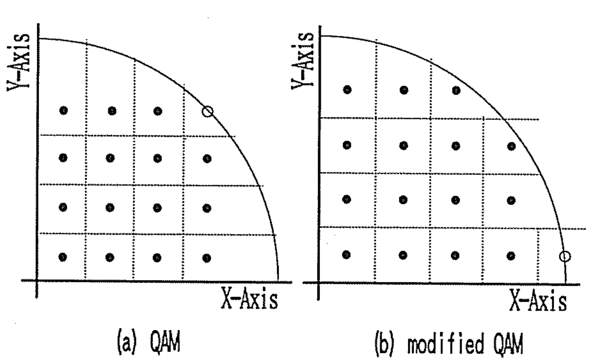


Fig. 1 64-state QAM and modified QAM.

2. 2 Characteristics of Gray-Code Table

Gray-code table consisting of $n+m$ bits has $2^{(n+m)}$ codes as shown in Fig. 3, mapped on two dimensional plane with 2^n by 2^m. In Gray-code table, first n-bit indicates the vertical coordinate of vertical codeline and the rest of m-bit indicates the horizontal coordinate of horizontal codeline. Let us call first n-bit upper-bit and rest of m-bit lower bit. There are two main characteristics of Gray-code table.

(a) Upper-bit (n-bit) and lower-bit (m-bit) of the arbitrary code must equal to the same side of vertical and horizontal codelines of Gray-code table, respectively.

(b) The set of codes having Hamming distance of 1 (dH = 1) only exists on the vertical and horizontal codeline with the number of n and m on each codeline, respectively.

Figure 3 shows the characteristics of Gray-code table, which we just have mentioned. The circled codes are the set of the codes having dH = 1 with the code, (111 110), and its upper-bit (111) and lower-bit (110) are shown with italic letter.

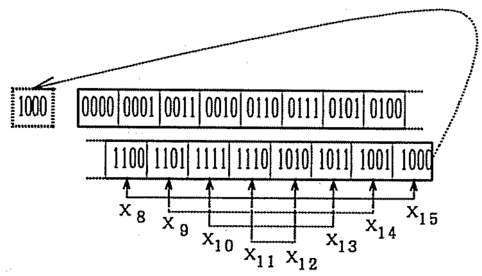


Fig. 2 Cyclic Gray-code and its character where Hamming distance of x_8 and x_{15} is unity. Thus, x_9 and x_{14}, x_{10} and x_{13}, x_{11} and x_{12} have Hamming distance of unity.

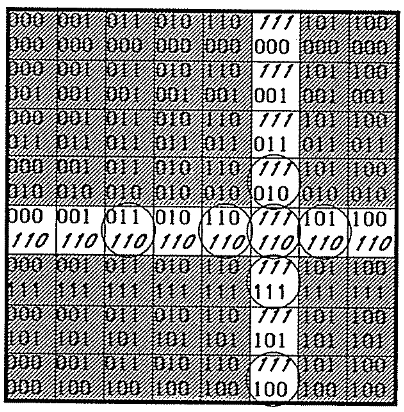


Fig. 3 64-state Gray-code table and two main characteristics of Gray-code table.

3. Code Mapping

In this section, a code mapping algorithms for 2-D rectangular and hexagonal lattice based constellations is described. In order to clarify our method, 64 and 256-state modified QAM systems and 128-state cross-APK system for the case of rectangular lattice are shown in Sect. 3. 1. 64-state Triangular-Shaped Signal Set for the case of hexagonal lattice is also shown in Sect. 3. 2. Reformation algorithm is given as follows:

(1) Find transferring codes by comparison of signal constellation diagram and the same states of Gray-code table. If the constellation is based on a hexagonal-lattice, shift the horizontal codelines of the table to suit for the constellation before making the decision.

(2) Decide conditions between the transferring codes and the codes effected by the transferring codes. If the table was shifted, then shift back to normal table form in order to decide all conditions.

(3) Make all conditions of the codes to satisfy the characteristics of Gray-code table (i.e. if the condition satisfies the characteristics of Gray-code table, then the transferring code will have Hamming distance of unity with the neighboring codes after reformation of Gray-code table). If the conditions dissatisfy the characteristics of Gray-code table, do step (2) until minimize number of those dissatisfactory conditions.

