

Functional relations and universality for several types of multiple zeta functions

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Abstract

Firstly, we prove a functional relation for the Tornheim double zeta function. Using this functional relation, we obtain simple proofs of some known formulas for special values of Tornheim and Euler-Zagier double zeta functions.

Secondly, we obtain functional relations for Witten zeta functions by using a double L-values relation. By these functional relations, we obtain new proofs of known results on the Tornheim double zeta function, the Euler-Zagier double zeta function, their alternating and character analogues.

Thirdly, we define λ -joint, a' -joint, (λ, λ) -joint, (λ, a') -joint and (a', a') -joint t -universality of Lerch zeta functions and consider the relations among those. Next we show the existence of (λ, λ) -joint t -universality. We also show the existence of λ -joint, a' -joint, (λ, a') -joint and (a', a') -joint t -universality by using inversion formulas.

Fourthly, we show the following theorems. Suppose $0 < a_l < 1$ are algebraically independent numbers and $0 < \lambda_l \leq 1$ for $1 \leq l \leq m$. Then we have the joint t -universality for Lerch zeta functions $L(\lambda_l, a_l, s)$ for $1 \leq l \leq m$. Next we generalize Lerch zeta functions, and obtain the joint t -universality for them. In addition, we show examples of the non-existence of the joint t -universality for Lerch zeta functions and generalized Lerch zeta functions.

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Introduction

This paper is divided into two chapters.

Chapter 1 is devoted to functional and value relations and explicit evaluation formulae for several types of multiple zeta functions, especially, the Tornheim double zeta function and Witten zeta functions. Multiple zeta has along history. In the paper [6] published in 1775, Euler studied the double zeta series (see (0.0.2)). More than forty years earlier, he found the famous formula $\zeta(2) = \pi^2/6$ and its generalization for even n

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \cdots = \frac{|B_n|(2\pi)^n}{2 \cdot n!}, \quad (0.0.1)$$

where B_n is the Bernoulli number. The paper [6] examines relations between the double series and the series (0.0.1), which is the value of the Riemann zeta function. The paper [6] is the origin of the study of multiple zeta values. However, after Euler's research, there came very long inactive period in the study of multiple zeta values. In the twentieth century, this series regained the interest of many mathematicians. At first some sporadic research was done for its own sake, or under the motivation of analytic number theory. In 1990s, it turned out that the multiple zeta values are closely related to many other branch of mathematics, for example, arithmetic geometry, Galois representations, invariants for knots, quantum groups, etc.

Now we define multiple zeta values. For $m_1, \dots, m_n \in \mathbb{N}$, $m_1 \geq 2$, the multiple zeta values (Euler-Zagier multiple sum) is defined by the series

$$\zeta(m_1, m_2, \dots, m_n) := \sum_{k_1 > k_2 > \cdots > k_n > 0} \frac{1}{k_1^{m_1} k_2^{m_2} \cdots k_n^{m_n}}. \quad (0.0.2)$$

For these series, the next formulae are well-known :

$$\zeta(\underbrace{2, 2, \dots, 2}_n) = \frac{\pi^{2n}}{(2n+1)!}, \quad \zeta(\underbrace{3, 1, 3, 1, \dots, 3, 1}_{2n}) = \frac{2\pi^{4n}}{(4n+2)!}.$$

Apart from a small number of such formulae, there are few known examples. At present, the main interest of research is to find various relations among the multiple zeta values, rather than the values themselves. For instance, we have the following relation, which is called the sum formula.

Theorem 0.0.1. *Fix n and m satisfying $1 \leq n \leq m - 1$. Then we have*

$$\sum_{\substack{m_1 \geq 2, m_2, \dots, m_n \geq 1 \\ m_1 + \cdots + m_n = m}} \zeta(m_1, \dots, m_n) = \zeta(m).$$

There are many other linear relations, for example, duality, Hoffman's relation and Ohno's relation (see [11]). These relations, except for duality, are deduced from two types of so called shuffle product formulae, written by a product of multiple zeta values as a linear combination of other such values. The first type of formulae are called harmonic product formulae. The simplest example is the relation

$$\zeta(m_1)\zeta(m_2) = \zeta(m_1, m_2) + \zeta(m_2, m_1) + \zeta(m_1 + m_2). \quad (0.0.3)$$

The second type of formulae are called shuffle product formulae. The simplest example is the relation

$$\zeta(m_1)\zeta(m_2) = \sum_{j=0}^{m_1-1} \binom{m_2-1+j}{j} \zeta(m_2+j, m_1-j) + \sum_{j=0}^{m_2-1} \binom{m_1-1+j}{j} \zeta(m_1+j, m_2-j). \quad (0.0.4)$$

Although the double shuffle relation arises very naturally, they do not account for all the linear relations. As a “regularization” of the double shuffle relation, there are so called generalized double shuffle relations. For these relations, the following conjecture is well-known.

Conjecture 0.0.2. *Any linear relation among multiple zeta values is a consequence of generalized double shuffle relations.*

On the other hand, we can consider that multiple zeta values are the special values of multiple zeta function, $\zeta(s_1, \dots, s_n)$, $s_1, \dots, s_n \in \mathbb{C}$. The multiple zeta functions have been studied by many mathematicians (see [27]). Based on these studies, some years ago, Matsumoto suggested the following problem.

Are the known relations for multiple zeta values valid only at positive integers,
or valid at other values?

As a trivial answer, we have the harmonic product formula (0.0.3), which holds for any $s_1, s_2 \in \mathbb{C}$ except for the singular points. In [52], Tsumura proved a non-trivial functional relation (1.1.2) for the Tornheim double zeta function, not for the Euler-Zagier zeta function. Recently, many functional relations for Witten zeta functions are obtained by Komori, Matsumoto, Tsumura and the author.

Chapter 2 is devoted to the universality theory for several types of multiple zeta functions. The study of the distribution of the values of the Riemann zeta function $\zeta(\sigma + it)$ for fixed σ and variable $t > 0$ was investigated by H. Bohr. In 1914, he showed the following denseness theorem, as a joint work with Courant.

Theorem 0.0.3 ([42, Theorem 11.9]). *Let σ_0 be a fixed number in the range $1/2 < \sigma < 1$. Then the values which $\zeta(s)$ takes on $\sigma = \sigma_0$, $t > 0$, are everywhere dense in the whole plane.*

This theorem of Bohr was the first remarkable denseness result for the Riemann zeta function and it was generalized by S.M.Voronin in 1972. He proved that if s_1, s_2, \dots, s_m are distinct points lying in the strip $1/2 < \sigma < 1$, and $h > 0$ is an arbitrary fixed number then the sequence

$$(\zeta(s_1 + inh), \zeta(s_2 + inh), \dots, \zeta(s_m + inh)) \quad n \in \mathbb{N}$$

is dense in \mathbb{C}^m . He also obtained that the sequence

$$(\zeta(s_0 + inh), \zeta'(s_0 + inh), \dots, \zeta^{(m-1)}(s_0 + inh)) \quad n \in \mathbb{N}$$

is dense in \mathbb{C}^m for any fixed s_0 such that $1/2 < \Re(s_0) \leq 1$. The question on differential properties of the Riemann zeta function was raised by D.Hilbert in 1900 during the International Congress of Mathematicians. He noted that an algebraic-differential independence of $\zeta(s)$ can be proved by the algebraic-differential independence of the gamma function $\Gamma(s)$ and functional equation of $\zeta(s)$. As an generalization of this mention of Hilbert, we obtain the following theorem by using the above Voronin’s theorem.

Theorem 0.0.4 ([16, Theorem 6.6.3]). *Let F_k , $k = 0, 1, \dots, n$, be continuous functions, and let*

$$\sum_{k=0}^n s^k F_k(\zeta(s), \zeta'(s), \dots, \zeta^{(j-1)}(s)) = 0$$

be valid identically for $s \in \mathbb{C}$. Then F_k , for $k = 0, 1, \dots, n$.

A natural next step is to study the situation on infinite dimensional spaces, namely on functional spaces. Concerning this problem, in 1975, S.M.Voronin showed the next theorem, which is now called the universality. We prepare some notation for universality. By $\text{meas}\{A\}$ we denote the Lebesgue measure of the set A , and, for $T > 0$, we use the notation

$$\nu_T^\tau\{\dots\} := \frac{1}{T} \text{meas}\{\tau \in [0, T] : \dots\}$$

where in place of dots some condition satisfied by τ is to be written. Let $D := \{s \in \mathbb{C} : 1/2 < \Re(s) < 1\}$, K and K_1, \dots, K_m ($m \geq 2$) be compact subsets of the strip D with connected complements.

Theorem 0.0.5 ([16, Theorem 6.5.2]). *Let $f(s)$ be a non-vanishing function analytic in the interior of K and continuous on K . Then for every $\varepsilon > 0$*

$$\liminf_{T \rightarrow \infty} \nu_T^\tau \left\{ \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0.$$

Roughly speaking, this theorem means that any non-vanishing analytic function can be uniformly approximated by the Riemann zeta function $\zeta(s)$. Note that the original statement of Voronin in [54] is actually, weaker. The above is the formulation by Reich [37].

As a generalization of Theorem 0.0.5, Voronin also proved the next theorem, that means a collection of Dirichlet L -functions of non-equivalent characters uniformly approximate simultaneously non-vanishing analytic functions. In slightly different form this was also establish by Gonek [7] and Bagchi [3], independently (all these papers are unpublished doctoral theses).

Theorem 0.0.6 ([39, Theorem 1.10]). *For $1 \leq l \leq m$, let $\chi_1 \pmod{q_1}, \dots, \chi_m \pmod{q_m}$ be pairwise non-equivalent Dirichlet characters, and $f_l(s)$ be a non-vanishing function analytic in the interior of K_l and continuous on K_l . Then for every $\varepsilon > 0$*

$$\liminf_{T \rightarrow \infty} \nu_T^\tau \left\{ \sup_{1 \leq l \leq m} \sup_{s \in K_l} |L(s + i\tau, \chi_l) - f_l(s)| < \varepsilon \right\} > 0.$$

We call this type of results the joint universality. By using this theorem, we obtain the following theorem, joint functional independence.

Theorem 0.0.7 ([16, Theorem 6.6.3]). *Let F_k , $k = 0, 1, \dots, n$, be continuous functions, and let*

$$\sum_{k=0}^n s^k F_k \left(L(s, \chi_1), \dots, L^{(j-1)}(s, \chi_1), \dots, L(s, \chi_m), \dots, L^{(j-1)}(s, \chi_m) \right) = 0$$

be valid identically for $s \in \mathbb{C}$. Then F_k , for $k = 0, 1, \dots, n$.

Afterwards many mathematicians consider generalizations of universality, for example Lerch zeta functions, Hecke L -functions, Rakin-Selberg L -functions (see [20] and [39]).

Section 1.1 is based on References of [30]. We prove a functional relation for the Tornheim double zeta function, and obtain simple proofs of some known formulas for special values of Tornheim and Euler-Zagier double zeta functions.

Section 1.2 is mainly based on Reference of [33]. Section 1.2.7 and 1.2.8 is based on [32] and [29], respectively. We obtain functional relations for Witten zeta functions and obtain new proofs of known results on several types double and triple zeta function.

Section 2.1 is based on Reference of [34]. We define several types t -universality for Lerch zeta functions and consider the relations among those. Next we show the existence of all of these joint t -universality.

Section 2.2 is based on Reference of [35]. We show joint t -universality for Lerch zeta function, generalize Lerch zeta functions, and obtain the joint t -universality for them. Moreover, we show examples of the non-existence of the joint t -universality for Lerch zeta functions and generalized Lerch zeta functions.

Therefore we can say that Chapter 1 is the study finding the functional relation for multiple zeta function, on the contrary, Chapter 2 is the universality theorem, which can show the non-existence of functional relation for multiple zeta function.

1 Functional and value relations and explicit evaluation formulae for several types of multiple zeta functions

1.1 Tornheim double zeta function

1.1.1 Introduction for Tornheim double zeta function

Definition 1.1.1. *The Tornheim double zeta function $T(s, t, u)$, for $s, t, u \in \mathbb{C}$, $\Re(s + u) > 1$, $\Re(t + u) > 1$ and $\Re(s + t + u) > 2$, is defined by*

$$T(s, t, u) := \sum_{m, n=1}^{\infty} \frac{1}{m^s n^t (m+n)^u}. \quad (1.1.1)$$

This function $T(s, t, u)$ is a generalization of the Riemann zeta function $\zeta(s)$. The function $T(s, t, u)$ is continued meromorphically to \mathbb{C}^3 in [21]. By the definition, we have

$$T(s, t, u) = T(t, s, u), \quad T(s, t, 0) = \zeta(s)\zeta(t).$$

The case of $t = 0$, that is $T(s, 0, u)$, is called the Euler-Zagier double zeta function [57].

The values $T(a, b, c)$ for $a, b, c \in \mathbb{N}$ were first investigated by Tornheim [43] in 1950 and later Mordell [28] in 1958. Tornheim [43, Theorem 7] showed that $T(a, b, c)$ can be expressed as a polynomial in $\{\zeta(j) \mid 2 \leq j \leq a + b + c\}$ with rational coefficients when $a + b + c$ is odd, and that the same is true for $T(2r, 2r, 2r)$ and $T(2r - 1, 2r, 2r + 1)$ [43, Theorem 8], but he did not give the coefficients. Mordell [28, Theorem III] proved that $T(2r, 2r, 2r) = k_r \pi^{6r}$ for some rational number k_r .

In 1985 Subbarao and Sitaramachandrarao [40, Theorem 4.1] explicitly determined $T(2p, 2q, 2r) + T(2q, 2r, 2p) + T(2r, 2p, 2q)$ ($p, q, r \in \mathbb{N}$). Then, by taking $p = q = r$, they gave an explicit formula for $T(2r, 2r, 2r)$ ($r \in \mathbb{N}$) [40, Remark 3.1]. In 1996 Huard, Williams and Zhang [10, Theorems 1,2,3] determined $T(r, 0, N - r)$ ($r \in \mathbb{N}$, $N \in 2\mathbb{N} + 1$, $1 \leq r \leq N - 2$), $T(r, s, N - r - s)$ ($r, s \in \mathbb{N} \cup \{0\}$, $N \in 2\mathbb{N} + 1$, $1 \leq r + s \leq N - 1$, $0 \leq r, s \leq N - 2$) and $T(r, r, r)$ ($r \in \mathbb{N}$). In 2002 Tsumura [44, Theorem 1] proved that $T(p, q, r) + (-1)^p T(p, r, q) + (-1)^{p+r} T(r, q, p)$ is a polynomial in $\{\zeta(k) \mid 2 \leq k \leq p + q + r\}$ with rational coefficients for $p, q, r \in \mathbb{N} \cup \{0\}$ with $p + q \geq 2$ and $r \geq 2$.

Recently, Espinosa and Moll provide an explicit formula for $T(s, t, u)$, $s, t, u \in \mathbb{R}$ in terms of integrals involving Hurwitz zeta functions (see [5, Proposition 2.1 and Theorem 2.4]). In 2006 Tsumura [52, Theorem 4.5] proved the following functional relation:

$$\begin{aligned} & T(a, b, s) + (-1)^b T(b, s, a) + (-1)^a T(s, a, b) \\ &= 2 \sum_{\substack{j=0 \\ j \equiv a \pmod{2}}}^a (2^{1-a+j} - 1) \zeta(a - j) \sum_{l=0}^{j/2} \frac{(i\pi)^{2l}}{(2l)!} \binom{b-1+j-2l}{j-2l} \zeta(b+j+s-2l) \\ &- 4 \sum_{\substack{j=0 \\ j \equiv a \pmod{2}}}^a (2^{1-a+j} - 1) \zeta(a - j) \sum_{l=0}^{(j-1)/2} \frac{(i\pi)^{2l}}{(2l+1)!} \sum_{\substack{k=0 \\ k \equiv b \pmod{2}}}^b \zeta(b-k) \binom{k-1+j-2l}{j-2l-1} \zeta(k+j+s-2l) \end{aligned} \quad (1.1.2)$$

(where (2) means mod 2), for $a, b \in \mathbb{N} \cup \{0\}$, $b \geq 2$, $s \in \mathbb{C}$ except for the singular points of each side of this formula. In this section, we prove the following result.

Theorem 1.1.2. *For all $a, b \in \mathbb{N}$ and $s \in \mathbb{C}$ except for the singular points, we have*

$$\begin{aligned} & T(a, b, s) + (-1)^b T(b, s, a) + (-1)^a T(s, a, b) \\ &= \frac{2}{a!b!} \sum_{k=0}^{\max(a,b)/2} \left\{ a \binom{b}{2k} + b \binom{a}{2k} \right\} (a+b-2k-1)!(2k)!\zeta(2k)\zeta(a+b+s-2k). \end{aligned} \quad (1.1.3)$$

This functional relation is considerably simpler than that of Tsumura. However, in [25], it is shown that Tsumura's expression coincides with the author's one. In Section 1.1.2 we prove the functional relation 1.1.3. Moreover, in Section 1.1.3, we obtain new proofs of formulas for the special values of $T(a, b, c)$ mentioned in the introduction by using the functional relation (1.1.3).

1.1.2 Proof of Theorem 1.1.2 and new proofs of known formulas

We denote by $B_j(x)$ the Bernoulli polynomial of order j defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{j=0}^{\infty} B_j(x) \frac{t^j}{j!}, \quad |t| < 2\pi. \quad (1.1.4)$$

It is known (see [1, p. 266, (22) and p. 267, (24)]) that

$$B_{2j} := B_{2j}(0) = (-1)^{j+1} 2(2j)! (2\pi)^{-2j} \zeta(2j), \quad j \in \mathbb{N}, \quad (1.1.5)$$

$$B_j(x) = -\frac{j!}{(2\pi i)^j} \lim_{K \rightarrow \infty} \sum_{\substack{k=-K \\ k \neq 0}}^K \frac{e^{2\pi i k x}}{k^j}, \quad j \in \mathbb{N}. \quad (1.1.6)$$

For $k \in \mathbb{Z}$, $j \in \mathbb{N}$ we have

$$\int_0^1 e^{-2\pi i k x} B_j(x) dx = \begin{cases} 0 & k = 0, \\ -(2\pi i k)^{-j} j! & k \neq 0. \end{cases} \quad (1.1.7)$$

In fact, the case of $k = 0$ is obvious, and in the case of $k \neq 0$, we have (1.1.7) by using (1.1.6). Next we quote Carlitz's formula [1, p. 276 19.(b)], for $p + q \geq 2$, which is

$$B_p(x) B_q(x) = \sum_{k=0}^{\max(p,q)/2} \left\{ p \binom{q}{2k} + q \binom{p}{2k} \right\} \frac{B_{2k} B_{p+q-2k}(x)}{p+q-2k} - (-1)^p \frac{p! q!}{(p+q)!} B_{p+q}. \quad (1.1.8)$$

We adopt a simple proof of Theorem 1.1.2, which is obtained in [31].

Proof of Theorem 1.1.2. Firstly, we assume $\Re(s) > 1$. We have

$$\begin{aligned} \lim_{K \rightarrow \infty} \int_0^1 \sum_{m=1}^K \frac{e^{2\pi i m x}}{m^a} \sum_{n=1}^K \frac{e^{2\pi i n x}}{n^b} \sum_{l=1}^K \frac{e^{-2\pi i l x}}{l^s} dx &= T(a, b, s), \\ \lim_{K \rightarrow \infty} \int_0^1 \sum_{m=-K}^{-1} \frac{e^{2\pi i m x}}{m^a} \sum_{n=1}^K \frac{e^{2\pi i n x}}{n^b} \sum_{l=1}^K \frac{e^{-2\pi i l x}}{l^s} dx &= (-1)^a T(s, a, b), \\ \lim_{K \rightarrow \infty} \int_0^1 \sum_{m=1}^K \frac{e^{2\pi i m x}}{m^a} \sum_{n=-K}^{-1} \frac{e^{2\pi i n x}}{n^b} \sum_{l=1}^K \frac{e^{-2\pi i l x}}{l^s} dx &= (-1)^b T(b, s, a), \\ \lim_{K \rightarrow \infty} \int_0^1 \sum_{m=-K}^{-1} \frac{e^{2\pi i m x}}{m^a} \sum_{n=-K}^{-1} \frac{e^{2\pi i n x}}{n^b} \sum_{l=1}^K \frac{e^{-2\pi i l x}}{l^s} dx &= 0. \end{aligned}$$

Therefore we obtain

$$T(a, b, s) + (-1)^b T(b, s, a) + (-1)^a T(s, a, b) = \int_0^1 \lim_{K \rightarrow \infty} \sum_{\substack{m=-K \\ m \neq 0}}^K \frac{e^{2\pi i m x}}{m^a} \sum_{\substack{k=-K \\ k \neq 0}}^K \frac{e^{2\pi i k x}}{k^b} \sum_{l=1}^K \frac{e^{-2\pi i l x}}{l^s} dx.$$

Changing the order of limitation and integration is justified by bounded convergence. By using (1.1.5), (1.1.6), (1.1.7) and (1.1.8), we obtain (1.1.3) in this region. By analytic continuation, we have (1.1.3) for all $a, b \in \mathbb{N}$, and $c \in \mathbb{C}$ except for singular points of each side of this formula. \square

Next, from our theorem we deduce formulas for the special values of $T(a, b, c)$ ($a, b, c \in \mathbb{N}$) mentioned in the introduction. By taking $a = 2p$, $b = 2q$, $s = 2r$ in (1.1.3), we have

$$\begin{aligned} & T(2p, 2q, 2r) + T(2q, 2r, 2p) + T(2r, 2p, 2q) \\ &= \frac{2}{(2p)!(2q)!} \sum_{k=0}^{\max(p,q)} \left\{ 2p \binom{2q}{2k} + 2q \binom{2p}{2k} \right\} (2p + 2q - 2k - 1)! (2k)! \zeta(2k) \zeta(2p + 2q + 2r - 2k). \end{aligned}$$

This formula coincides with [40, Theorem 4.1] (There is a misprint in [40, Theorem 4.1], “min” is to be replaced by “max”.) Putting $a = b = s = r$ in (1.1.3) we have, after easy computations of binomial coefficients,

$$T(r, r, r) = \frac{4}{1 + 2(-1)^r} \sum_{k=0}^{r/2} \binom{2r - 2k - 1}{2k - 1} \zeta(2k) \zeta(3r - 2k).$$

This formula is [10, Theorem 3].

For $a, b, c \in \mathbb{N}$, we define $N(a, b, c)$ as half of the right-hand side of (1.1.3). We recall the harmonic product formula

$$T(a, 0, b) + T(b, 0, a) = \zeta(a)\zeta(b) - \zeta(a + b).$$

Putting $s = 0$ in (1.1.3) and multiplying $(-1)^a$, we have

$$(-1)^a \zeta(a)\zeta(b) + (-1)^{a+b} T(b, 0, a) + T(a, 0, b) = 2(-1)^a N(a, b, 0).$$

When $a + b \in 2\mathbb{N} + 1$, we can remove $T(b, 0, a)$ by summing the above two formulas. Hence we have

$$T(a, 0, b) = -\frac{\zeta(a + b)}{2} + \frac{(1 + (-1)^b)}{2} \zeta(a)\zeta(b) + (-1)^a N(a, b, 0) \quad (1.1.9)$$

for all $a, b \geq 2$, $a + b \in 2\mathbb{N} + 1$. Next by changing the variables in (1.1.3), we obtain

$$\begin{cases} (-1)^b T(a, b, c) + T(b, c, a) + (-1)^c T(c, a, b) = 2N(b, c, a), \\ (-1)^a T(a, b, c) + (-1)^c T(b, c, a) + T(c, a, b) = 2N(c, a, b). \end{cases}$$

In the case of $a + b + c \in 2\mathbb{N} + 1$, we can remove $T(b, c, a)$ and $T(c, a, b)$ by multiplying the former formula by $(-1)^b$ and the latter formula by $(-1)^a$, and summing the resulting formulas. Therefore we have

$$T(a, b, c) = (-1)^b N(b, c, a) + (-1)^a N(c, a, b), \quad a + b + c \in 2\mathbb{N} + 1. \quad (1.1.10)$$

By the definition of Tornheim double zeta function, we obtain

$$T(1, 1, c) = 2T(1, 0, c + 1).$$

Hence we can calculate $T(1, 0, c + 1)$, if $c + 1 \in 2\mathbb{N}$. Therefore we obtain another proof of [10, Theorem 1,2]. Moreover we obtain

$$T(p, q, r) + (-1)^p T(p, r, q) + (-1)^{p+r} T(r, q, p) = 2(-1)^p N(p, r, q)$$

by taking $a = p$, $b = r$ and $s = q$ in (1.1.3), and multiplying $(-1)^p$. Hence we obtain another proof of [44, Theorem 1], because $N(p, q, r)$ is a polynomial in $\{\zeta(k) \mid 2 \leq k \leq p + q + r\}$ with rational coefficients for $p, q, r \in \mathbb{N} \cup \{0\}$ with $p + q \geq 2$ and $r \geq 2$.

1.2 Multiple L-values and Witten zeta functions

1.2.1 Introduction for multiple L-values and Witten zeta functions

Definition 1.2.1. For $s_1, s_2, s_3, s_4, s_5, s_6 \in \mathbb{C}$ and $0 < \alpha, \beta, \gamma, \delta, \eta, \theta \leq 1$, we define generalized Witten zeta functions for $sl(2)$, $sl(3)$ and $sl(4)$ by

$$\begin{aligned} \text{Li}(s_1; \alpha) &:= \sum_{n=1}^{\infty} \frac{e^{2\pi i n \alpha}}{n^{s_1}}, \\ \zeta_{sl(3)}(s_1, s_2, s_3; \alpha, \beta, \gamma) &:= \sum_{l, m=1}^{\infty} \frac{e^{2\pi i l \alpha} e^{2\pi i m \beta} e^{2\pi i (l+m) \gamma}}{l^{s_1} m^{s_2} (l+m)^{s_3}}, \\ \zeta_{sl(4)}(s_1, s_2, s_3, s_4, s_5, s_6; \alpha, \beta, \gamma, \delta, \eta, \theta) &:= \sum_{l, m, n=1}^{\infty} \frac{e^{2\pi i l \alpha} e^{2\pi i m \beta} e^{2\pi i n \gamma} e^{2\pi i (l+m) \delta} e^{2\pi i (m+n) \eta} e^{2\pi i (l+m+n) \theta}}{l^{s_1} m^{s_2} n^{s_3} (l+m)^{s_4} (m+n)^{s_5} (l+m+n)^{s_6}}, \end{aligned}$$

in the region of convergence. Originally, Zagier [57] defined Witten zeta functions by

$$\zeta_g(s) := \sum_{\rho} (\dim \rho)^{-s},$$

where $s \in \mathbb{C}$ and ρ runs over all finite dimensional irreducible representations of a certain semisimple Lie algebra g .

The values of $\zeta_g(2k)$ for $k \in \mathbb{N}$ had been studied by Witten [55] before Zagier's work in order to calculate the volumes of certain moduli spaces. As a generalization of $\zeta_g(s)$, Matsumoto [22] defined the Witten zeta function for $g = so(5)$ of several complex variables and proved its analytic continuation. Then Matsumoto and Tsumura [24] introduced Witten zeta function of several variables for $g = sl(n)$ ($n \geq 2$).

In the case of $\alpha = \beta = \gamma = \delta = \eta = \theta = 1$, we write them as $\zeta(s_1)$, $\zeta_{sl(3)}(s_1, s_2, s_3)$ and $\zeta_{sl(4)}(s_1, s_2, s_3, s_4, s_5, s_6)$, respectively, which coincide with the zeta function introduced by Matsumoto and Tsumura [24]. Needless to say, $\zeta(s_1)$ is the Riemann zeta function. The function $\zeta_{sl(3)}(s_1, s_2, s_3)$ is called the Tornheim double zeta function (in Section 1.1. we write this $T(s, t, u)$).

As a triple analogue of $\zeta_{sl(3)}(s_1, s_2, s_3)$, the function $\zeta_{sl(4)}(s_1, s_2, s_3, 0, 0, s_6)$ is called the Mordell-Zagier triple zeta function, continued analytically, and a functional relation has been obtained in [25]. Functional relations for $\zeta_{sl(4)}(s_1, s_2, s_3, s_4, s_5, s_6)$ were proved in Matsumoto and Tsumura [24]. Afterwards, the author [32] showed a new proof of these functional relations and another type of functional relation for $\zeta_{sl(4)}(s_1, s_2, s_3, s_4, s_5, s_6)$. In [12], Komori, Matsumoto and Tsumura discuss these functions as zeta-functions of root systems. In [13], independently of the present work of the author, they prove functional relations for Witten zeta functions of other Lie algebra.

On the other hand, Euler-Zagier multiple zeta values, the case of double zeta of which is written by $\zeta_{sl(3)}(0, a, b)$, has been investigated by a lot of mathematicians (see for example Hoffman's web page). These values are related to knot theory, cohomology of motives and so on. Arakawa and Kaneko defined multiple L-values, the case of double L-values of which is written by $\zeta_{sl(3)}(0, a, b; 1, \beta, \gamma)$. They studied the regularized double shuffle and the derivation relations of multiple L-values and gave some applications in [2]. Afterwards Terhune gave some evaluations for double L-values in [41].

In this Section, we obtain functional relations for generalized Witten zeta functions. All functional relations for Witten zeta functions proved in this paper are derived from the double L-value relation (1.2.2). We can obtain new functional relations since double L-value relation (1.2.2) deduce the Bernoulli polynomial formula (1.1.8), which is a key of the proof of functional relations showed in [30], [31] and [32]. By multiplying the double L-value relation (1.2.2) by some functions and integrating the resulting expression, we obtain all functional relations for generalized Witten zeta functions in this paper (see the proofs of Lemma 1.2.11, Theorems 1.2.7, 1.2.21, 1.2.24, 1.2.26 and 1.2.27). Hence we may guess that "All of functional relations for Witten zeta functions are constructed by multiple L-value relations." As an evidence, we will show that a lot of known results on double L-values and Witten zeta functions are induced from (1.2.2).

This section is divided into 6 subsections. In Section 1.2.2, we prove (1.2.2). We show the functional relation for $\zeta_{sl(3)}(s_1, s_2, s_3; \alpha, \beta, \gamma)$ which is a generalization of [30, Theorem 1.2] and [31, Theorem 3.2] in Theorem 1.2.7. By using Theorem 1.2.7, we obtain new proofs of Tornheim [43, Theorem 2] (resp. Tsumura [51, Theorem 2] and [46, Theorem 3.4]), in Proposition 1.2.9 (resp. Proposition 1.2.10). In Section 1.2.4, we obtain functional relations for $\zeta_{sl(3)}(s_1, s_2, s_3)$ with characters in Lemmas 1.2.11, 1.2.12, 1.2.13 and Theorem 1.2.14. By these theorems, we can obtain explicit expression for double L-values and Tornheim double zeta with character. For example, we can obtain new proofs of Terhune [41, Theorem 1] and Tsumura [48, Theorem 3.1], (see Propositions 1.2.15 and 1.2.18). In Section 1.2.5, we show a functional relation for Witten zeta functions attached to $so(5)$ in Theorems 1.2.21 and 1.2.24. In Section 1.2.6, we obtain functional relations (1.2.44) and (1.2.45), which are generalizations of [24, Theorem 5.9 and 5.10] and [32, Theorems 2.1 and 2.3]. In Theorem 1.2.27, we obtain functional relation (1.2.46).

1.2.2 Double L-values

We define double L-values by

$$L(a, b; x, y) := \lim_{R \rightarrow \infty} \sum_{\substack{R \geq m_1 + m_2 \\ m_1, m_2 = 1}} \frac{e^{2\pi i m_1 x} e^{2\pi i m_2 y}}{(m_1 + m_2)^a m_2^b}, \quad a, b \in \mathbb{N}, \quad x, y \in \mathbb{R}. \quad (1.2.1)$$

These and their multiple sum versions have been already defined in Arakawa and Kaneko [2] for the case of $x, y \in \mathbb{Q}$. In the case of $a \geq 2$, $L(a, b; x, y)$ are absolutely convergent. As for the case $a = 1$, we have the following criterion.

Proposition 1.2.2. *For $x \in \mathbb{R} \setminus \mathbb{Z}$, the series $L(1, b; x, y)$ is uniformly convergent in the wider sense in $(0, 1)$.*

Proof. We can obtain this proposition by modifying the proof of [2, Proposition 1.1], which is based on Abel's summation. \square

For the sake of simplicity, we say that the index set $(a, b; x, y)$ for which the series $L(a, b; x, y)$ is convergent is *admissible*. The next double L-values relation plays an important role in this section.

Theorem 1.2.3. *For any admissible index, we have*

$$\begin{aligned} & \text{Li}(a; x) \text{Li}(b; x - y) + (-1)^b L(a, b; x, y) + (-1)^a L(b, a; x - y, -y) \\ &= \sum_{j=1}^a \binom{a+b-j-1}{a-j} \text{Li}(a+b-j; x-y) (\text{Li}(j; y) + (-1)^j \text{Li}(j; -y)) \\ & \quad + \sum_{j=1}^b \binom{a+b-j-1}{b-j} \text{Li}(a+b-j; x) (\text{Li}(j; -y) + (-1)^j \text{Li}(j; y)) \\ & \quad - \binom{a+b-1}{a} \text{Li}(a+b; x-y) - \binom{a+b-1}{b} \text{Li}(a+b; x). \end{aligned} \quad (1.2.2)$$

We remark that when $y \rightarrow 0$ the term $j = 1$ of the first sum on the right-hand side of (1.2.2) and the term $j = 1$ of the second sum are cancelled.

Proof. We use the identity

$$\begin{aligned} & \frac{(-1)^b e^{2\pi i r x} e^{2\pi i k y}}{r^u (k+r)^a k^{b+v}} = \\ & \sum_{j=1}^b \binom{a+b-j-1}{b-j} \frac{(-1)^j e^{2\pi i r x} e^{2\pi i k y}}{r^{a+b-j+u} k^{j+v}} + \sum_{j=1}^a \binom{a+b-j-1}{a-j} \frac{e^{2\pi i r x} e^{2\pi i k y}}{r^{a+b-j+u} (k+r)^j k^v} \end{aligned}$$

where $k, r \in \mathbb{N}$, $u, v \in \mathbb{C}$, $\Re(u) > 1$ and $\Re(v) > 1$ which follows from Huard, Williams and Zhang [10, (1.12)]. Summing on k and r , we have

$$\begin{aligned} (-1)^b \zeta_{sl(3)}(u, b+v, a; x, y, 1) &= \sum_{j=1}^b \binom{a+b-j-1}{b-j} (-1)^j \text{Li}(a+b-j+u; x) \text{Li}(j+v; y) \\ & \quad + \sum_{j=1}^a \binom{a+b-j-1}{a-j} \zeta_{sl(3)}(a+b-j+u, v, j; x, y, 1). \end{aligned} \quad (1.2.3)$$

By analytic continuation ([31, Theorem 2.1]), we find that the above formula is valid in the case of $u = v = 0$. We may regard that the formula with $u = v = 0$ is a kind of shuffle product formula. Next applying the harmonic product formula

$$\text{Li}(s; x-y) \text{Li}(t; y) = L(s, t; x-y, x) + L(t, s; y, x) + \text{Li}(s+t; x) \quad (1.2.4)$$

to the case $u = v = 0$ in (1.2.3) and changing the order of sum involving the term $L(a+b-j, j; x-y, x)$, we have

$$\begin{aligned} (-1)^b L(a, b; x, y) &= \sum_{j=1}^b \binom{a+b-j-1}{b-j} (-1)^j \text{Li}(a+b-j; x) \text{Li}(j; y) \\ &\quad + \sum_{j=1}^a \binom{a+b-j-1}{a-j} (\text{Li}(a+b-j; x-y) \text{Li}(j; y) - \text{Li}(a+b; x)) \\ &\quad - \sum_{h=0}^{a-1} \binom{b-1+h}{h} L(b+h, a-h; x-y, x). \end{aligned}$$

By changing variables and parameters as $x \mapsto x-y$ and $y \mapsto -y$ in the above formula, we have

$$\begin{aligned} (-1)^a L(b, a; x-y, -y) &= \sum_{j=1}^a \binom{a+b-j-1}{a-j} (-1)^j \text{Li}(a+b-j; x-y) \text{Li}(j; -y) \\ &\quad + \sum_{j=1}^b \binom{a+b-j-1}{b-j} (\text{Li}(a+b-j; x) L(j; -y) - \text{Li}(a+b; x-y)) \\ &\quad - \sum_{h=0}^{b-1} \binom{a-1+h}{h} L(a+h, b-h; x, x-y). \end{aligned}$$

By using the shuffle product formula of Arakawa and Kaneko [2, p.972, (8)] (in [2], this equation is shown for only $x, y \in \mathbb{Q}$, but this formula holds for $x, y \in \mathbb{R}$ by continuity of double L-values for admissible index which are derived by uniform convergence in the wider sense in $(0,1)$, and denseness of \mathbb{Q} in \mathbb{R}), we have

$$\begin{aligned} \text{Li}(a; x) \text{Li}(b; x-y) &= \\ &= \sum_{h=0}^{a-1} \binom{b-1+h}{h} \text{Li}(b+h, a-h; x-y, x) + \sum_{h=0}^{b-1} \binom{a-1+h}{h} \text{Li}(b+h, a-h; x, x-y), \end{aligned}$$

and applying the well-known formula $\sum_{j=1}^b \binom{a+b-j-1}{b-j} = \binom{a+b-1}{a}$, we obtain the theorem. \square

We write by $K(a, b; x, y)$ the right hand side of (1.2.2). In the case of $(x, y) = (1, 1)$, the next proposition has already been proved by Huard, Williams and Zhang [10, Theorem 1].