(4) Omit the dissatisfactory conditions given by step (3). Then make Gray-code table with consideration of the conditions and by deciding upper- and lower-bit using cyclic Gray-code.

(5) Reform the Gray-code table. If the constellation is based on hexagonal lattice, shift the table after reformation.

3. 1 Mapping for Rectangular Lattice Based Constellation

Both modified QAM and cross-APK systems do not have square-shaped constellations as QAM. This leads the constellations unable to adapt Gray-code mapping used for QAM system as mentioned before. In the previous section, we analyzed the characteristics of the Gray-code table. Now we can reform the Gray-code table suited for the non-square constellation with limited conditions in order to have minimum difference of bits with neighboring symbols after reformation. Figure 4 shows 64-state Gray-code table and the reformation of the table for the same states of modified QAM. For this case, the code mapping algorithm is as follows; there are four codes to be transferred which can be easily found by comparison of the Gray-code table and the constellation diagram (step (1)). Their conditions are also shown in the figure (step (2)). There are two kinds of condition, upper- and lower-bit conditions. The upper-bit condition indicates the condition of upper-bit in horizontal codeline to have a minimum Hamming distance after reformation because all lower-bits are the same (character (a)). The lower-bit condition indicates the condition of lower-bit. All conditions satisfy table characters (step (3)). This is because the codes indicated by upper-bit and lower-bit conditions are on the same belonging horizontal and vertical codelines, respectively (character (a)), and number of the codes are four on each codelines (character (b)). Thus, upper-bit conditions are satisfactory conditions and so are lower-bit conditions. Upper- and lower-codes can be easily found by cyclic Gray-code (step (4)). Thus, all neighboring symbols differ by one bit. This is the same case as mapping of QAM.

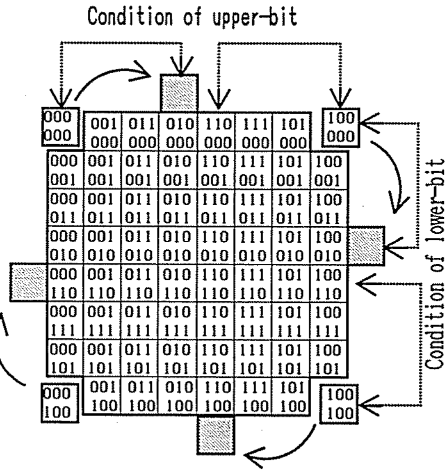


Fig. 4 Reformation of 64-state Gray-code table for modified QAM with limited conditions of upper- and lower-bit.

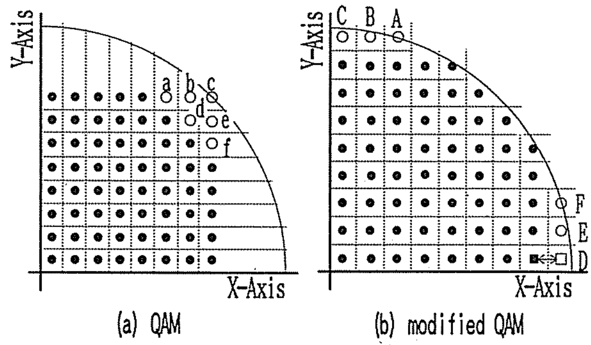
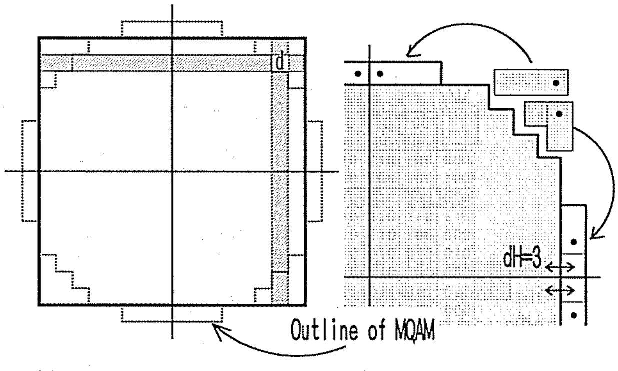


Fig. 5 256-state QAM and modified QAM.