Proposition 1.2.4. *Suppose $a+b \in 2\mathbb{N}+1$, and (x, y) is equal to one of $(1, 1)$, $(1/2, 1)$, $(1, 1/2)$, or $(1/2, 1/2)$. For admissible indices, we have*

$$\begin{aligned} 2L(a, b; x, y) &= \text{Li}(a; -x) \text{Li}(b; x-y) - \text{Li}(a+b; -y) \\ &\quad - (-1)^b \text{Li}(a; x) \text{Li}(b; x-y) + (-1)^b K(a, b; x, y). \end{aligned} \tag{1.2.5}$$

Proof. Multiplying (1.2.2) by $(-1)^b$, we have

$$(-1)^b \text{Li}(a; x) \text{Li}(b; x-y) + L(a, b; x, y) + (-1)^{a+b} L(b, a; x-y, -y) = (-1)^b K(a, b; x, y).$$

By summing the above formula and the harmonic product formula (1.2.4) with changing the variables as $s \mapsto a$ and $t \mapsto b$ and the parameters as $x-y \mapsto -x$ and $y \mapsto x-y$, we can remove $L(b, a; x-y, -y)$. In these cases, we have $L(a, b; x, y) = L(a, b; -x, -y)$. Hence we obtain this proposition. \square

Remark 1.2.5. *By taking $a = b$ in (1.2.2), we can obtain an explicit formula for $L(a, a; x, 1)$. We can also obtain an explicit evaluation formula for $L(a, a; y, 2y)$ by putting $a = b$ in (1.2.4).*

1.2.3 $sl(3)$

Our main result in this section is Theorem 1.2.7 below. The special case $\beta = \gamma = 1$ of this theorem was first obtained in Tsumura [52, Theorem 4.5] (see (1.1.2)) and [30, Theorem 1.2] (see (1.1.3)), and then, the case $\beta = 1$ was proved in [31, Theorem 3.2]. Because of (1.2.13) with the parameter β , which is proved by (1.2.2), we can obtain functional relations (1.2.25), (1.2.40), (1.2.42) and (1.2.46), for example. This is a novel point of the present paper ; we have only obtained the case of $\beta = 1$,

$$\begin{aligned} & \zeta_{sl(3)}(a, b, s; 1, 1, \gamma) + (-1)^b \zeta_{sl(3)}(b, s, a; 1, \gamma, 1) + (-1)^a \zeta_{sl(3)}(s, a, b; \gamma, 1, 1) \\ &= 2 \sum_{j=0}^{\max\{a,b\}/2} \left\{ \binom{a+b-2j-1}{a-2j} + \binom{a+b-2j-1}{b-2j} \right\} \zeta(2j) \text{Li}(a+b+s-j; \gamma), \end{aligned} \quad (1.2.6)$$

in [30, Theorem 1.2] and [31, Theorem 3.2], since these are based on (1.1.8), which is weaker than (actually deduced from) (1.2.2) (see Lemma 1.2.22). Firstly, we will continue $\zeta_{sl(3)}(s, t, u; \alpha, \beta, \gamma)$ meromorphically.

Theorem 1.2.6. *Let $k \in \mathbb{N} \cup \{0\}$. The function $\zeta_{sl(3)}(s_1, s_2, s_3; \alpha, \beta, \gamma)$ can be continued meromorphically to \mathbb{C}^3 , and all of its singularities are located on the subsets of \mathbb{C}^3 defined by the following equations ;*

$$s_2 = 1 - k, \quad \text{if } \alpha + \gamma \not\equiv 1, \beta + \gamma \equiv 1 \pmod{1}, \quad (1.2.7)$$

$$s_1 = 1 - k, \quad \text{if } \alpha + \gamma \equiv 1, \beta + \gamma \not\equiv 1 \pmod{1}, \quad (1.2.8)$$

$$\text{no singularity} \quad \text{if } \alpha + \gamma \not\equiv 1, \beta + \gamma \not\equiv 1 \pmod{1}. \quad (1.2.9)$$

Proof. For $0 < \alpha < 1$, we recall that $\text{Li}(s; \alpha)$ is analytically continuable to an entire function. We will use the Mellin-Barnes formula

$$(1 + \lambda)^{-s} = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(s+z)\Gamma(-z)}{\Gamma(s)} \lambda^z dz, \quad (1.2.10)$$

where $\Re(s) > 0$, $|\arg \lambda| < \pi$, $\lambda \neq 0$, $c \in \mathbb{R}$ with $-\Re(s) < c < 0$ and the path (c) of integration is the vertical line $\Re(z) = c$. Assume $s_j \in \mathbb{C}$ with $\Re(s_j) > 1$ ($j = 1, 2, 3$). Then $\zeta_{sl(3)}(s_1, s_2, s_3; \alpha, \beta, \gamma)$ is convergent absolutely. Let $(l+m)^{-s_3} = l^{-s_3}(1+m/l)^{-s_3}$ and substitute (1.2.10) with $\lambda = l/m$. Assume $-\Re(s_2) < c < 0$. Then we have

$$\begin{aligned} & \zeta_{sl(3)}(s_1, s_2, s_3; \alpha, \beta, \gamma) \\ &= \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(s_3+z)\Gamma(-z)}{\Gamma(s_3)} \sum_{l=1}^{\infty} \frac{e^{2\pi i l(\alpha+\gamma)}}{l^{s_1+s_3}} \sum_{m=1}^{\infty} \frac{e^{2\pi i m(\beta+\gamma)}}{m^{s_2}} \left(\frac{l}{m}\right)^z dz \\ &= \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(s_3+z)\Gamma(-z)}{\Gamma(s_3)} \text{Li}(s_1+s_3-z; \alpha+\gamma) \text{Li}(s_2+z; \beta+\gamma) dz. \end{aligned} \quad (1.2.11)$$

Hence, by considering singularities of $\Gamma(s)$, $\text{Li}(s; \alpha + \gamma)$ and $\text{Li}(s; \beta + \gamma)$, we see that the possible singularities of the integrand of (1.2.11) are determined by $z = -s_3 - k$, $z = k$ ($k \in \mathbb{N} \cup \{0\}$), $z = s_1 + s_3 - 1$ and $z = 1 - s_2$.

Now we shift the path $\Re(z) = c$ to $\Re(z) = M - \varepsilon$ for a sufficiently large $M \in \mathbb{N}$ and a sufficiently small positive $\varepsilon \in \mathbb{R}$. In the case of $\alpha + \gamma \not\equiv 1$, all the relevant singularities are $z = r$ ($0 \leq r \leq M - 1$). Counting their residues, and using the relations

$$\begin{aligned} & \frac{(-1)^k \Gamma(s+k)}{k! \Gamma(s)} = (-1)^k \binom{s+k-1}{k} = \binom{-s}{k}, \\ & \Gamma(-k-\delta) = \frac{\Gamma(1-\delta)}{(-\delta) \cdots (-k-\delta)} = -\frac{(-1)^k}{k!} \left(\frac{1}{\delta} + O(1) \right), \quad \delta \rightarrow 0, \end{aligned}$$

$$\frac{\Gamma(s_3 + k)}{\Gamma(s_3)} \operatorname{Res}_{z=k} \Gamma(-z) = -\binom{-s_3}{k}.$$

for $k \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} & \zeta_{sl(3)}(s_1, s_2, s_3; \alpha, \beta, \gamma) \\ &= \sum_{k=0}^{M-1} \binom{-s_3}{k} \operatorname{Li}(s_1 + s_3 - k; \alpha + \gamma) \operatorname{Li}(s_2 + k; \beta + \gamma) \\ & \quad + \frac{1}{2\pi i} \int_{(M-\varepsilon)} \frac{\Gamma(s_3 + z) \Gamma(-z)}{\Gamma(s_3)} \operatorname{Li}(s_1 + s_3 - z; \alpha + \gamma) \operatorname{Li}(s_2 + z; \beta + \gamma) dz. \end{aligned} \quad (1.2.12)$$

Since M can be taken arbitrarily large, (1.2.12) implies the meromorphic continuation of the function $\zeta_{sl(3)}(s_1, s_2, s_3; \alpha, \beta, \gamma)$ to \mathbb{C}^3 . In the case of $\alpha + \gamma \not\equiv 1 \pmod{1}$, we have (1.2.7) and (1.2.9) by the first term of the right-hand side of (1.2.12). In the case of $\beta + \gamma \not\equiv 1 \pmod{1}$, we also obtain (1.2.12) by using the following formula

$$\zeta_{sl(3)}(s_1, s_2, s_3; \alpha, \beta, \gamma) = \zeta_{sl(3)}(s_2, s_1, s_3; \beta, \alpha, \gamma).$$

□

Theorem 1.2.7. *For all $a, b \in \mathbb{N}$ and $s \in \mathbb{C}$ except for the singular points, we have*

$$\begin{aligned} & \zeta_{sl(3)}(a, b, s; 1, \beta, \gamma) + (-1)^b \zeta_{sl(3)}(b, s, a; -\beta, \gamma, 1) + (-1)^a \zeta_{sl(3)}(s, a, b; \gamma, 1, \beta) \\ &= \sum_{j=1}^a \binom{a+b-j-1}{a-j} \operatorname{Li}(a+b+s-j; \beta + \gamma) (\operatorname{Li}(j; -\beta) + (-1)^j \operatorname{Li}(j; \beta)) \\ & \quad + \sum_{j=1}^b \binom{a+b-j-1}{b-j} \operatorname{Li}(a+b+s-j; \gamma) (\operatorname{Li}(j; \beta) + (-1)^j \operatorname{Li}(j; -\beta)) \\ & \quad - \binom{a+b-1}{a} \operatorname{Li}(a+b+s; \beta + \gamma) - \binom{a+b-1}{b} \operatorname{Li}(a+b+s; \gamma). \end{aligned} \quad (1.2.13)$$

Remark 1.2.8. *When $\beta \rightarrow 0$ the term $j = 1$ of the first sum on the right-hand side of (1.2.2) and the term $j = 1$ of the second sum are cancelled. Hence we find that (1.2.13) coincides with (1.2.6) when $\beta \rightarrow 0$.*

Proof. Firstly, we assume $a, b \geq 2$, and $\Re(s) > 1$. Change the parameter as $-y \mapsto \beta$ in (1.2.2), multiply by $\sum_{n=1}^{\infty} e^{2\pi i n(\gamma-x)} n^{-s}$ and integrate from 0 to 1 with respect to x . Then we have

$$\begin{aligned} & \int_0^1 \sum_{l=1}^{\infty} \frac{e^{2\pi i l x}}{l^a} \sum_{m=1}^{\infty} \frac{e^{2\pi i m(x+\beta)}}{m^b} \sum_{n=1}^{\infty} \frac{e^{2\pi i n(\gamma-x)}}{n^s} dx = \sum_{l,m=1}^{\infty} \frac{e^{2\pi i m \beta} e^{2\pi i (l+m)\gamma}}{l^a m^b (l+m)^s}, \\ & \int_0^1 \sum_{l,m=1}^{\infty} \frac{e^{2\pi i m x} e^{-2\pi i l \beta}}{(m+l)^a l^b} \sum_{n=1}^{\infty} \frac{e^{2\pi i n(\gamma-x)}}{n^s} dx = \sum_{l,m=1}^{\infty} \frac{e^{-2\pi i l \beta} e^{2\pi i m \gamma}}{(l+m)^a l^b m^s}, \\ & \int_0^1 \sum_{l=1}^{\infty} \frac{e^{2\pi i l(x+\beta)}}{l^{a+b-j}} \sum_{n=1}^{\infty} \frac{e^{2\pi i n(\gamma-x)}}{n^s} dx = \sum_{l=1}^{\infty} \frac{e^{2\pi i l(\beta+\gamma)}}{l^{a+b+s-j}}, \end{aligned}$$

which are $\zeta_{sl(3)}(a, b, s; 1, \beta, \gamma)$ and $\zeta_{sl(3)}(b, s, a; -\beta, \gamma, 1)$ on the left-hand side of (1.2.13) and $\operatorname{Li}(a+b+s-j; \beta + \gamma)$ on the right-hand side of (1.2.13). We can also obtain $\zeta_{sl(3)}(s, a, b; \gamma, 1, \beta)$ and $\operatorname{Li}(a+b+s-j; \gamma)$, similarly. Hence we have (1.2.13) in this region.

Next we consider the case of $a = 1, b \geq 2$. We define $K(a, b, s; \beta, \gamma)$ by the right-hand side of (1.2.13). We modify some basic properties Huard, Williams and Zhang [10, (1.5)] proved by easy computations, for $\alpha, \beta, \gamma \in \mathbb{R}$ and $s, t, u \in \mathbb{C}$ except for the singular points, which are

$$\begin{cases} \zeta_{sl(3)}(s, t-1, u+1; \alpha, \beta, \gamma) + \zeta_{sl(3)}(s-1, t, u+1; \alpha, \beta, \gamma) = \zeta_{sl(3)}(s, t, u; \alpha, \beta, \gamma), \\ \zeta_{sl(3)}(s, t+1, u-1; \alpha, \beta, \gamma) - \zeta_{sl(3)}(s-1, t+1, u; \alpha, \beta, \gamma) = \zeta_{sl(3)}(s, t, u; \alpha, \beta, \gamma), \\ \zeta_{sl(3)}(s+1, t, u-1; \alpha, \beta, \gamma) - \zeta_{sl(3)}(s+1, t-1, u; \alpha, \beta, \gamma) = \zeta_{sl(3)}(s, t, u; \alpha, \beta, \gamma). \end{cases} \quad (1.2.14)$$

For $b \geq 2$, we have

$$\begin{aligned}
& K(2, b, s; \beta, \gamma) \\
&= \zeta_{sl(3)}(2, b, s; 1, \beta, \gamma) + (-1)^b \zeta_{sl(3)}(b, s, 2; -\beta, \gamma, 1) + (-1)^2 \zeta_{sl(3)}(s, 2, b; \gamma, 1, \beta) \\
&= \zeta_{sl(3)}(1, b, s+1; 1, \beta, \gamma) + (-1)^b \zeta_{sl(3)}(b, s+1, 1; -\beta, \gamma, 1) + (-1) \zeta_{sl(3)}(s+1, 1, b; \gamma, 1, \beta) \\
&\quad + K(2, b-1, s+1; \beta, \gamma)
\end{aligned}$$

by (1.2.14) and the result in the case $a, b \geq 2$ which we have already shown. Hence we have to show

$$K(2, b, s; \beta, \gamma) = K(1, b, s+1; \beta, \gamma) + K(2, b-1, s+1; \beta, \gamma), \quad b \geq 2. \quad (1.2.15)$$

Especially, we must treat four binomial coefficients of the right-hand side of (1.2.13). We can see that the first binomial coefficients appear from the formula

$$\binom{2+b-j-1}{2-j} = \binom{1+b-j-1}{1-j} + \binom{2+b-1-j-1}{2-j},$$

which is easily deduced by Pascal's triangle. The other binomial coefficients on the right-hand side of (1.2.13) appear similarly. Hence we obtain (1.2.15). We can prove (1.2.13) for the cases of $a \geq 2$, $b = 1$ and $a = b = 1$ in the same way. \square

By (1.2.13), we obtain the following proposition, which has already been proved by Tornheim.

Proposition 1.2.9 ([43, Theorem 2]). *For $3 \leq N \in \mathbb{N}$, we have*

$$\zeta_{sl(3)}(1, N-2, 1) = \frac{N+1}{2} \zeta(N) - \frac{1}{2} \sum_{j=2}^{N-j} \zeta(j) \zeta(N-j), \quad (1.2.16)$$

$$\zeta_{sl(3)}(1, 1, N-2) = 2\zeta_{sl(3)}(1, 0, N-1) = (N-1) \zeta(N) - \sum_{j=2}^{N-j} \zeta(j) \zeta(N-j). \quad (1.2.17)$$

Proof. The following proof is a simplified version of [43, Theorem 2]. By (1.2.14), we have

$$\zeta_{sl(3)}(j, 0, N-j) + \zeta_{sl(3)}(j-1, 1, N-j) = \zeta_{sl(3)}(j, 1, N-j-1).$$

Summing on j from 2 to $N-2$, we have

$$\sum_{j=2}^{N-2} \zeta_{sl(3)}(j, 0, N-j) + \zeta_{sl(3)}(1, 1, N-2) = \zeta_{sl(3)}(N-2, 1, 1).$$

From the harmonic product formula, we have

$$2 \sum_{j=2}^{N-2} \zeta_{sl(3)}(j, 0, N-j) = \sum_{j=2}^{N-2} \zeta(j) \zeta(N-j) - (N-3) \zeta(N).$$

Taking $a = b = \beta = \gamma = 1$ and $s = N-2$ in (1.2.13), we obtain

$$2\zeta_{sl(3)}(N-2, 1, 1) - \zeta_{sl(3)}(1, 1, N-2) = 2\zeta(N).$$

Therefore we have the explicit evaluation formulae for $\zeta_{sl(3)}(N-2, 1, 1)$ and $\zeta_{sl(3)}(1, 1, N-2)$ from the above three formulae. We can also obtain evaluation formula for $\zeta_{sl(3)}(1, 0, N-1)$ by (1.2.14). \square

Explicit evaluation formulae for $\zeta_{sl(3)}(a, b, c)$, $\zeta_{sl(3)}(a, b, c; 1/2, 1, 1)$ and $\zeta_{sl(3)}(a, b, c; 1, 1, 1/2)$ for $a, b, c \in \mathbb{N}$, $a+b+c \in 2\mathbb{N}+1$, have already proved in Huard, Williams and Zhang [10, Theorem 2] (see also the proof of [30, (3.2)]), Tsumura [51, Theorem 2] and [46, Theorem 3.4], respectively. We give new and simple proofs and evaluation formulae for $\zeta_{sl(3)}(a, b, c; 1/2, 1, 1)$, $\zeta_{sl(3)}(a, b, c; 1, 1/2, 1)$ and $\zeta_{sl(3)}(a, b, c; 1, 1, 1/2)$ in the next proposition.

Proposition 1.2.10. For $a, b, c \in \mathbb{N}$, $a + b + c \in 2\mathbb{N} + 1$, we have

$$2\zeta_{sl(3)}(a, b, c; 1/2, 1, 1) = (-1)^a K(c, a, b; 1/2, 1) + (-1)^b K(b, c, a; 1, 1/2), \quad (1.2.18)$$

$$2\zeta_{sl(3)}(a, b, c; 1, 1/2, 1) = (-1)^a K(c, a, b; 1, 1/2) + (-1)^b K(b, c, a; 1/2, 1/2), \quad (1.2.19)$$

$$2\zeta_{sl(3)}(a, b, c; 1, 1, 1/2) = (-1)^a K(c, a, b; 1/2, 1/2) + (-1)^b K(b, c, a; 1/2, 1). \quad (1.2.20)$$

Proof. Putting $\beta = 1/2$, $\gamma = 1$, and $\beta = 1$, $\gamma = 1/2$, and $\beta = \gamma = 1/2$ in (1.2.13), and changing variables, we have the following six formulae ;

$$\begin{aligned} & \zeta_{sl(3)}(b, c, a; 1, 1/2, 1) + (-1)^c \zeta_{sl(3)}(c, a, b; 1/2, 1, 1) + (-1)^b \zeta_{sl(3)}(a, b, c; 1, 1, 1/2) \\ & = K(b, c, a; 1/2, 1), \end{aligned}$$

$$\begin{aligned} & \zeta_{sl(3)}(c, a, b; 1, 1/2, 1) + (-1)^a \zeta_{sl(3)}(a, b, c; 1/2, 1, 1) + (-1)^c \zeta_{sl(3)}(b, c, a; 1, 1, 1/2) \\ & = K(c, a, b; 1/2, 1), \end{aligned}$$

$$\begin{aligned} & \zeta_{sl(3)}(b, c, a; 1, 1, 1/2) + (-1)^c \zeta_{sl(3)}(c, a, b; 1, 1/2, 1) + (-1)^b \zeta_{sl(3)}(a, b, c; 1/2, 1, 1) \\ & = K(b, c, a; 1, 1/2), \end{aligned}$$

$$\begin{aligned} & \zeta_{sl(3)}(c, a, b; 1, 1, 1/2) + (-1)^a \zeta_{sl(3)}(a, b, c; 1, 1/2, 1) + (-1)^c \zeta_{sl(3)}(b, c, a; 1/2, 1, 1) \\ & = K(c, a, b; 1, 1/2), \end{aligned}$$

$$\begin{aligned} & \zeta_{sl(3)}(b, c, a; 1/2, 1, 1) + (-1)^c \zeta_{sl(3)}(c, a, b; 1, 1, 1/2) + (-1)^b \zeta_{sl(3)}(a, b, c; 1, 1/2, 1) \\ & = K(b, c, a; 1/2, 1/2), \end{aligned}$$

$$\begin{aligned} & \zeta_{sl(3)}(c, a, b; 1/2, 1, 1) + (-1)^a \zeta_{sl(3)}(a, b, c; 1, 1, 1/2) + (-1)^c \zeta_{sl(3)}(b, c, a; 1, 1/2, 1) \\ & = K(c, a, b; 1/2, 1/2). \end{aligned}$$

Multiplying the second formula by $(-1)^a$ and the third formula by $(-1)^b$, and summing the resulting formulae, we can remove $\zeta_{sl(3)}(c, a, b; 1, 1/2, 1)$ and $\zeta_{sl(3)}(b, c, a; 1, 1, 1/2)$ since $a + b + c \in 2\mathbb{N} + 1$. Hence we obtain (1.2.18). Similarly, we can also obtain (1.2.19) (resp. (1.2.20)), by multiplying the fourth (resp. sixth) formula by $(-1)^a$ and fifth (resp. first) formula by $(-1)^b$ and summing the resulting formulae. Needless to say, we can prove (1.2.18) by (1.2.19) and $\zeta_{sl(3)}(a, b, c; 1/2, 1, 1) = \zeta_{sl(3)}(b, a, c; 1, 1/2, 1)$. But we adopt the above proof intentionally, since we want to show the symmetry in the proof. \square

1.2.4 $sl(3)$ with characters

We define $L^\varphi(s; \alpha)$ and $L_{sl(3)}^{\varphi, \chi, \psi}(s_1, s_2, s_3; \alpha, \beta, \gamma)$ by

$$L^\varphi(s; \alpha) := \sum_{n=1}^{\infty} \frac{e^{2\pi i n \alpha} \varphi(n)}{n^s},$$

$$L_{sl(3)}^{\varphi, \chi, \psi}(s_1, s_2, s_3; \alpha, \beta, \gamma) := \sum_{l, m=1}^{\infty} \frac{e^{2\pi i l \alpha} e^{2\pi i m \beta} e^{2\pi i (l+m) \gamma} \varphi(l) \chi(m) \psi(l+m)}{l^{s_1} m^{s_2} (l+m)^{s_3}},$$

where φ , χ and ψ are primitive Dirichlet characters of conductor g , f and h , respectively. Next we consider analytic continuation for $L_{sl(3)}^{\varphi, \chi, \psi}(s_1, s_2, s_3; \alpha, \beta, \gamma)$. In the case of $\psi = 1$ (trivial character) and $\alpha = \beta = \gamma = 1$, Wu has continued this function analytically in [56, Theorem 1]. We denote the Gauss sum by $\tau(\chi) := \sum_{l=1}^f \bar{\chi}(l) e^{2\pi i l / f}$, and recall the well-known formula

$$e^{2\pi i n \beta} \chi(n) = \frac{e^{2\pi i n \beta}}{\tau(\bar{\chi})} \sum_{l=1}^f \bar{\chi}(l) e^{2\pi i n l / f}, \quad (1.2.21)$$

which holds for non-trivial and primitive characters. By this formula, $L_{sl(3)}^{\varphi, \chi, \psi}(s_1, s_2, s_3; \alpha, \beta, \gamma)$ can be written as

$$\frac{1}{\tau(\bar{\varphi})\tau(\bar{\chi})\tau(\bar{\psi})} \sum_{j_1=1}^g \sum_{j_2=1}^f \sum_{j_3=1}^h \bar{\varphi}(j_1)\bar{\chi}(j_2)\bar{\psi}(j_3) \sum_{l,m=1}^{\infty} \frac{e^{2\pi il(\alpha+j_1/g)} e^{2\pi im(\beta+j_2/f)} e^{2\pi i(l+m)(\gamma+j_3/h)}}{l^{s_1} m^{s_2} (l+m)^{s_3}}$$

in the region of convergence (see [26, Section 2]). Since the existence of the analytic continuation of the infinite series on the right-hand side has been shown in Theorem 1.2.6 ([31, Theorem 2.1]), we also obtain the existence of the analytic continuation for $L_{sl(3)}^{\varphi, \chi, \psi}(s_1, s_2, s_3; \alpha, \beta, \gamma)$. Firstly, we will show a functional relation with a Dirichlet character ψ .

Lemma 1.2.11. *For all $a, b \in \mathbb{N}$ and $s \in \mathbb{C}$ except for the singular points, we have*

$$\begin{aligned} & L_{sl(3)}^{1,1,\psi}(a, b, s; 1, \beta, \gamma) + (-1)^b L_{sl(3)}^{1,\psi,1}(b, s, a; -\beta, \gamma, 1) + (-1)^a L_{sl(3)}^{\psi,1,1}(s, a, b; \gamma, 1, \beta) \\ &= \sum_{j=1}^a \binom{a+b-j-1}{a-j} L^\psi(a+b+s-j; \beta+\gamma) (\text{Li}(j; -\beta) + (-1)^j \text{Li}(j; \beta)) \\ & \quad + \sum_{j=1}^b \binom{a+b-j-1}{b-j} L^\psi(a+b+s-j; \gamma) (\text{Li}(j; \beta) + (-1)^j \text{Li}(j; -\beta)) \\ & \quad - \binom{a+b-1}{a} L^\psi(a+b+s; \beta+\gamma) - \binom{a+b-1}{b} L^\psi(a+b+s; \gamma). \end{aligned} \tag{1.2.22}$$

Proof. Firstly, we assume $a, b \geq 2$, and $\Re(s) > 1$. Change the parameter as $-y \mapsto \beta$ in (1.2.2), multiply by $\sum_{n=1}^{\infty} \psi(n) e^{2\pi in(\gamma-x)} n^{-s}$ and integrate from 0 to 1 with respect to x . Then we have

$$\int_0^1 \sum_{l=1}^{\infty} \frac{e^{2\pi ilx}}{l^a} \sum_{m=1}^{\infty} \frac{e^{2\pi im(x+\beta)}}{m^b} \sum_{n=1}^{\infty} \frac{\psi(n) e^{2\pi in(\gamma-x)}}{n^s} dx = \sum_{l,m=1}^{\infty} \frac{\psi(l+m) e^{2\pi im\beta} e^{2\pi i(l+m)\gamma}}{l^a m^b (l+m)^s},$$

which is the first term on the left-hand side of (1.2.22). We obtain the other terms of (1.2.22), similarly. Hence we have (1.2.22) in this region. In the case of $a, b \leq 2$, we can prove this Lemma similarly to the proof of Theorem 1.2.7. \square

In the case of Dirichlet characters with conductor $f = 2, 4$, the following lemma seems to coincide with Tsumura [49, Theorem 4.7] and [50, Proposition 4.3], while in the general cases, it seems to coincide with Tsumura [53, Theorem 3.1], though we have not checked rigorously.

Lemma 1.2.12. *Suppose χ is a non-trivial and primitive Dirichlet character of conductor f . For all $a, b \in \mathbb{N}$ and $s \in \mathbb{C}$ except for the singular points, we have*

$$\begin{aligned} & L_{sl(3)}^{1,\chi,\psi}(a, b, s; 1, \beta, \gamma) + (-1)^b \chi(-1) L_{sl(3)}^{\chi,\psi,1}(b, s, a; -\beta, \gamma, 1) + (-1)^a L_{sl(3)}^{\psi,1,\chi}(s, a, b; \gamma, 1, \beta) \\ &= \frac{1}{\tau(\bar{\chi})} \sum_{l=1}^f \bar{\chi}(m) \sum_{j=1}^a \binom{a+b-j-1}{a-j} L^\psi(a+b+s-j; \beta+\gamma+l/f) \\ & \quad \times (\text{Li}(j; -\beta-l/f) + (-1)^j \text{Li}(j; \beta+l/f)) \\ & \quad + \sum_{j=1}^b \binom{a+b-j-1}{b-j} L^\psi(a+b+s-j; \gamma) (L^\chi(j; \beta) + (-1)^j \chi(-1) L^\chi(j; -\beta)) \\ & \quad - \binom{a+b-1}{a} L^{\chi\psi}(a+b+s; \beta+\gamma). \end{aligned} \tag{1.2.23}$$

Proof. Change the parameter as $\beta \mapsto l/f + \beta$ in (1.2.22) and multiply these formulae by $\bar{\chi}(l)/\tau(\bar{\chi})$. Summing the resulting formulae on l from 1 to f , and using

$$e^{2\pi i n \beta} \sum_{l=1}^f \bar{\chi}(l) e^{2\pi i(-l)n/f} = e^{2\pi i n \beta} \chi(-n) \tau(\chi) = e^{2\pi i n \beta} \chi(-1) \sum_{l=1}^f \bar{\chi}(l) e^{2\pi i l n / f},$$

we obtain this lemma. \square

Hereafter we also use simplified symbols

$$L_{sl(3)}^{\varphi, \chi, \psi}(s_1, s_2, s_3) := L_{sl(3)}^{\varphi, \chi, \psi}(s_1, s_2, s_3; 1, 1, 1), \quad L^X(s) := L^X(s; 1).$$

Taking parameters $-\beta = \gamma = m/g$ in (1.2.22) and multiply these formulae by $\bar{\varphi}(m)/\tau(\bar{\varphi})$, and summing the resulting formulae on m from 1 to g , we obtain the following lemma.

Lemma 1.2.13. *Suppose φ is a non-trivial and primitive Dirichlet character of conductor g . For all $a, b \in \mathbb{N}$ and $s \in \mathbb{C}$ except for the singular points, we have*

$$\begin{aligned} & L_{sl(3)}^{\varphi, 1, \psi}(a, b, s) + (-1)^b L_{sl(3)}^{1, \psi, \varphi}(b, s, a) + (-1)^a \varphi(-1) L_{sl(3)}^{\psi, \varphi, 1}(s, a, b) \\ &= \sum_{j=1}^a \binom{a+b-j-1}{a-j} L^\psi(a+b+s-j) (L^\varphi(j) + (-1)^j \varphi(-1) L^\varphi(j)) \\ &+ \frac{1}{\tau(\bar{\varphi})} \sum_{m=1}^g \bar{\varphi}(k) \sum_{j=1}^b \binom{a+b-j-1}{b-j} L^\psi(a+b+s-j; m/g) \\ &\quad \times (\text{Li}(j; -m/g) + (-1)^j \text{Li}(j; m/g)) \\ &- \binom{a+b-1}{b} L^{\varphi\psi}(a+b+s). \end{aligned} \tag{1.2.24}$$

Taking parameters $-\beta = \gamma = m/g$ in (1.2.23) and multiply these formulae by $\bar{\varphi}(m)/\tau(\bar{\varphi})$. Summing the resulting formulae on m from 1 to g , we obtain the following theorem.

Theorem 1.2.14. *Suppose χ and φ are non-trivial and primitive Dirichlet characters of conductors f and g , respectively. For all $a, b \in \mathbb{N}$ and $s \in \mathbb{C}$ except for the singular points, we have*

$$\begin{aligned} & L_{sl(3)}^{\varphi, \chi, \psi}(a, b, s) + (-1)^b \chi(-1) L_{sl(3)}^{\chi, \psi, \varphi}(b, s, a) + (-1)^a \varphi(-1) L_{sl(3)}^{\psi, \varphi, \chi}(s, a, b) \\ &= \frac{1}{\tau(\bar{\chi})} \sum_{l=1}^f \bar{\chi}(l) \sum_{j=1}^a \binom{a+b-j-1}{a-j} L^\psi(a+b+s-j; l/f) \\ &\quad \times (L^\varphi(j; -l/f) + (-1)^j \varphi(-1) L^\varphi(j; l/f)) \\ &+ \frac{1}{\tau(\bar{\varphi})} \sum_{m=1}^g \bar{\varphi}(m) \sum_{j=1}^b \binom{a+b-j-1}{b-j} L^\psi(a+b+s-j; m/g) \\ &\quad \times (L^X(j; -m/g) + (-1)^j \chi(-1) L^X(j; m/g)). \end{aligned} \tag{1.2.25}$$

Next we will show explicit evaluation formulae for double L-values with characters, similarly to Proposition 1.2.4. We define $D_1(a, b, s; 1, 1, \psi)$ by the right-hand side of (1.2.22) with $\beta = \gamma = 1$, $D_2(a, b, s; 1, \chi, \psi)$ by the right-hand side of (1.2.23) with $\beta = \gamma = 1$, $D_3(a, b, s; \varphi, 1, \psi)$ and $D_4(a, b, s; \varphi, \chi, \psi)$ by the right-hand side of (1.2.24) and (1.2.25), respectively.

Proposition 1.2.15. *If χ is non-trivial and primitive and $\chi(-1) = (-1)^{a+b-1}$, for any admissible index, we have*

$$2L_{sl(3)}^{1, 1, \chi}(0, a, b) = (1 - (-1)^a) \zeta(a) L^X(b) - L^X(a+b) + (-1)^a D_2(a, b, 0; 1, \chi, 1) \tag{1.2.26}$$

$$2L_{sl(3)}^{1,\chi,1}(0, b, a) = (1 + (-1)^a)\zeta(a)L^\chi(b) - L^\chi(a+b) - (-1)^a D_2(a, b, 0; 1, \chi, 1). \quad (1.2.27)$$

If φ and χ are non-trivial and primitive and $\chi(-1)\varphi(-1) = (-1)^{a+b-1}$, for any admissible index, we have

$$2L_{sl(3)}^{1,\varphi,\chi}(0, a, b) = (1 - (-1)^a\varphi(-1))L^\varphi(a)L^\chi(b) - L^{\varphi\chi}(a+b) + (-1)^a\varphi(-1)D_4(a, b, 0; \varphi, \chi, 1). \quad (1.2.28)$$

Proof. We recall the harmonic product formula

$$L_{sl(3)}^{1,\varphi,\chi}(0, a, b) + L_{sl(3)}^{1,\chi,\varphi}(0, b, a) + L^{\varphi\chi}(a+b) = L^\varphi(a)L^\chi(b). \quad (1.2.29)$$

By putting $s = 0$ in (1.2.23), multiplying it by $(-1)^a$, adding the resulting expression and (1.2.29), we can remove $L_{sl(3)}^{1,\chi,1}(0, b, a)$. Hence we have (1.2.26). By (1.2.29) and (1.2.26), we obtain (1.2.27). By putting $s = 0$ in (1.2.25), multiplying it by $(-1)^a\varphi(-1)$, and summing the resulting expression and (1.2.29), we obtain (1.2.28). \square

For l and m , let $R_{l,m}$ be the set of all polynomials in convergent series of the form $\sum_{n=1}^{\infty} e^{2\pi i n \xi_l} n^{-j}$ (where ξ_l 's are l th roots of unit and $j \in \mathbb{N}$) with $\mathbb{Q}(e^{2\pi i/m})$ coefficients. By (1.2.21) and Proposition 1.2.15, we can immediately obtain the following proposition, which has already proved by Terhune.

Proposition 1.2.16 ([41, Theorem 1]). *Let χ and φ be Dirichlet characters of conductor f and h , respectively. Set $m = \text{lcm}(f, h)$ and $M = \text{lcm}(f, h, \phi(f), \phi(h))$ where lcm means the least common multiple, ϕ is the Euler totient function. For any admissible index, if $\chi(-1)\varphi(-1) = (-1)^{a+b+1}$, then $L_{sl(3)}^{1,\varphi,\chi}(0, a, b) \in R_{M,m}$.*

Next we will show explicit evaluate formulae for the Tornheim double zeta function with characters similarly to Proposition 1.2.10.

Proposition 1.2.17. *If φ , χ and ψ are non-trivial and primitive character and $\varphi(-1)\chi(-1)\psi(-1) = (-1)^{a+b+c+1}$, then we have*

$$2L_{sl(3)}^{\varphi,\chi,\psi}(a, b, c) = (-1)^b\chi(-1)D_4(b, c, a; \chi, \psi, \varphi) + (-1)^a\varphi(-1)D_4(c, a, b; \psi, \varphi, \chi). \quad (1.2.30)$$

Proof. By changing parameters and characters in (1.2.25), we have

$$\begin{aligned} L_{sl(3)}^{\chi,\psi,\varphi}(b, c, a) + (-1)^c\psi(-1)L_{sl(3)}^{\psi,\varphi,\chi}(c, a, b) + (-1)^b\chi(-1)L_{sl(3)}^{\varphi,\chi,\psi}(a, b, c) &= D_4(b, c, a; \chi, \psi, \varphi), \\ L_{sl(3)}^{\psi,\varphi,\chi}(c, a, b) + (-1)^a\varphi(-1)L_{sl(3)}^{\varphi,\chi,\psi}(a, b, c) + (-1)^c\psi(-1)L_{sl(3)}^{\chi,\psi,\varphi}(b, c, a) &= D_4(c, a, b; \psi, \varphi, \chi). \end{aligned}$$

In the case of $\varphi(-1)\chi(-1)\psi(-1) = (-1)^{a+b+c+1}$, we can remove $L_{sl(3)}^{\chi,\psi,\varphi}(b, c, a)$ and $L_{sl(3)}^{\psi,\varphi,\chi}(c, a, b)$ by multiplying the former formula by $(-1)^b\chi(-1)$ and the latter formula by $(-1)^a\varphi(-1)$. Hence we obtain (1.2.30). \square

We can obtain the following proposition, which should be compared with Tsumura [48, Theorem 3.1], similarly to the above.