For the case of 256-states, there are 24 codes to be transferred. Figure 5 shows the constellation diagram of modified QAM and QAM, the first quadrant are only shown. Transferring codes are labeled *a* to *f* in the figure. Deciding conditions are difficult for this case as there exist codes having no place to be transferred with a requirement of dH = 1. Code *d* in Fig. 6 is the one in the first quadrant as *d* has the set of *dH* = 1 codes only exist on the belonging codeline shown



(a) Dissaticefyed code, d (b) Reforming of Gray-code

Fig. 6 Mapping for the case of 256-state modified QAM.

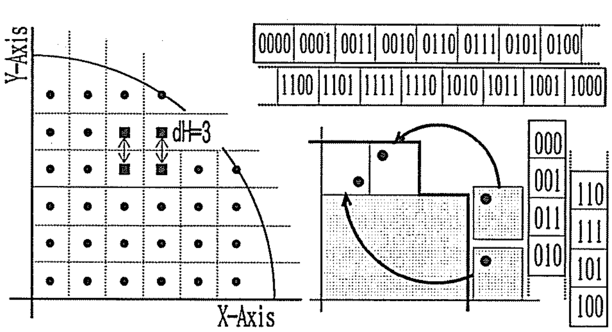


Fig. 7 128-state Cross-APK and reformation of Gray-code table with its upper- and lower-bit.

with the dashed lines. This gives the existent of $dH > 1$ codes. So 4 out of 24 codes have $dH = 3$ after reformation (See Fig. 5 (b)). But for the rest of codes, we can have conditions satisfying characteristics of Gray-code table, so suited table can be easily found by the cyclic Gray-code. The upper- and lower-code for this case are as same as the one shown in Fig. 2.

Signal constellation diagram of 128-states cross-APK is shown in Fig. 7 with reformation of table and its upper- and lower-code of the table. This is the case that the bit-length of upper-bit and lower-bit are different.

3.2 Mapping for Hexagonal Lattice Based Constellation

For the 2-D hexagonal case, we use a 1/2-bit shifted table. The constellation diagram, shifted table and outline of TSSS in the table diagram are shown in Fig. 8 with shifting directions of horizontal codelines. Since we shifted the table in order to suit with the constellation, there exist at least two $dH = 2$ codes with each code where number of neighboring codes are six. Mapping algorithm is the same as the rectangular case except for this shifting. Reformation of the table is shown in Fig. 9 with code *A*, which has no place to be transferred with limitation of $dH = 1$. Thus, *A* will

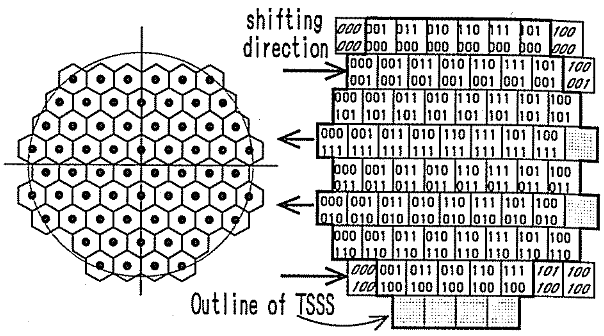


Fig. 8 64-state TSSS and 1/2-bit shifted Gray-code table.

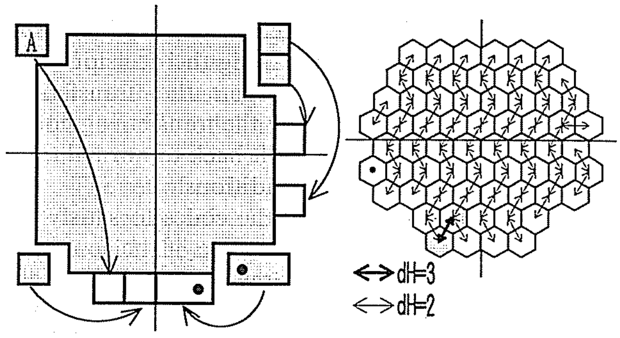
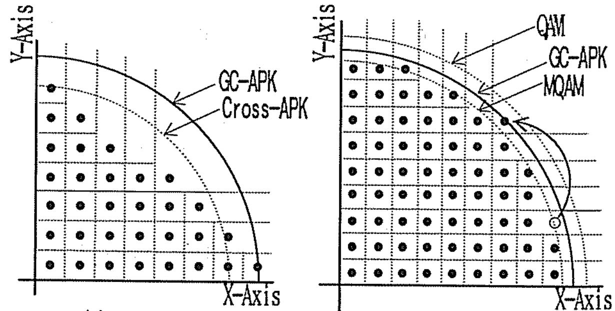


Fig. 9 Reformation of the table and relations of Hamming branches.