Proposition 1.2.18. *If ψ is primitive and $\psi(-1) = (-1)^{a+b+c+1}$, then we have*

$$2L_{sl(3)}^{1,1,\psi}(a, b, c) = (-1)^b D_2(b, c, a; 1, \psi, 1) + (-1)^a D_1(c, a, b; 1, 1, \psi), \quad (1.2.31)$$

$$2L_{sl(3)}^{1,\psi,1}(a, b, c) = (-1)^b\psi(-1)D_3(b, c, a; \psi, 1, 1) + (-1)^a D_1(c, a, b; 1, 1, \psi). \quad (1.2.32)$$

If $\chi(-1)$ and $\psi(-1)$ are primitive and $\chi(-1)\psi(-1) = (-1)^{a+b+c+1}$, then we have

$$2L_{sl(3)}^{1,\chi,\psi}(a, b, c) = (-1)^b\chi(-1)D_4(b, c, a; \chi, \psi, 1) + (-1)^a D_3(c, a, b; \psi, 1, \chi), \quad (1.2.33)$$

$$2L_{sl(3)}^{\chi,\psi,1}(a, b, c) = (-1)^b\psi(-1)D_3(b, c, a; \psi, 1, \chi) + (-1)^a\chi(-1)D_2(c, a, b; 1, \chi, \psi). \quad (1.2.34)$$

1.2.5 $so(5)$

For $s, t, u \in \mathbb{C}$, $\Re(s) > 1$, $\Re(t) > 1$, $\Re(u) > 1$, we define four functions ;

$$\begin{aligned} T_{E1}(s, t, u) &:= \sum_{l,m=1}^{\infty} \frac{1}{(2l)^s m^t (2l+m)^u}, & T_{E2}(s, t, u) &:= \sum_{l,m=1}^{\infty} \frac{1}{l^s (2m)^t (l+2m)^u}, \\ T_{OO}(s, t, u) &:= \sum_{l,m=1}^{\infty} \frac{1}{(2l-1)^s (2m-1)^t (2l+2m-2)^u}, \\ T_{AE}(s, t, u) &:= \sum_{\substack{l,m,n=1 \\ 2n=l+m}}^{\infty} \frac{1}{l^s m^t n^u} = 2^{-s-t} \zeta_{sl(3)}(s, t, u) + 2^u T_{OO}(s, t, u). \end{aligned}$$

Proposition 1.2.19. *For $s, t, u \in \mathbb{C}$ except for singular points, we have*

$$T_{E1}(s, t, u) = 2^{-1} \zeta_{sl(3)}(s, t, u) + 2^{-1} \zeta_{sl(3)}(s, t, u; 1/2, 1, 1), \quad (1.2.35)$$

$$T_{E2}(s, t, u) = 2^{-1} \zeta_{sl(3)}(s, t, u) + 2^{-1} \zeta_{sl(3)}(s, t, u; 1, 1/2, 1), \quad (1.2.36)$$

$$T_{OO}(s, t, u) = (2^{-1} - 2^{-s-t-u}) \zeta_{sl(3)}(s, t, u) + 2^{-1} \zeta_{sl(3)}(s, t, u; 1, 1, 1/2), \quad (1.2.37)$$

$$T_{AE}(s, t, u) = 2^{u-1} \zeta_{sl(3)}(s, t, u) + 2^{u-1} \zeta_{sl(3)}(s, t, u; 1, 1, 1/2). \quad (1.2.38)$$

Especially, in the case of $a, b, c \in \mathbb{N}$, $a + b + c \in 2\mathbb{N} + 1$, $T_{E1}(a, b, c)$, $T_{E2}(a, b, c)$, $T_{OO}(a, b, c)$ and $T_{AE}(a, b, c)$ are polynomials in $\{\zeta(k) \mid 2 \leq k \leq a + b + c\}$ with rational coefficients.

Proof. The first, second and fourth formulae are obvious. By the definition of $\zeta_{sl(3)}(s, t, u; \alpha, \beta, \gamma)$ and $T_{OO}(s, t, u)$, we immediately obtain

$$\begin{aligned} &\zeta_{sl(3)}(s, t, u) + \zeta_{sl(3)}(s, t, u; 1/2, 1, 1) + \zeta_{sl(3)}(s, t, u; 1, 1/2, 1) + \zeta_{sl(3)}(s, t, u; 1, 1, 1/2) \\ &= 2^{2-s-t-u} \zeta_{sl(3)}(s, t, u), \\ &\zeta_{sl(3)}(s, t, u) - \zeta_{sl(3)}(s, t, u; 1/2, 1, 1) - \zeta_{sl(3)}(s, t, u; 1, 1/2, 1) + \zeta_{sl(3)}(s, t, u; 1, 1, 1/2) \\ &= 4T_{OO}(s, t, u). \end{aligned}$$

By summing the above formulae, we obtain the third formula of this proposition. \square

For $s, t, u, v \in \mathbb{C}$, we define the Witten zeta function attached to $so(5)$ by

$$\zeta_{so(5)}(s, t, u, v) := \sum_{l,m=1}^{\infty} \frac{1}{l^s m^t (l+m)^u (l+2m)^v},$$

in the region of convergence. We recall that Matsumoto defined this function and continued analytically in [22, Theorem 3]. The following proposition has already shown by Tsumura. We can show this proposition in a simpler way, since the evaluation formulae of $T_{E1}(a, b, c)$, $T_{E2}(a, b, c)$ and $\zeta_{sl(3)}(a, b, c)$ for $a + b + c \in 2\mathbb{N} + 1$ are obtained much simpler.

Proposition 1.2.20 ([47, Theorem]). *Suppose that $a, b, c, d \in \mathbb{N}$ and $a + b + c + d \in 2\mathbb{N} + 1$. Then $\zeta_{so(5)}(a, b, c, d)$ can be expressed as a rational linear combination of products of Riemann's zeta values at positive integers.*

Proof. We quote [47, (10)],

$$\begin{aligned} \zeta_{so(5)}(a, b, c, d) &= (-1)^c \sum_{j=1}^c \binom{c+d-j-1}{c-j} (-1)^j \zeta_{so(5)}(a, b+c+d-j, j, 0) \\ &\quad + (-1)^c \sum_{j=1}^d \binom{c+d-j-1}{d-j} (-1)^j \zeta_{so(5)}(a, b+c+d-j, 0, j). \end{aligned} \quad (1.2.39)$$

Hence we obtain this proposition by Propositions 1.2.10 and 1.2.19. \square

Next we will prove a functional relation for two complex variables.

Theorem 1.2.21. *For $a, b \in \mathbb{N}$ and $s, t \in \mathbb{C}$ except for singular points, we have*

$$\begin{aligned}
& \zeta_{so(5)}(a, b, s, t) + (-1)^b \zeta_{so(5)}(t, b, s, a) + (-1)^a \zeta_{so(5)}(a, s, b, t) \\
&= \sum_{j=1}^a \binom{a+b-j-1}{a-j} (T_{AE}(j, t, a+b+s-j) + 2^{a+b+s-j} (-1)^j T_{E1}(a+b+s-j, j, t)) \\
&+ \sum_{j=1}^b \binom{a+b-j-1}{b-j} (\zeta_{sl(3)}(a+b+s-j, j, t) + (-1)^j \zeta_{sl(3)}(j, t, a+b+s-j)) \\
&- \binom{a+b-1}{a} 2^{-t} \zeta(a+b+s+t) - \binom{a+b-1}{b} \zeta(a+b+s+t).
\end{aligned} \tag{1.2.40}$$

Proof. Put $\beta = \gamma = x$ in (1.2.13), multiply by $\sum_{n=1}^{\infty} e^{-2\pi i n x} n^{-t}$ and integrate from 0 to 1 with respect to x . For $\Re(s) > 1$ and $\Re(t) > 1$, we have

$$\begin{aligned}
& \int_0^1 \sum_{l,m=1}^{\infty} \frac{e^{2\pi i l x} e^{2\pi i m(2x)}}{l^a m^b (l+m)^s} \sum_{n=1}^{\infty} \frac{e^{-2\pi i n x}}{n^t} dx = \sum_{l,m=1}^{\infty} \frac{1}{l^a m^b (l+m)^s (l+2m)^t} \\
& \int_0^1 \sum_{l=1}^{\infty} \frac{e^{2\pi i l(2x)}}{l^{a+b+s-j}} \sum_{m=1}^{\infty} \frac{e^{-2\pi i m x}}{m^j} \sum_{n=1}^{\infty} \frac{e^{-2\pi i n x}}{n^t} dx = \sum_{\substack{l,m,n=1 \\ 2l=m+n}}^{\infty} \frac{1}{l^{a+b+s-j} m^j n^t}, \\
& \int_0^1 \sum_{l=1}^{\infty} \frac{e^{2\pi i l(2x)}}{l^{a+b+s-j}} \sum_{m=1}^{\infty} \frac{e^{2\pi i m x}}{m^j} \sum_{n=1}^{\infty} \frac{e^{-2\pi i n x}}{n^t} dx = \sum_{l,m,n=1}^{\infty} \frac{1}{l^{a+b+s-j} m^j (2l+m)^t}, \\
& \int_0^1 \sum_{l=1}^{\infty} \frac{e^{2\pi i l(2x)}}{l^{a+b+s}} \sum_{n=1}^{\infty} \frac{e^{-2\pi i n x}}{n^t} dx = \sum_{l=1}^{\infty} \frac{1}{l^{a+b+s} (2l)^t},
\end{aligned}$$

which are $\zeta_{so(5)}(a, b, s, t)$, $T_{AE}(j, t, a+b+s-j)$, $T_{E1}(a+b+s-j, j, t)$ and $2^{-t} \zeta(a+b+s+t)$ in (1.2.40). The other terms of (1.2.40) are obtained similarly. By analytic continuation, we have (1.2.40) except for the singular points. \square

Lemma 1.2.22. *For any admissible index, we have*

$$\begin{aligned}
& (\text{Li}(c; x) + (-1)^c \text{Li}(c; -x)) (\text{Li}(d; x) + (-1)^d \text{Li}(d; -x)) + ((-1)^c + (-1)^d) \zeta(c+d) \\
&= 2 \sum_{j=0}^c \binom{c+d-2j-1}{c-2j} \zeta(2j) (\text{Li}(c+d-2j; x) + (-1)^{c+d} \text{Li}(c+d-2j; -x)) \\
&+ 2 \sum_{j=0}^d \binom{c+d-2j-1}{d-2j} \zeta(2j) (\text{Li}(c+d-2j; x) + (-1)^{c+d} \text{Li}(c+d-2j; -x)).
\end{aligned} \tag{1.2.41}$$

Proof. By the harmonic product formula, we have

$$\begin{aligned}
& (\text{Li}(c; x) + (-1)^c \text{Li}(c; -x)) (\text{Li}(d; x) + (-1)^d \text{Li}(d; -x)) \\
&= \text{Li}(c; x) \text{Li}(d; x) + (-1)^d (L(c, d; x, 0) + L(d, c; -x, 0) + \zeta(c+d)) \\
&+ (-1)^c (L(c, d; -x, 0) + L(d, c; x, 0) + \zeta(c+d)) + (-1)^{c+d} \text{Li}(c; -x) \text{Li}(d; -x).
\end{aligned}$$

Using (1.2.2) and $\zeta(0) = -1/2$, we obtain this lemma. \square

Remark 1.2.23. We denote by $B_j(x)$ the Bernoulli polynomial of order j , which is defined by (1.1.4). By (1.2.41), we can immediately prove (1.1.8). This formula is the key of the proof of [30, Theorem 1.2], which means that a product of Bernoulli polynomials is a linear combination of Bernoulli polynomials. We may regard that this is a special case of a well-known fact, which is, the set of multiple zeta or (L) -values is closed with respect to the operations of the harmonic product and the shuffle product.

At the end of this section, we show the following functional relations for one complex variable.

Theorem 1.2.24. For $a, b, c \in \mathbb{N}$ and $s \in \mathbb{C}$ except for singular points, we have

$$\begin{aligned}
& \zeta_{so(5)}(a, b, s, c) + (-1)^b \zeta_{so(5)}(c, b, s, a) + (-1)^a \zeta_{so(5)}(a, s, b, c) + (-1)^{b+c} \zeta_{so(5)}(c, s, b, a) \\
&= 2 \sum_{d=0}^a \binom{a+b-d-1}{a-d} \sum_{j=0}^{\max(c,d)/2} \left\{ \binom{c+d-2j-1}{c-2j} + \binom{c+d-2j-1}{d-2j} \right\} \\
&\quad \times 2^{2j-c-d} \zeta(2j) \zeta(a+b+c+s-2j) \\
&+ 2 \sum_{d=0}^b (-1)^d \binom{a+b-j-1}{b-d} \sum_{j=0}^{\max(c,d)/2} \left\{ \binom{c+d-2j-1}{c-2j} + \binom{c+d-2j-1}{d-2j} \right\} \\
&\quad \times \zeta(2j) \zeta(a+b+c+s-2j).
\end{aligned} \tag{1.2.42}$$

Proof. Firstly, we assume $\Re(s) > 1$. Taking parameters $\beta = \gamma = -x$ in (1.2.13), multiplying by $(\sum_{n=1}^{\infty} e^{2\pi i n x} n^{-c} + (-1)^c \sum_{n=1}^{\infty} e^{-2\pi i n x} n^{-c})$, and integrating 0 to 1 with respect to x , we obtain

$$\begin{aligned}
& \int_0^1 \sum_{l,m=1}^{\infty} \frac{e^{-2\pi i l x} e^{-2\pi i (2m)x}}{l^a m^b (l+m)^s} \sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{n^c} dx = \sum_{l,m=1}^{\infty} \frac{1}{l^a m^b (l+m)^s (l+2m)^c}, \\
& \int_0^1 \sum_{m=1}^{\infty} \frac{e^{-2\pi i m (2x)}}{m^{a+b+s-d}} \left(\sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{n^{c+d-2j}} + (-1)^{c+d} \sum_{n=1}^{\infty} \frac{e^{-2\pi i n x}}{n^{c+d-2j}} \right) dx = \sum_{m=1}^{\infty} \frac{1}{m^{a+b+s-d} (2m)^{c+d-2j}},
\end{aligned}$$

which are $\zeta_{so(5)}(a, b, s, c)$ and $2^{2j-c-d} \zeta(2j) \zeta(a+b+c+s-2j)$ in (1.2.42). Changing the order of summation and integration is justified as follows. We can change the order by the absolute convergence in the case of $c \geq 2$. We recall the well-known formula

$$\sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{n} = -\log(2 \sin \pi x) - i\pi B_1(x), \quad 0 < x < 1. \tag{1.2.43}$$

We define $f(x)$ by the right-hand side of (1.2.43). By (1.2.43), for a sufficiently large integer N , there exists an $M > 0$ such that

$$\left| \sum_{n=1}^N \frac{e^{2\pi i n x}}{n} \right| \leq |f(x)| + M, \quad 0 < x < 1.$$

From this fact and the well-known formula

$$\int_0^1 \log(2 \sin \pi x) dx = \log 2 - \log 2 = 0,$$

we see that we can use Lebesgue's bounded convergence theorem. The other terms of (1.2.42) are obtained similarly. By (1.2.41), $\zeta(0) = -1/2$ and the argument similar to the proof of Theorem 1.2.21, we obtain this theorem. \square

1.2.6 $sl(4)$

We recall that Matsumoto continued $\zeta_{sl(4)}(s_1, s_2, s_3, s_4, s_5, s_6)$ analytically in [22, Theorem 3]. Afterwards Matsumoto and Tsumura determined its singularities more closely in [24, Theorem 3.5]. We define the function

$$Z(a, b, s, t, u; \beta, \gamma, \delta, \eta) := \sum_{\substack{l, m, n, k=1 \\ l+m=n+k}}^{\infty} \frac{e^{2\pi i m \beta} e^{2\pi i n \gamma} e^{2\pi i k \delta} e^{2\pi i (l+m) \eta}}{l^a m^b n^s k^t (l+m)^u}.$$

Lemma 1.2.25. *For $a, b \in \mathbb{N}$, $s, t, u \in \mathbb{C}$, $\Re(s) > 1$, $\Re(t) > 1$, $\Re(u) > 1$, we have*

$$\begin{aligned} Z(a, b, s, t, u; \alpha, \gamma, \delta, \eta) &= \zeta_{sl(3)}(a+s, b+t, u; \gamma, \beta+\delta, \eta) \\ &\quad + \zeta_{sl(4)}(a, 0, t, s, b, u; 1, 1, \delta, \gamma, \beta, \eta) + \zeta_{sl(4)}(b, 0, s, t, a, u; \beta, 1, \gamma, \delta, 1, \eta). \end{aligned}$$

Proof. In this region, we have

$$\begin{aligned} Z(a, b, s, t, u; \alpha, \gamma, \delta, \eta) &= \sum_{h=1}^{\infty} \sum_{\substack{l, m=1 \\ l+m=h}}^{\infty} \frac{e^{2\pi i m \beta}}{l^a m^b} \sum_{\substack{n, k=1 \\ n+k=h}}^{\infty} \frac{e^{2\pi i n \gamma} e^{2\pi i k \delta} e^{2\pi i h \eta}}{n^s k^t h^u} \\ &= \sum_{l, n=1}^{\infty} \frac{1}{l^a n^s} \sum_{h > \max\{l, n\}} \frac{e^{2\pi i (h-l) \beta} e^{2\pi i n \gamma} e^{2\pi i (h-n) \delta} e^{2\pi i h \eta}}{(h-l)^b (h-n)^t h^u}. \end{aligned}$$

We separate the right-hand side of the above formula as $\sum_{l=n} + \sum_{l < n} + \sum_{l > n}$. In the case of $l = n$, we have

$$\sum_{l=n} = \sum_{n=1}^{\infty} \sum_{h > n} \frac{e^{2\pi i n \gamma} e^{2\pi i (h-n) (\beta+\delta)} e^{2\pi i h \eta}}{n^{a+s} (h-n)^{b+t} h^u} = \zeta_{sl(3)}(a+s, b+t, u; \gamma, \beta+\delta, \eta).$$

By putting $n = l + j$ and $h = l + j + k$, we have

$$\sum_{l < n} = \sum_{l, j, k=1}^{\infty} \frac{e^{2\pi i (j+k) \beta} e^{2\pi i (l+j) \gamma} e^{2\pi i k \delta} e^{2\pi i (l+j+k) \eta}}{l^a (l+j)^s (j+k)^b k^t (l+j+k)^u} = \zeta_{sl(4)}(a, 0, t, s, b, u; 1, 1, \delta, \gamma, \beta, \eta).$$

By putting $l = n + j$ and $h = n + j + k$, we have

$$\sum_{l > n} = \sum_{n, j, k=1}^{\infty} \frac{e^{2\pi i k \beta} e^{2\pi i n \gamma} e^{2\pi i (j+k) \delta} e^{2\pi i (n+j+k) \eta}}{(n+j)^a n^s k^b (j+k)^t (n+j+k)^u} = \zeta_{sl(4)}(b, 0, s, t, a, u; \beta, 1, \gamma, \delta, 1, \eta).$$

Therefore by these formulae, we obtain this lemma. \square

In the case of $\beta = 1$, the next Theorem coincides with [32, Theorem 2.1 and Theorem 2.3]. Therefore it also contains [32, Theorem 3.4] which is a functional relation for $\zeta_{sl(4)}(a, b, c, 0, 0, s)$.

Theorem 1.2.26. *For $a, b \in \mathbb{N}$, $s, t, u \in \mathbb{C}$, $\Re(s) > 1$, $\Re(t) > 1$, $\Re(u) > 1$, we have*

$$\begin{aligned} &\zeta_{sl(4)}(a, b, t, s, 0, u; 1, \beta, \delta, \gamma, 1, \eta) \\ &\quad + (-1)^b \zeta_{sl(4)}(b, s, t, a, u, 0; -\beta, \gamma, \delta, 1, \eta, 1) + (-1)^a \zeta_{sl(4)}(a, s, t, b, u, 0; 1, \gamma, \delta, \beta, \eta, 1) \\ &= \sum_{j=1}^a \binom{a+b-j-1}{a-j} \zeta_{sl(3)}(a+b+s-j, t, u; \beta+\gamma, \delta, \eta) (\text{Li}(j; -\beta) + (-1)^j \text{Li}(j; \beta)) \\ &\quad + \sum_{j=1}^b \binom{a+b-j-1}{b-j} \zeta_{sl(3)}(a+b+s-j, t, u; \gamma, \delta, \eta) (\text{Li}(j; \beta) + (-1)^j \text{Li}(j; -\beta)) \\ &\quad - \binom{a+b-1}{a} \zeta_{sl(3)}(a+b+s, t, u; \beta+\gamma, \delta, \eta) - \binom{a+b-1}{b} \zeta_{sl(3)}(a+b+s, t, u; \gamma, \delta, \eta), \end{aligned} \tag{1.2.44}$$

and

$$\begin{aligned}
& Z(a, b, s, t, u; \beta, \gamma, \delta, \eta) + (-1)^b \zeta_{sl(4)}(b, t, u, 0, s, a; -\beta, \delta, \eta, 1, \gamma, 1) \\
& + (-1)^a \zeta_{sl(4)}(a, u, t, 0, s, b; 1, \eta, \delta, 1, \gamma, \beta) \\
& = \sum_{j=1}^a \binom{a+b-j-1}{a-j} \zeta_{sl(3)}(t, u, a+b+s-j; \delta, \eta, \beta+\gamma) (\text{Li}(j; -\beta) + (-1)^j \text{Li}(j; \beta)) \\
& + \sum_{j=1}^b \binom{a+b-j-1}{b-j} \zeta_{sl(3)}(t, u, a+b+s-j; \delta, \eta, \gamma) (\text{Li}(j; \beta) + (-1)^j \text{Li}(j; -\beta)) \\
& - \binom{a+b-1}{a} \zeta_{sl(3)}(t, u, a+b+s; \delta, \eta, \beta+\gamma) - \binom{a+b-1}{b} \zeta_{sl(3)}(t, u, a+b+s; \delta, \eta, \gamma).
\end{aligned} \tag{1.2.45}$$

Proof. Change the parameter as $x \mapsto x + \gamma$ in (1.2.13), multiply by two functions $\sum_{n=1}^{\infty} e^{2\pi i n(x+\delta)} n^{-t}$ and $\sum_{k=1}^{\infty} e^{2\pi i n(\eta-x)} k^{-u}$ and integrate from 0 to 1 with respect to x . We have

$$\begin{aligned}
& \int_0^1 \sum_{l,m=1}^{\infty} \frac{e^{2\pi i m \beta} e^{2\pi i(l+m)(x+\gamma)}}{l^a m^b (l+m)^s} \sum_{n=1}^{\infty} \frac{e^{2\pi i n(x+\delta)}}{n^t} \sum_{k=1}^{\infty} \frac{e^{2\pi i k(\eta-x)}}{k^u} dx \\
& = \sum_{l,m,n=1}^{\infty} \frac{e^{2\pi i m \beta} e^{2\pi i n \delta} e^{2\pi i(l+m)\gamma} e^{2\pi i(l+m+n)\eta}}{l^a m^b n^t (l+m)^s (l+m+n)^u}, \\
& \int_0^1 \sum_{l,m=1}^{\infty} \frac{e^{-2\pi i l \beta} e^{2\pi i m(x+\gamma)}}{l^b m^s (l+m)^a} \sum_{n=1}^{\infty} \frac{e^{2\pi i n(x+\delta)}}{n^t} \sum_{k=1}^{\infty} \frac{e^{2\pi i k(\eta-x)}}{k^u} dx = \sum_{l,m,n=1}^{\infty} \frac{e^{-2\pi i l \beta} e^{2\pi i m \gamma} e^{2\pi i n \delta} e^{2\pi i(m+n)\eta}}{l^b m^s n^t (l+m)^a (m+n)^u}, \\
& \int_0^1 \sum_{l=1}^{\infty} \frac{e^{2\pi i l(x+\beta+\gamma)}}{l^{a+b+s-j}} \sum_{n=1}^{\infty} \frac{e^{2\pi i n(x+\delta)}}{n^t} \sum_{k=1}^{\infty} \frac{e^{2\pi i k(\eta-x)}}{k^u} dx = \sum_{l,n=1}^{\infty} \frac{e^{2\pi i l(\beta+\gamma)} e^{2\pi i n \delta} e^{2\pi i(l+n)\eta}}{l^{a+b+s-j} n^t (l+n)^u},
\end{aligned}$$

which are $\zeta_{sl(4)}(a, b, t, s, 0, u; 1, \beta, \delta, \gamma, 1, \eta)$, $\zeta_{sl(4)}(b, s, t, a, u, 0; -\beta, \gamma, \delta, 1, \eta, 1)$ and $\zeta_{sl(3)}(a+b+s-j, t, u; \beta+\gamma, \delta, \eta)$ in (1.2.44). The other terms of (1.2.44) are obtained in the same way.

Next change the parameter as $x \mapsto x + \gamma$ in (1.2.13), multiply by $\sum_{n=1}^{\infty} e^{2\pi i n(\delta-x)} n^{-t}$ and $\sum_{k=1}^{\infty} e^{2\pi i n(\eta-x)} k^{-u}$ and integrate from 0 to 1 with respect to x . We obtain

$$\int_0^1 \sum_{l,m=1}^{\infty} \frac{e^{2\pi i m \beta} e^{2\pi i(l+m)(x+\gamma)}}{l^a m^b (l+m)^s} \sum_{n=1}^{\infty} \frac{e^{2\pi i n(\delta-x)}}{n^t} \sum_{k=1}^{\infty} \frac{e^{2\pi i k(\eta-x)}}{k^u} dx = \sum_{\substack{l,m,n,k=1 \\ l+m=n+k}}^{\infty} \frac{e^{2\pi i m \beta} e^{2\pi i n \gamma} e^{2\pi i k \delta} e^{2\pi i(l+m)\eta}}{l^a m^b n^s k^t (l+m)^u},$$

which is $Z(a, b, s, t, u; \beta, \gamma, \delta, \eta)$ in (1.2.45). The other terms of (1.2.45) are obtained similarly. Hence we obtain (1.2.44) and (1.2.45). \square

The following functional relation holds for three complex variables.

Theorem 1.2.27. For $a, b, c \in \mathbb{N}$, $s, t, u \in \mathbb{C}$, $\Re(s) > 1$, $\Re(t) > 1$, $\Re(u) > 1$, we have

$$\begin{aligned}
& \zeta_{sl(4)}(a, b, t, s, c, u; 1, 1, \delta, \gamma, 1, \eta) + (-1)^b \zeta_{sl(4)}(s, b, c, a, t, u; \gamma, 1, 1, 1, \delta, \eta) \\
& + (-1)^a \zeta_{sl(4)}(a, s, t, b, u, c; 1, \gamma, \delta, 1, \eta, 1) + (-1)^{b+c} \zeta_{sl(4)}(s, t, c, u, b, a; \gamma, \delta, 1, \eta, 1, 1) \\
& = 2 \sum_{d=0}^a \binom{a+b-d-1}{a-d} \sum_{j=0}^{\max(c,d)/2} \left\{ \binom{c+d-2j-1}{c-2j} + \binom{c+d-2j-1}{d-2j} \right\} \\
& \quad \times \zeta(2j) \zeta_{sl(3)}(a+b+s-d, t, u+c+d-2j; \gamma, \delta, \eta) \\
& + 2 \sum_{d=0}^b (-1)^d \binom{a+b-d-1}{b-d} \sum_{j=0}^{\max(c,d)/2} \left\{ \binom{c+d-2j-1}{c-2j} + \binom{c+d-2j-1}{d-2j} \right\} \\
& \quad \times \zeta(2j) \zeta_{sl(3)}(a+b+s-d, t+c+d-2j, u; \gamma, \delta, \eta).
\end{aligned} \tag{1.2.46}$$

Proof. Change parameters as $\beta \mapsto -x$ and $\delta \mapsto \delta - x$ in (1.2.44), multiply by $(\sum_{h=1}^{\infty} e^{2\pi i h x} h^{-c} + (-1)^c \sum_{h=1}^{\infty} e^{-2\pi i h x} h^{-c})$, and integrate from 0 to 1 with respect to x . We have

$$\begin{aligned} & \int_0^1 \sum_{l,m,n=1}^{\infty} \frac{e^{-2\pi i m x} e^{2\pi i n(\delta-x)} e^{2\pi i(l+m)\gamma} e^{2\pi i(l+m+n)\eta}}{l^a m^b n^t (l+m)^s (l+m+n)^u} \sum_{h=1}^{\infty} \frac{e^{2\pi i h x}}{h^c} dx \\ &= \sum_{l,m,n=1}^{\infty} \frac{e^{2\pi i n \delta} e^{2\pi i(l+m)\gamma} e^{2\pi i(l+m+n)\eta}}{l^a m^b n^t (l+m)^s (m+n)^c (l+m+n)^u}, \end{aligned}$$

which is $\zeta_{sl(4)}(a, b, t, s, c, u; 1, 1, \delta, \gamma, 1, \eta)$ in (1.2.46). By using (1.2.41) and the manner similar to the proof of Theorem 1.2.24, we can obtain this theorem. \square

In [9, Proposition 8.5], Gunnells and Sczech gave explicit formula for $\zeta_{sl(4)}(2k, 2k, 2k, 2k, 2k, 2k)$. In [12, Theorem 4.4], Komori, Matsumoto and Tsumura proved this functional relation in the case of $a, b, c \in 2\mathbb{N}$ and $\gamma = \delta = \eta = 1$. These results should be compared with Theorem 1.2.27.

1.2.7 Mordell-Tornheim and related triple zeta-functions

Firstly, we define the functions

$$\begin{aligned} \zeta_{MT3}(s_1, s_2, s_3, s_4) &:= \zeta_{sl(4)}(s_1, s_2, s_3, 0, 0, s_4), \\ H(s_1, s_2, s_3, s_4) &:= \zeta_{sl(4)}(s_1, 0, s_2, s_3, s_4, 0). \end{aligned}$$

We remark that parameters s_1, s_2, s_3 of $\zeta_{MT3}(s_1, s_2, s_3, s_4)$ is commutative and

$$H(s_1, s_2, s_3, s_4) = H(s_2, s_1, s_4, s_3).$$

We denote $K_1(a, b, s, u)$ and $K_2(a, b, s, t)$ by the right hand side of (1.2.44) and (1.2.45) with putting $t = 0$ and $u = 0$, respectively. Taking $t = 0$, $\gamma = \delta = \theta = 1$ in (1.2.44), we have

$$\zeta_{MT3}(a, b, s, u) = K_1(a, b, s, u) - (-1)^b H(b, s, a, u) - (-1)^a H(a, s, b, u). \quad (1.2.47)$$

Taking $u = 0$, $\gamma = \delta = \theta = 1$ in (1.2.45), we have

$$\begin{aligned} K_2(a, b, s, t) &= \zeta(a+s)\zeta(b+t) + H(a, t, s, b) + H(b, s, t, a) \\ &\quad + (-1)^b \zeta_{MT3}(b, s, t, a) + (-1)^a \zeta_{MT3}(a, s, t, b). \end{aligned} \quad (1.2.48)$$

By changing variables in (1.2.47), we have

$$\begin{aligned} \zeta_{MT3}(c, b, s, a) &= K_1(c, b, s, a) - (-1)^b H(b, s, c, a) - (-1)^c H(c, s, b, a), \\ \zeta_{MT3}(a, c, s, b) &= K_1(a, c, s, b) - (-1)^c H(c, s, a, b) - (-1)^a H(a, s, c, b). \end{aligned}$$

Substituting the resulting expressions into (1.2.48) with $t = c \in \mathbb{N}$, and using $H(c, s, b, a) = H(s, c, a, b)$, we have the following functional relation.

Theorem 1.2.28. *For $a, b, c \in \mathbb{N}$, $s \in \mathbb{C}$, except for singular points, we have*

$$\begin{aligned} & H(a, s, c, b) - H(a, c, s, b) + (-1)^{a+c} H(c, s, a, b) + (-1)^{c+b} H(s, c, a, b) \\ &= (-1)^b K_1(c, b, s, a) + (-1)^a K_1(a, c, s, b) + \zeta(a+s)\zeta(b+c) - K_2(a, b, s, c). \end{aligned} \quad (1.2.49)$$

By this theorem, we obtain the following corollary.

Corollary 1.2.29. *For $a \in \mathbb{N}$, we have*

$$2H(a, a, a, a) = 2(-1)^a K_1(a, a, a, a) - K_2(a, a, a, a) + (-1)^a (\zeta(2a))^2. \quad (1.2.50)$$

Next we show a functional relation for $\zeta_{MT3}(a, b, c, s)$. This functional relation essentially coincides with the functional relation obtained in ([25, Theorem 4.5])

Theorem 1.2.30. *For $a, b, c \in \mathbb{N}$ and $s \in \mathbb{C}$, except for singular points, we have*

$$\begin{aligned} & \zeta_{MT3}(a, b, c, s) - (-1)^{a+b}\zeta_{MT3}(a, b, s, c) - (-1)^{b+c}\zeta_{MT3}(c, b, s, a) - (-1)^{a+c}\zeta_{MT3}(c, a, s, b) \\ &= (-1)^{a+b}K_1(a, b, s, c) + K_1(a, b, c, s) - (-1)^bK_2(a, c, b, s) - (-1)^aK_2(b, c, a, s) \\ &+ ((-1)^a + (-1)^b)\zeta(a+b)\zeta(c+s). \end{aligned} \quad (1.2.51)$$

Proof. By taking $u = c$ in (1.2.47) and multiplying $(-1)^{a+b}$, we have

$$(-1)^{a+b}\zeta_{MT3}(a, b, s, c) = (-1)^{a+b}K_1(a, b, s, c) - (-1)^aH(b, s, a, c) - (-1)^bH(a, s, b, c). \quad (1.2.52)$$

Taking $s = c$ and $u = s$ in (1.2.47), we have

$$\zeta_{MT3}(a, b, c, s) = K_1(a, b, c, s) - (-1)^bH(b, c, a, s) - (-1)^aH(a, c, b, s). \quad (1.2.53)$$

By changing variables $(a, b, s, t) \rightarrow (a, c, b, s)$ and $(a, b, s, t) \rightarrow (b, c, a, s)$ in (1.2.48), we have

$$\begin{aligned} & H(a, s, b, c) + H(b, c, a, s) = \\ & K_2(a, c, b, s) - \zeta(a+b)\zeta(c+s) - (-1)^c\zeta_{MT3}(c, b, s, a) - (-1)^a\zeta_{MT3}(a, b, s, c), \\ & H(b, s, a, c) + H(a, c, b, s) = \\ & K_2(b, c, a, s) - \zeta(a+b)\zeta(c+s) - (-1)^c\zeta_{MT3}(c, a, s, b) - (-1)^b\zeta_{MT3}(b, a, s, c), \end{aligned}$$

respectively, Hence substituting the resulting expressions into (1.2.52)+(1.2.53), we have this functional relation. \square

We define $M(a, b, c, d)$, $a, b, c, d \in \mathbb{N}$ by the right hand side of (1.2.51). We show an explicit formula for $\zeta_{MT3}(a, b, c, d)$.

Proposition 1.2.31. *For $a, b, c, d \in \mathbb{N}$, $a + b + c + d \in 2\mathbb{N}$, we have*

$$\begin{aligned} & 4\zeta_{MT3}(a, b, c, d) = \\ & M(a, b, c, d) - (-1)^{b+c}M(b, c, d, a) - (-1)^{a+c}M(c, d, a, b) - (-1)^{a+b}M(d, a, b, c) \end{aligned} \quad (1.2.54)$$

Proof. By changing variables in (1.2.51), we have

$$\begin{aligned} M(b, c, d, a) &= \zeta_{MT3}(b, c, d, a) - (-1)^{c+d}\zeta_{MT3}(c, d, a, b) \\ &\quad - (-1)^{d+b}\zeta_{MT3}(d, a, b, c) - (-1)^{b+c}\zeta_{MT3}(a, b, c, d), \\ M(c, d, a, b) &= \zeta_{MT3}(c, d, a, b) - (-1)^{d+a}\zeta_{MT3}(d, a, b, c) \\ &\quad - (-1)^{a+c}\zeta_{MT3}(a, b, c, d) - (-1)^{c+d}\zeta_{MT3}(b, c, d, a), \\ M(d, a, b, c) &= \zeta_{MT3}(d, a, b, c) - (-1)^{a+b}\zeta_{MT3}(a, b, c, d) \\ &\quad - (-1)^{b+d}\zeta_{MT3}(b, c, d, a) - (-1)^{d+a}\zeta_{MT3}(c, d, a, b). \end{aligned}$$

Multiply $(-1)^{b+c}$, $(-1)^{a+c}$, $(-1)^{a+b}$ on the both side of above three equations, respectively and sum them up. By using (1.2.51) in the case of $s = d$, we obtain (1.2.54). \square

1.2.8 Bernoulli Numbers and Multiple Zeta Values

Above these subsections, we only deal with double and triple zeta values. But in this subsection, we consider the Multiple Zeta Values (Euler-Zagier multiple sum)

$$\zeta(m_1, m_2, \dots, m_n) := \sum_{k_1 > k_2 > \dots > k_n > 0} \frac{1}{k_1^{m_1} k_2^{m_2} \dots k_n^{m_n}}, \quad (1.2.55)$$

and give an explicit formula for $\zeta(2m, 2m, \dots, 2m)$ (Corollary 1.2.34). Simultaneously, we will show an apparently new expression of Bernoulli numbers $B_n := B_n(0)$. In Gould [8], there are many formulas about for Bernoulli numbers $B_n := B_n(0)$. For example,

$$B_n = \sum_{k=0}^n \frac{1}{k+1} \sum_{j=0}^k (-1)^j \binom{k}{j} j^n,$$

and

$$B_n = \sum_{j=0}^n (-1)^j \binom{n+1}{j+1} \frac{n!}{(n+j)!} \sum_{k=0}^j (-1)^{j-k} \binom{j}{k} k^{n+j}.$$

Gould ended the paper [8] with the following conjecture.

The writer has seen no formula for B_n which does not require at least two actual summations. All the formulas we have quoted here are of this type.

However, we show an expression for B_n which needs only “one” summation (Corollary 1.2.33).

After I had completed my proof of the following result, Professor Masanobu Kaneko kindly informed me that the same result is included in an unpublished paper [58].