(a) 128-state (b) 256-state

Fig. 10 Gray-coded modified QAM.

have $dH = 3$ after shifting has been implemented where $dH = 3$ in normal table diagram. The result is also shown in the figure with relations of Hamming branches for the cases of branches having more than one. The Hamming branch is defined as Hamming distance between an arbitrary code and its neighboring code in *j*th direction where *j* is 6 for the hexagonal case.

3.3 Gray-Code APK System

For the case of rectangular lattice based constellation, although our technique can achieve most of symbols having $dH = 1$ with neighboring symbols, there still remains the codes having $dH > 1$. However, our technique enables the constellation having all neighboring codes differ only by one bit by remodifying constellation (Gray-coded APK : GC-APK). For example, we can have such constellation by remaining the symbol d in the Fig. 5(a) which would have the $dH = 3$ after reformation. This gives the threshold level decrease by 3.8%. Figure10 shows GC-APK for 128- and 256-states along with their peak signal power.

4. Bit Error Performance

Table 1 shows the number of Hamming branch and average Hamming distance of cross-APK and modified QAM for the case of 2-D rectangular lattice based constellations and TSSS for the case of hexagonal lattice based constellation. Average Hamming distance is defined as follows;

Table 1 The Number of Hamming branch for the case of $dH_j = 1$ to 5 and average Hamming distance of TSSS, cross-APK and modified QAM.

	64-states		128-states	256-states		512-states	1024-states	
	TSSS	MQAM	CROSS-APK	TSSS	MQAM	CROSS-APK	TSSS	MQAM
dHj=1	109	108	216	464	472	760	1926	1880
dHj=2	53	-	-	223	-	-	959	-
dHj=3	1	-	8	6	4	16	30	8
dHj=4	-	-	-	2	-	-	26	-
dHj=5	-	-	-	-	-	-	10	-
dH_{ave} [bit]	1.3374	1.0	1.0714	1.3381	1.0168	1.0412	1.3852	1.0084

Table 2 Threshold levels, A, and expressions of probability of error of GC-APK.

	128-states	256-states	512-states	1024-states
threshold level	$A=\frac{1}{\sqrt{226}}A_0$	$A=\frac{1}{\sqrt{338}}A_0$	$A=\frac{1}{\sqrt{962}}A_0$	$A=\frac{1}{\sqrt{1370}}A_0$
BEP	$Pe=\frac{119}{32}P_1$	$Pe=\frac{242}{64}P_1$	$Pe=\frac{496}{128}P_1$	$Pe=\frac{997}{256}P_1$
threshold decrease	15.3%	3.8%	21.7%	2.4%
performance decrease	1.24 [dB]	0.32 [dB]	1.70 [dB]	0.20 [dB]

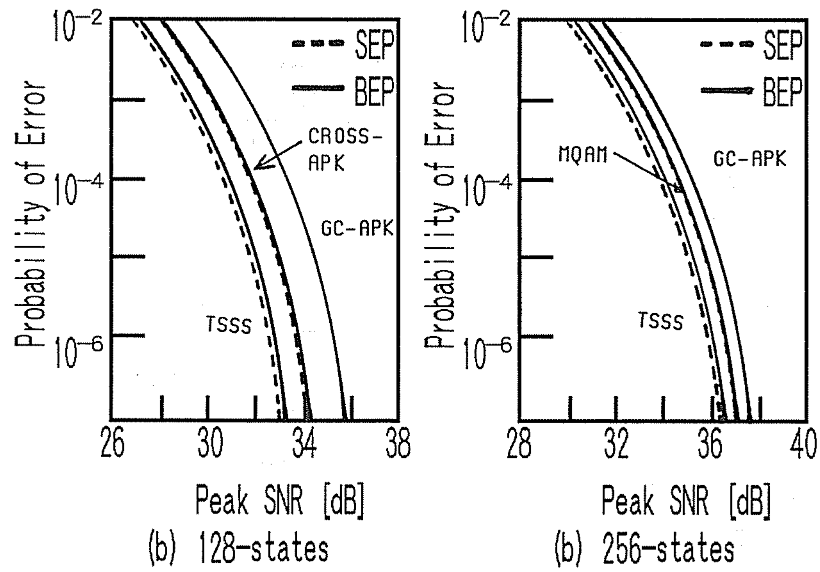


Fig. 11 BEP of MQAM, Cross-APK, TSSS and GC-APK along with their SEP.