Theorem 1.2.32. *Let l, m and n be positive integers, x be an indeterminate element. Then*

$$\begin{aligned} & \frac{(-1)^{m+1} i}{(2\pi x)^m} \sum' \delta \exp((\pm\omega_{2m} \pm \omega_{2m}^2 \pm \dots \pm \omega_{2m}^m) \pi i x) \\ &= 1 + \sum_{n=1}^{\infty} (-1)^n \zeta(\underbrace{2m, 2m, \dots, 2m}_n) x^{2mn}, \end{aligned} \quad (1.2.56)$$

where

$$\omega_m^l = \exp(2\pi i l / m) \quad (1 \leq l \leq m).$$

The symbol \sum' means that all cases of \pm are taken, that is to say, it contains 2^m cases. And δ is defined by

$$\delta := \begin{cases} 1 & \text{if } -1 \text{ appears even times,} \\ -1 & \text{if } -1 \text{ appears odd times.} \end{cases}$$

Proof. Using

$$\sin \pi \omega_{2m}^l x = \frac{\exp(i\pi \omega_{2m}^l x) - \exp(-i\pi \omega_{2m}^l x)}{2i}$$

and

$$\prod_{l=1}^m \omega_{2m}^l = \omega_{2m}^{m(m+1)/2} = i^{m+1},$$

we obtain

$$\prod_{l=1}^m \frac{\sin \pi \omega_{2m}^l x}{\pi \omega_{2m}^l x} = \frac{(-1)^{m+1} i}{(2\pi x)^m} \sum' \delta \exp((\pm\omega_{2m} \pm \omega_{2m}^2 \pm \dots \pm \omega_{2m}^m) \pi i x).$$

On the other hand, using

$$\frac{\sin \pi t}{\pi t} = \prod_{n=1}^{\infty} \left(1 - \frac{t^2}{n^2}\right),$$

we have

$$\begin{aligned}
\prod_{l=1}^m \frac{\sin \pi \omega_{2m}^l x}{\pi \omega_{2m}^l x} &= \prod_{l=1}^m \prod_{n=1}^{\infty} \left(1 - \frac{\omega_m^l x^2}{n^2} \right) = \prod_{n=1}^{\infty} \left(1 - \frac{x^{2m}}{n^{2m}} \right) \\
&= 1 - \left(\sum_{k=1}^{\infty} \frac{1}{k^{2m}} \right) x^{2m} + \left(\sum_{k_1 > k_2 > 0} \frac{1}{k_1^{2m} k_2^{2m}} \right) x^{4m} - \left(\sum_{k_1 > k_2 > k_3 > 0} \frac{1}{k_1^{2m} k_2^{2m} k_3^{2m}} \right) x^{6m} + \dots \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n \zeta(\underbrace{2m, 2m, \dots, 2m}_n) x^{2mn}.
\end{aligned}$$

□

By (1.1.5) and (1.2.56)

Corollary 1.2.33. *we have*

$$B_{2m} = \frac{-2i^{3m+1}(2m)!}{2^{3m}(3m)!} \sum' \delta(\pm \omega_{2m} \pm \omega_{2m}^2 \pm \dots \pm \omega_{2m}^m)^{3m}. \quad (1.2.57)$$

This formula includes only “one” summation .

Corollary 1.2.34. *Using (1.2.56), we obtain*

$$\zeta(\underbrace{2m, 2m, \dots, 2m}_n) = \frac{(-1)^{(n+1)(m+1)} i^{m+1} \pi^{2nm}}{2^m (2nm + m)!} \sum' \delta(\pm \omega_{2m} \pm \omega_{2m}^2 \pm \dots \pm \omega_{2m}^m)^{2nm+m}. \quad (1.2.58)$$

Example 1.2.35.

$$\zeta(\underbrace{2, 2, \dots, 2}_n) = \frac{(-1)^{2n+2} (-1) \pi^{2n}}{2(2n+1)!} \sum' \delta(\pm (-1))^{2n+1} = \frac{\pi^{2n}}{(2n+1)!}. \quad (1.2.59)$$

$$\zeta(\underbrace{4, 4, \dots, 4}_n) = \frac{(-1)^{3n+3} (-i) \pi^{4n}}{4(4n+2)!} \sum' \delta(\pm i \pm (-1))^{4n+2} = \frac{2^{2n+1} \pi^{4n}}{(4n+2)!}. \quad (1.2.60)$$

Remark 1.2.36. *We can obtain (1.2.60) by another method . Note that*

$$\frac{2}{\pi^2 t^2} \sin\left(\frac{1+i}{2} \pi t\right) \sin\left(\frac{1-i}{2} \pi t\right) = \frac{\cosh \pi t - \cos \pi t}{\pi^2 t^2} = \sum_{n=0}^{\infty} \frac{2\pi^{4n} t^{4n}}{(4n+2)!}.$$

On the other hand, we have

$$\begin{aligned}
\frac{2}{\pi^2 t^2} \sin\left(\frac{1+i}{2} \pi t\right) \sin\left(\frac{1-i}{2} \pi t\right) &= \prod_{n=1}^{\infty} \left(1 - \frac{it^2}{2n^2} \right) \times \prod_{n=1}^{\infty} \left(1 + \frac{it^2}{2n^2} \right) = \prod_{n=1}^{\infty} \left(1 + \frac{t^4}{4n^4} \right) \\
&= 1 + \left(\sum_{k=1}^{\infty} \frac{4^{-1}}{k^4} \right) t^4 + \left(\sum_{k_1 > k_2 > 0} \frac{4^{-2}}{k_1^4 k_2^4} \right) t^8 + \left(\sum_{k_1 > k_2 > k_3 > 0} \frac{4^{-3}}{k_1^4 k_2^4 k_3^4} \right) t^{12} + \dots \\
&= 1 + \sum_{n=1}^{\infty} 4^{-n} \zeta(\underbrace{4, 4, \dots, 4}_n) t^{4n}.
\end{aligned}$$

Hence (1.2.60) follows.

2 Universality for several types of multiple zeta functions

2.1 Applications of inversion formulas to the joint t -universality of Lerch zeta functions

2.1.1 Introduction for applications of inversion formulas to the joint universality

Definition 2.1.1. *The Lerch zeta function $L(\lambda, a, s)$, for $\lambda \in \mathbb{R}$, $0 < a \leq 1$ and $\Re(s) > 1$, is defined by the series*

$$L(\lambda, a, s) := \sum_{n=0}^{\infty} \frac{e^{2\pi i \lambda n}}{(n+a)^s}. \quad (2.1.1)$$

We call the parameter in the numerator the λ -part parameter and call the parameter in the denominator the a -part parameter. In this case, the λ -part parameter is λ , and the a -part parameter is a .

When $\lambda \in \mathbb{Z}$, the Lerch zeta function $L(\lambda, a, s)$ reduces to the Hurwitz zeta function $\zeta(s, a)$. If $\lambda \notin \mathbb{Z}$, the function $L(\lambda, a, s)$ is analytically continuable to an entire function. But the function $\zeta(s, a)$ is analytically continuable to a meromorphic function, which has a simple pole at $s = 1$. We prepare some notation for t -universality.

By $\text{meas}\{A\}$ we denote the Lebesgue measure of the set A , and, for $T > 0$, we use the notation

$$\nu_T^t\{\dots\} := \frac{1}{T} \text{meas}\{t \in [0, T]; \dots\}$$

where in place of dots some condition satisfied by t is to be written. Let $D := \{s \in \mathbb{C}; 1/2 < \Re(s) < 1\}$. The next theorem is proved by A. Laurinćikas and K. Matsumoto in [17] (see also [16, p.122, Theorem 3.1]).

Theorem 2.1.2 (Joint t -universality [17, Theorem 1]). *Let a_1, \dots, a_m be transcendental and algebraically independent numbers, $\lambda_1 = b_1/q_1, \dots, \lambda_m = b_m/q_m$, $(b_1, q_1) = 1, \dots, (b_m, q_m) = 1$, where q_1, \dots, q_m are distinct positive integers and b_1, \dots, b_m are positive integers with $b_1 < q_1, \dots, b_m < q_m$. Let K_1, \dots, K_m be compact subsets of the strip D , with connected complements, and for $1 \leq l \leq m$, let $f_l(s)$ be a continuous function on K_l which is analytic in the interior of K_l . Then for every $\varepsilon > 0$ it holds that*

$$\liminf_{T \rightarrow \infty} \nu_T^t \left\{ \sup_{1 \leq l \leq m} \sup_{s \in K_l} |L(\lambda_l, a_l, s + it) - f_l(s)| < \varepsilon \right\} > 0.$$

As an application of Theorem 2.1.2, we obtain the following theorem.

Theorem 2.1.3 ([17, Theorem 2]). *Let λ_l, a_l be as in Theorem 2.1.2, and F_k , $0 \leq k \leq n$ be continuous functions on \mathbb{C}^{Nm} . Suppose*

$$\sum_{k=0}^n s^k F_k(L(\lambda_1, a_1, s), \dots, L(\lambda_m, a_m, s), L'(\lambda_1, a_1, s), \dots, L'(\lambda_m, a_m, s), \dots, L^{(N-1)}(\lambda_1, a_1, s), \dots, L^{(N-1)}(\lambda_m, a_m, s)) = 0$$

identically for all $s \in \mathbb{C}$. Then $F_k \equiv 0$, $0 \leq k \leq n$.

Remark. The assumption of [17, Theorem 1] is that a_1, \dots, a_m are merely transcendental. But actually we need the assumption that a_1, \dots, a_m are not only transcendental but also algebraically independent (see [18]).

This theorem means that joint functional independence of Lerch zeta functions is implied by joint t -universality. However we obtain new types of t -universality in Theorem 2.1.16 below by the inversion formula (2.1.2) which is a relation of Lerch zeta functions.

This section can be divided into three subsections. In Section 2.1.2, we recall the inversion formula (2.1.2) in Lemma 2.1.4. Next we consider λ -joint and a' -joint t -universality, which are defined by

Definitions 2.1.6 and 2.1.7. By formula (2.1.2), we can obtain a theorem of t -universality in Theorem 2.1.8. In Section 2.1.3, we consider (λ, λ) -joint, (λ, a') -joint and (a', a') -joint t -universality, which are defined by Definitions 2.1.10, 2.1.11 and 2.1.12. By formula (2.1.2), we can obtain theorems of t -universality in Theorems 2.1.13 and 2.1.14. In Section 2.1.4, we show the existence of (λ, λ) -joint t -universality in Theorem 2.1.15, by modifying the proof of [15, Theorem 1]. In Theorem 2.1.2, λ_l are rational numbers, but we can also obtain the case of when λ_l are irrational numbers. And we establish the existence of λ -joint, a' -joint, (λ, a') -joint and (a', a') -joint t -universality by chart (2.1.11). Finally, as an application of Theorem 2.1.15, we show the joint functional independence in Theorem 2.2.15.

2.1.2 Joint t -universality

Firstly, we quote the following inversion formula [36, Theorem 2.2]. Let $i = \sqrt{-1}$, and

$$\omega_m^j = \exp(2\pi i j/m), \quad j, m \in \mathbb{N}, \quad 0 \leq j \leq m-1.$$

Lemma 2.1.4 (The inversion formula [36, Theorem 2.2]). *We have*

$$L\left(m\lambda, a + \frac{j}{m}, s\right) = m^{s-1} e^{-2\pi i \lambda j} \sum_{n=0}^{m-1} \omega_m^{-jn} L\left(\lambda + \frac{n}{m}, ma, s\right). \quad (2.1.2)$$

Proof. We give a proof for the convenience of readers. If $J \in \mathbb{N}$, we have

$$\sum_{n=0}^{m-1} (\omega_m^j)^n (\omega_m^n)^J = \begin{cases} m & j + J \equiv 0 \pmod{m}, \\ 0 & \text{otherwise.} \end{cases}$$

From this formula, we have

$$m^{-1} \sum_{n=0}^{m-1} \omega_m^{-jn} \sum_{h=0}^{\infty} \frac{\omega_m^{nh} z^{h/m}}{(h+ma)^s} = \sum_{h=0}^{\infty} \frac{z^{(mh+j)/m}}{(mh+ma+j)^s}, \quad z := e^{2\pi i m \lambda}.$$

We obtain (2.1.2) by the above equation. \square

Now we consider t -universality for Lerch zeta functions. We put $0 < R < 1/4$, $K := \{s; |s-3/4| \leq R\}$ and k be a positive integer. We denote by U the set of functions continuous on the disk K and analytic in the interior of K .

Definition 2.1.5 (t -universality).

The t -universality for $L(\lambda, a, s)$ is the following property : Let $\lambda \in \mathbb{R}$ and $a > 0$ be fixed and $f(s) \in U$. Then for every $\varepsilon > 0$

$$\liminf_{T \rightarrow \infty} \nu_T^t \left\{ \max_{s \in K} |k^{it} L(\lambda, a, s+it) - f(s)| < \varepsilon \right\} > 0. \quad (2.1.3)$$

Next we define joint t -universality for the parameter which is related to the λ -part. For convenience we use the symbol

$$\max_{n=0}^{m-1} := \max_{0 \leq n \leq m-1}.$$

Definition 2.1.6 (λ -joint t -universality).

The λ -joint t -universality for $\{L(\lambda + n/m, a, s)\}_n$ is the following property : Let $\lambda \in \mathbb{R}$ and $a > 0$ be fixed and $f_n(s) \in U$. Then for every $\varepsilon > 0$

$$\liminf_{T \rightarrow \infty} \nu_T^t \left\{ \max_{n=0}^{m-1} \max_{s \in K} |k^{it} L\left(\lambda + \frac{n}{m}, a, s+it\right) - f_n(s)| < \varepsilon \right\} > 0. \quad (2.1.4)$$

Further, we prepare the following definition.

Definition 2.1.7 (a' -joint t -universality).

The a' -joint t -universality for $\{L(\lambda, a + j/m, s)\}_j$ is the following property : Let $\lambda \in \mathbb{R}$ and $a > 0$ be fixed and $f(s) \in U$. Then for every $\varepsilon > 0$

$$\liminf_{T \rightarrow \infty} \nu_T^t \left\{ \max_{j=0}^{m-1} \max_{s \in K} \left| k^{it} L\left(\lambda, a + \frac{j}{m}, s + it\right) - f(s) \right| < \varepsilon \right\} > 0. \quad (2.1.5)$$

We note that, in (2.1.5), all the functions which are to be approximated are the same $f(s)$ for $0 \leq j \leq m-1$. Under definitions 2.1.6 and 2.1.7, we have the next theorem.

Theorem 2.1.8. We have for $0 \leq j \leq m-1$,

$$\begin{aligned} & \text{if the set } \{L(\lambda + n/m, ma, s)\}_n \text{ has } \lambda\text{-joint } t\text{-universality,} \\ & \implies \text{the set } \{L(m\lambda, a + j/m, s)\}_j \text{ has } a'\text{-joint } t\text{-universality.} \end{aligned}$$

Epecially if we put $a = 1/m$ and $ma + j = l = 1, 2, \dots, m$, we have for $1 \leq l \leq m$,

$$\begin{aligned} & \text{if the set } \{L(\lambda + n/m, 1, s)\}_n \text{ has } \lambda\text{-joint } t\text{-universality} \\ & \implies \text{the set } \{L(m\lambda, l/m, s)\}_l \text{ has } a'\text{-joint } t\text{-universality.} \end{aligned}$$

Proof. To simplify the argument, we consider the case of $k = 1$. We can take

$$g(s) := m^{s-1} e^{-2\pi i \lambda j} \sum_{n=0}^{m-1} \omega_m^{-jn} f_n(s) \in U,$$

$$f_0(s) = m^{1-s} e^{2\pi i \lambda j} g(s), \quad f_j(s) = 0, \quad 1 \leq j \leq m-1$$

without loss of generality. For every $\varepsilon > 0$, we have

$$\liminf_{T \rightarrow \infty} \nu_T^t \left\{ \max_{n=0}^{m-1} \max_{s \in K} \left| m^{it} L\left(\lambda + \frac{n}{m}, ma, s + it\right) - f_n(s) \right| < \varepsilon \right\} > 0, \quad (2.1.6)$$

because $\{L(\lambda + n/m, ma, s)\}_n$ has λ -joint t -universality. By formula (2.1.2), for any t which satisfies (2.1.6), we have

$$\begin{aligned} & \max_{j=0}^{m-1} \max_{s \in K} \left| L\left(\lambda, a + \frac{j}{m}, s + it\right) - g(s) \right| \\ & \leq \sum_{j=0}^{m-1} \max_{s \in K} \left| m^{s-1} e^{-2\pi i \lambda j} \sum_{n=0}^{m-1} \omega_m^{-jn} \left\{ m^{it} L\left(\lambda + \frac{n}{m}, ma, s + it\right) - f_n(s) \right\} \right| \\ & \leq \sum_{j=0}^{m-1} \sum_{n=0}^{m-1} \max_{s \in K} \left| m^{it} L\left(\lambda + \frac{n}{m}, ma, s + it\right) - f_n(s) \right| < m^2 \varepsilon. \end{aligned} \quad (2.1.7)$$

This formula implies that $\{L(\lambda, (a + j)/m, s)\}_j$ has a' -joint t -universality. □

We obtain the next corollary by removing $\max_{j=0}^{m-1}$ and $\sum_{j=0}^{m-1}$ in (2.1.7).

Corollary 2.1.9. We have for $0 \leq j \leq m-1$,

$$\begin{aligned} & \text{if the set } \{L(\lambda + n/m, ma, s)\}_n \text{ has } \lambda\text{-joint } t\text{-universality,} \\ & \implies \text{the function } L(m\lambda, a + j/m, s) \text{ has } t\text{-universality.} \end{aligned}$$

Epecially, we have for $1 \leq l \leq m$

$$\begin{aligned} & \text{if the set } \{L(\lambda + n/m, 1, s)\}_n \text{ has } \lambda\text{-joint } t\text{-universality,} \\ & \implies \text{the function } L(m\lambda, l/m, s) \text{ has } t\text{-universality.} \end{aligned}$$

Remark. If $\lambda \in \mathbb{Q}$, it is shown in [16, p.119, Theorem 2.3] that the function $L(\lambda, l/m, s)$ has universality by joint universality of Dirichlet L functions. If λ is not a rational number, the universality of $L(\lambda, l/m, s)$ is unknown.

2.1.3 Double joint t -universality

Let k , m_1 and m_2 be positive integers and satisfy $(m_1, m_2) = 1$ in this section. Firstly, we define the following double joint t -universality which is related to the λ -part.

Definition 2.1.10 ((λ, λ)-joint t -universality).

The (λ, λ) -joint t -universality for $\{L(\lambda + n_1/m_1 + n_2/m_2, a, s)\}_{n_1, n_2}$ is the following property : Let $\lambda \in \mathbb{R}$, $a > 0$ be fixed and $f_{n_1, n_2}(s) \in U$. Then for every $\varepsilon > 0$

$$\liminf_{T \rightarrow \infty} \nu_T^t \left\{ \max_{n_1=0}^{m_1-1} \max_{n_2=0}^{m_2-1} \max_{s \in K} \left| k^{it} L\left(\lambda + \frac{n_1}{m_1} + \frac{n_2}{m_2}, a, s + it\right) - f_{n_1, n_2}(s) \right| < \varepsilon \right\} > 0. \quad (2.1.8)$$

Next we define the double joint t -universality which is related to both the λ -part and the a -part.

Definition 2.1.11 ((λ, a')-joint t -universality).