$$dH_{\text{ave}}=\frac{\sum_{i=1}^{M}\sum_{j=1}^{N} dH_j}{MN} \quad (1)$$

where N is number of Hamming branch in arbitrary code, dH_j is Hamming branch in the jth direction and M is an alphabet size of the system. For rectangular based constellation, N is four for most of symbols and six for the case of hexagonal based constellation. Thus, the average Hamming distances are nothing less than 1 and 4/3 for rectangular and hexagonal based constellations, respectively. BEP can be easily derived by using this average Hamming distance which can be written as follows;

$$\text{BEP} = \text{SEP} \times dH_{\text{ave}} \quad (2)$$

Figure 11 shows the bit error performance (BEP) of the modified QAM, cross-APK, GC-APK and TSSS along with their SEP for the cases of 128- and 256-states, respectively. SEP except GC-APK are in Refs. (9) and (10). Threshold levels, A, and expressions of probability of error of GC-APK in the presence of additive Gaussian noise with variance σ_n are given in Table 2, where P_1 is given as follows;

$$P_1=\frac{1}{2}\text{erfc}\left(\frac{A}{\sqrt{2}\sigma_n}\right) \quad (3)$$

A_0 is defined as the maximum amplitude of received signal and erfc is the complement-error-function.

For the case of rectangular lattice based constellation, since our technique offers few increases of bits, SEP nearly equals to BEP. The performance of GC-APK is worse than the original constellation, especially for the case of re-modified constellation of cross-APK as the result of modifying a lot of symbols. For the case of 128-cross-APK, there are 8 codes to be modified where there are 4 codes in 256-MQAM (See Table 1). This causes the threshold level decrease. For example, the threshold decreases are 15.3% and 3. 8% for the cases of 128-cross-APK and 256-modified QAM, respectively. Thus, the performance losses against the original constellations are 1.24 dB and 0.32dB, respectively. The threshold and performance decrease of GC-APK against original constellation are also summarized in Table 2.

For the case of hexagonal lattice based constellation, as our technique achieves most of the constellations to have the $dH_{\text{ave}}= 1.33$ [bit], thus the BEP are 0.2 dB worse than SEP.

5. Conclusions and Discussions

The arrangement technique of Gray-code table for the 2-D rectangular lattice based constellation (such as modified QAM and cross APK) and hexagonal lattice based constellations (such as TSSS) has been discussed. With this technique, we can achieve the average Hamming distance nearly equals to 1 [bit] for the case of rectangular lattice based constellation and 4/3 [bit] for the case of hexagonal lattice based constellation, respectively. Thus, the BEP becomes nearly equal to SEP for the case of rectangular lattice based constellation and BEP are 0.2 dB worse than SEP for the case of hexagonal lattice based constellation. Furthermore, our technique provides a constellation based on rectangular-lattice which has all symbols differ by one-bit from neighboring symbols but the performance is worse than that of the original constellation as the result of threshold decrease. General case of reformation of 2-D signal constellation can be derived by using the Gray-code table which has an arbitral bit-shifting of horizontal codelines to match with such constellation.

Nowadays, coded modulation systems are being increasingly studied in digital communication systems. Especially the coded modulation systems based on lattice theory have become very popular with their explicit distinction between the design of codes and the design of constellations[(15)-(17)]. However, the mapping in these systems are also important[(16),(18),(19)]. Since the method proposed here is for uncoded modulation systems of 2-D rectangular and hexagonal based constellations, new mapping technique should be investigated for coded modulation systems of the considered constellations.

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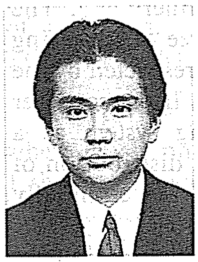

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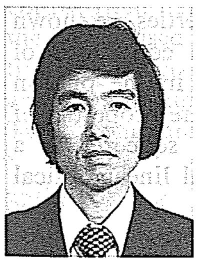

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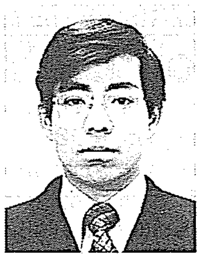

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