The (λ, a') -joint t -universality for $\{L(\lambda + n_1/m_1, a + j_2/m_2, s)\}_{n_1, j_2}$ is the following property : Let $\lambda \in \mathbb{R}$, $a > 0$ be fixed and $f_{n_1}(s) \in U$. Then for every $\varepsilon > 0$

$$\liminf_{T \rightarrow \infty} \nu_T^t \left\{ \max_{n_1=0}^{m_1-1} \max_{j_2=0}^{m_2-1} \max_{s \in K} \left| k^{it} L\left(\lambda + \frac{n_1}{m_1}, a + \frac{j_2}{m_2}, s + it\right) - f_{n_1}(s) \right| < \varepsilon \right\} > 0. \quad (2.1.9)$$

This is a generalization of λ -joint t -universality. We can also consider the case that the variable of λ -part is n_2/m_2 and the variable of a -part is j_1/m_1 . We call it (λ, a) -joint t -universality, too.

Finally, we define the double joint t -universality is related to the a -part.

Definition 2.1.12 ((a', a')-joint t -universality).

The (a', a') -joint t -universality for $\{L(\lambda, a + j_1/m_1 + j_2/m_2, s)\}_{j_1, j_2}$ is the following property : Let $\lambda \in \mathbb{R}$, $a > 0$ be fixed and $f(s) \in U$. Then for every $\varepsilon > 0$

$$\liminf_{T \rightarrow \infty} \nu_T^t \left\{ \max_{j_1=0}^{m_1-1} \max_{j_2=0}^{m_2-1} \max_{s \in K} \left| k^{it} L\left(\lambda, a + \frac{j_1}{m_1} + \frac{j_2}{m_2}, s + it\right) - f(s) \right| < \varepsilon \right\} > 0. \quad (2.1.10)$$

This is a generalization of a' -joint t -universality.

Since $(m_1, m_2) = 1$, we can see that

$$\{L(\lambda + m_1 n_1/m_1 + n_2/m_2, m_1 m_2 a, s)\}_{n_1, n_2} = \{L(\lambda + n_1/m_1 + n_2/m_2, m_1 m_2 a, s)\}_{n_1, n_2}$$

By using method similar to proof of Theorem 2.1.8 and Corollary 2.1.9, we have the next theorems.

Theorem 2.1.13. We have for $0 \leq j_2 \leq m_2 - 1$,

if the set $\{L(\lambda + n_1/m_1 + n_2/m_2, m_1 m_2 a, s)\}_{n_1, n_2}$ has (λ, λ) -joint universality,

$$\implies \begin{cases} \text{the set } \{L(m_2 \lambda + n_1/m_1, m_1 a + j_2/m_2, s)\}_{n_1, j_2} \text{ has } (\lambda, a')\text{-joint universality.} \\ \text{the set } \{L(m_2 \lambda + n_1/m_1, m_1 a + j_2/m_2, s)\}_{n_1} \text{ has } \lambda\text{-joint universality.} \end{cases}$$

Theorem 2.1.14. We have for $0 \leq j_1 \leq m_1 - 1$,

if the set $\{L(m_2 \lambda + n_1/m_1, m_1 a + j_2/m_2, s)\}_{n_1, j_2}$ has (λ, a') -joint universality,

$$\implies \begin{cases} \text{the set } \{L(m_1 m_2 \lambda, a + j_1/m_1 + j_2(m_1 m_2)^{-1}, s)\}_{j_1, j_2} \text{ has } (a', a')\text{-universality.} \\ \text{the set } \{L(m_1 m_2 \lambda, a + j_1/m_1 + j_2(m_1 m_2)^{-1}, s)\}_{j_2} \text{ has } a'\text{-universality.} \end{cases}$$

The next chart describes the relation among various types of universality. The arrows $\alpha \searrow \beta$ and $\alpha \downarrow \beta$ mean that we can obtain β by applying inversion formula (2.1.2) to α .

$$\begin{array}{ccccccc} (\lambda, \lambda) & & & & & & \\ \downarrow \text{Thm.} & \searrow \text{Thm.} & & & & & \\ 2.1.13 & 2.1.13 & & & & & \\ (\lambda, a') & \supset & \lambda & & & & \\ \downarrow \text{Thm.} & \searrow \text{Thm.} & \downarrow \text{Thm.} & \searrow \text{Cor.} & & & \\ 2.1.14 & 2.1.14 & 2.1.8 & 2.1.9 & & & \\ (a', a') & \supset & a' & \supset & t\text{-uni.} & & \\ \text{double} & & \text{joint} & & \text{single} & & \end{array} \quad (2.1.11)$$

2.1.4 The existence of joint t -universality

In this section, we show the existence of joint t -universality. Firstly, we will show (λ, λ) -joint universality in the following theorem.

Theorem 2.1.15. *Suppose $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ or $\lambda = n_3/m_3$, and $1 < m_1, m_2, m_3 \in \mathbb{N}$ are relatively prime. Let a be transcendental and $f_{n_1, n_2}(s) \in U$. Then for every $\varepsilon > 0$*

$$\liminf_{T \rightarrow \infty} \nu_T^t \left\{ \max_{n_1=0}^{m_1-1} \max_{n_2=0}^{m_2-1} \max_{s \in K} \left| L\left(\lambda + \frac{n_1}{m_1} + \frac{n_2}{m_2}, a, s + it\right) - f_{n_1, n_2}(s) \right| < \varepsilon \right\} > 0. \quad (2.1.12)$$

Proof. The proof of this theorem is based on [15, Theorem 1]. We have to check that [17, Lemma 1] and [17, Lemma 3] can be used in our present situation. We can confirm [17, Lemma 1] by [15, Lemma 2] and [15, Theorem 3] (these Lemma and theorem only need the transcendency of a). Next, in order to confirm [17, Lemma 3], we have to check conditions (a), (b) and (c) of [17, Lemma 3]. We need $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ or m_1, m_2, m_3 are relatively prime for condition (b). How to confirm assumptions (b) and (c) are the same as [17, Lemma 3]. To check condition (a), we modify the proof of [17, Lemma 6]. Firstly we put

$$\lambda_{n_1, n_2} := \lambda + \frac{n_1}{m_1} + \frac{n_2}{m_2}, \quad M := m_1 m_2.$$

Let $\mathfrak{B}(\mathbb{C})$ stands for the class of Borel sets of the space S . Similarly to [17, p. 221 (12)] let $\mu_{1,1}, \dots, \mu_{n_1, n_2}$ be complex measures on $(\mathbb{C}, \mathfrak{B}(\mathbb{C}))$ with compact supports contained in D such that

$$\sum_{n=1}^{\infty} \left| \sum_{n_1=0}^{m_1-1} \sum_{n_2=0}^{m_2-1} \int_{\mathbb{C}} \frac{e^{2\pi i \lambda_{n_1, n_2}}}{n^s} d\mu_{n_1, n_2}(s) \right| < \infty.$$

By $(m_1, m_2) = 1$ and Chinese Remainder Theorem, that is, $\mathbb{Z}/M\mathbb{Z} \cong \mathbb{Z}/m_1\mathbb{Z} \times \mathbb{Z}/m_2\mathbb{Z}$, we can rewrite it as

$$\sum_{n=1}^{\infty} \left| \sum_{k=1}^M \int_{\mathbb{C}} \frac{e^{2\pi i \lambda n} e^{2\pi i k n / M}}{n^s} d\mu_k(s) \right| < \infty,$$

where $\{d\mu_k\}$ is a rearrangement of $\{d\mu_{n_1, n_2}\}$. Hence we have

$$\sum_{\substack{n=1 \\ n \equiv r \pmod{M}}}^{\infty} \left| \sum_{k=1}^M \int_{\mathbb{C}} \frac{e^{2\pi i \lambda n} e^{2\pi i k r / M}}{n^s} d\mu_k(s) \right| < \infty, \quad 1 \leq r \leq M.$$

And let

$$\nu_r(A) := \sum_{k=1}^M e^{2\pi i k r / M} \mu_k(A), \quad A \in \mathfrak{B}(\mathbb{C}), \quad 1 \leq r \leq M.$$

We have

$$\sum_{\substack{n=1 \\ n \equiv r \pmod{M}}}^{\infty} \left| \int_{\mathbb{C}} e^{2\pi i \lambda n} n^{-s} d\nu_r(s) \right| < \infty, \quad 1 \leq r \leq M.$$

By the equality

$$\left| \int_{\mathbb{C}} e^{2\pi i \lambda n} n^{-s} d\nu_r(s) \right| = \left| e^{2\pi i \lambda n} \int_{\mathbb{C}} n^{-s} d\nu_r(s) \right| = \left| \int_{\mathbb{C}} n^{-s} d\nu_r(s) \right|,$$

similarly [17, p.221, (13)] we obtain

$$\sum_{\substack{n=1 \\ n \equiv r \pmod{M}}}^{\infty} \left| \int_{\mathbb{C}} n^{-s} d\nu_r(s) \right| < \infty, \quad 1 \leq r \leq M. \quad (2.1.13)$$

We continue the same argument as in the proof of [17, Lemma 6]. By $(m_1, m_2) = 1$, we have

$$\sum_{r=1}^M e^{2\pi i r (\lambda_{n_1, n_2} - \lambda_{n'_1, n'_2})} = \begin{cases} M & n_1 = n'_1, n_2 = n'_2 \\ 0 & \text{otherwise} \end{cases}. \quad (2.1.14)$$

By using (2.1.14) and modifying the proof of [17, Lemma 6], we find that condition (a) is valid. Hence we now verify [17, Lemma 3] in our case, and so, we obtain Theorem 2.1.15. \square

By Chart (2.1.11), we have the next theorem.

Theorem 2.1.16.

There exist Lerch zeta functions for which (λ, a') -joint, (resp. (a', a') -joint a' -joint and λ -joint) t -universality hold.

Remark. We can also consider triple t -joint universality, and by modifying the proof of Theorem 2.1.15, we obtain examples which satisfy $(\lambda, \lambda, \lambda)$ joint t -universality. Moreover, by the inversion formula (2.1.2), we obtain the following chart larger than (2.1.11).

$$\begin{array}{ccccccc} & & (\lambda, \lambda, \lambda) & & & & \\ & & \downarrow & \searrow & & & \\ (\lambda, \lambda, a') & \supset & & (\lambda, \lambda) & & & \\ & & \downarrow & \searrow & \downarrow & \searrow & \\ (\lambda, a', a') & \supset & & (\lambda, a') & \supset & \lambda & \\ & & \downarrow & \searrow & \downarrow & \searrow & \downarrow & \searrow \\ (a', a', a') & \supset & & (a', a') & \supset & a' & \supset & t\text{-uni.} \\ \text{triple} & & & \text{double} & & \text{joint} & & \text{single} \end{array} \quad (2.1.15)$$

At the end of this paper, we show the following theorem.

Theorem 2.1.17. *Let $\{\lambda_l\}$ be a rearrangement of $\{\lambda_{n_1, n_2}\}$, a be as in Theorem 2.1.15. Suppose F_k , $0 \leq k \leq n$ be continuous functions on \mathbb{C}^{Nm} . Suppose*

$$\sum_{k=0}^n s^k F_k(L(\lambda_1, a, s), \dots, L(\lambda_m, a, s), L'(\lambda_1, a, s), \dots, L'(\lambda_m, a, s), \dots, L^{(N-1)}(\lambda_1, a, s), \dots, L^{(N-1)}(\lambda_m, a, s)) = 0$$

identically for all $s \in \mathbb{C}$. Then $F_k \equiv 0$, $0 \leq k \leq n$.

Proof. The proof of this theorem is completely the same as the proof of [17, Theorem 2]. \square

2.2 The existence and the non-existence of joint t -universality of generalized Lerch zeta functions

2.2.1 Introduction for joint universality of generalized Lerch zeta functions

In this section, we will show two main theorems. Firstly we show the next theorem, which gives the universality under the assumption weaker than that in Theorem 2.1.2.

Theorem 2.2.1. *Suppose $0 < a_l < 1$ be algebraically independent numbers and $0 < \lambda_l \leq 1$ for $1 \leq l \leq m$. Let $f_l(s)$ be functions analytic in the interior of K_l and continuous on K_l . Then for every $\varepsilon > 0$ it holds that*

$$\liminf_{T \rightarrow \infty} \nu_T^T \left\{ \sup_{1 \leq l \leq m} \sup_{s \in K_l} |L(\lambda_l, a_l, s + i\tau) - f_l(s)| < \varepsilon \right\} > 0.$$

Next we introduce a generalization of Lerch zeta functions and consider their joint universality.

Definition 2.2.2. *The generalized Lerch zeta functions $\mathfrak{L}(\lambda, a, b, c; s)$, for $0 < \lambda \leq 1$, $0 < a \leq 1$, $0 < b \leq 1$, $\Re(s) > 1$ and $c \in \mathbb{C}$, is defined by*

$$\mathfrak{L}(\lambda, a, b, c; s) := \sum_{n=0}^{\infty} \frac{e^{2\pi i \lambda n}}{(n+a)^{s-c}(n+b)^c}. \quad (2.2.1)$$

The cases of $a = b$ or $c = 0$, these functions coincide with Lerch zeta functions. We remark that for $0 < \lambda \leq 1$, $\mathfrak{L}(\lambda, a, b, c; s)$ is meromorphic in the half-plane $\sigma > 0$, since

$$\mathfrak{L}(\lambda, a, b, c; s) = L(\lambda, a, s) + \sum_{n=0}^{\infty} \frac{e^{2\pi i \lambda n}}{(n+a)^{s-c}(n+b)^c} \left(1 - \frac{(n+b)^c}{(n+a)^c}\right),$$

and the series on the right hand side converges in the half-plane $\sigma > 0$. For $0 < \lambda < 1$, this series converges uniformly on any compact subset in the half-plane $\sigma > \sigma_0$ for any $\sigma_0 > 0$. The case of $b = 1$, $c = -1$ is

$$\sum_{n=0}^{\infty} \frac{e^{2\pi i \lambda n} (n+1)}{(n+a)^{s+1}} = \sum_{n_1, n_2=0}^{\infty} \frac{e^{2\pi i \lambda (n_1+n_2)}}{(n_1+n_2+a)^{s+1}}.$$

(See for example [38, p.85, (10)].) Hence $\mathfrak{L}(\lambda, a, b, c; s)$ contain a special case of Barnes double zeta functions. The following theorem gives a joint universality property of $\mathfrak{L}(\lambda, a, b, c; s)$. This is a partial solution of the problem of (joint) universality of multiple zeta functions presented in [19, Section 2].

Theorem 2.2.3. *Suppose $0 < a_l < 1$ be algebraically independent numbers and $0 < \lambda_l \leq 1$, $0 < b_l \leq 1$ for $1 \leq l \leq m$. Let $f_l(s)$ be functions analytic in the interior of K_l and continuous on K_l . Then for every $\varepsilon > 0$ it holds that*

$$\liminf_{T \rightarrow \infty} \nu_T^T \left\{ \sup_{1 \leq l \leq m} \sup_{s \in K_l} |\mathfrak{L}(\lambda_l, a_l, b_l, c; s + i\tau) - f_l(s)| < \varepsilon \right\} > 0.$$

This section is divided into six subsections. Section 2.2.2 is a preparation for the proof of these theorems. In Sections 2.2.3 and 2.2.4, we prove Theorems 2.2.1 and 2.2.3, respectively. We consider the proof of Theorems 2.2.1 and 2.2.3 in Section 2.2.5. We show examples of the non-existence of the joint t -universality for Lerch zeta functions in Section 2.2.6.

2.2.2 Preliminaries

In this section, we quote definitions and theorems from [14] and [17], and we omit the proof of those theorems. Denote by $H(D)$ the space of analytic on D functions equipped with the topology of uniform convergence on compacta. Let $\mathfrak{B}(S)$ stands for the class of Borel sets of the space S . Define on $(H^m(D), \mathfrak{B}(H^m(D)))$ the probability measure

$$P_{\underline{L}}^T(A) := \nu_T^T((L(\lambda_1, a_1, s + i\tau), \dots, L(\lambda_m, a_m, s + i\tau)) \in A), \quad A \in \mathfrak{B}(H^m(D)).$$

What we need is a limit theorem in the sense of weak convergence of probability measures for $P_{\underline{L}}^T$ as $T \rightarrow \infty$, with an explicit form of the limit measure. Denote by γ the unit circle on \mathbb{C} , and let

$$\Omega := \prod_{n=0}^{\infty} \gamma_n,$$

where $\gamma_n = \gamma$ for all $n \in \mathbb{N} \cup \{0\}$. With the product topology and pointwise multiplication the infinite dimensional torus Ω is a compact topological Abelian group. Denoting by m_{H_m} the probability Haar measure on $(\Omega^m, \mathfrak{B}(\Omega^m))$, where $\Omega^m := \Omega \times \dots \times \Omega$, we obtain a probability space $(\Omega^m, \mathfrak{B}(\Omega^m), m_{H_m})$.

Let $\omega_l(n)$ be the projection of $\omega_l \in \Omega$ to the coordinate space γ_n , and define on the probability space $(\Omega^m, \mathfrak{B}(\Omega^m), m_{H^m})$ the $H^m(D)$ -valued random element $\underline{L}(s, \underline{\omega})$ by

$$\underline{L}(s, \underline{\omega}) := (L(\lambda_1, a_1, s, \omega_1), \dots, L(\lambda_m, a_m, s, \omega_m)),$$

where

$$L(\lambda_l, a_l, s, \omega_l) := \sum_{n=0}^{\infty} \frac{e^{2\pi i \lambda_l n \omega_l(n)}}{(n + a_l)^s}, \quad s \in D, \quad \omega_l \in \Omega, \quad 1 \leq l \leq m.$$

The function $L(\lambda_l, a_l, s, \omega_l)$ is an $H(D)$ -valued random element. Let $P_{\underline{L}}$ stands for the distribution of the random element $\underline{L}(s, \underline{\omega})$, i.e.

$$P_{\underline{L}}(A) := m_{H^m}(\underline{\omega} \in \Omega^m; \underline{L}(s, \underline{\omega}) \in A), \quad A \in \mathfrak{B}(H^m(D)).$$

In [18, Theorem 1], the following lemma is proved in the case of $0 < \lambda_l < 1$. But we can prove the case of $0 < \lambda_l \leq 1$ similarly.

Lemma 2.2.4 ([18, Theorem 1]). *Suppose $0 < a_l < 1$ be algebraically independent numbers and $0 < \lambda_l \leq 1$ for $1 \leq l \leq m$. The probability measure $P_{\underline{L}}^T$ converges weakly to $P_{\underline{L}}$ as $T \rightarrow \infty$.*

Next we consider the support of the measure P . We recall that the minimal closed set $S_P \subseteq H^m(D)$ such that $P(S_P) = 1$ is called the support of P . The set S_P consists of all $\underline{f} \in H^m(D)$ such that for every neighborhood V of \underline{f} the inequality $P(V) > 0$ is satisfied. The support of the distribution of the random element X is called the support of X and is denoted by S_X .

Lemma 2.2.5 ([17, Lemma 2]). *Let $\{X_n\}$ be a sequence of independent $H^m(D)$ -valued random elements, and suppose that the series $\sum_{n=1}^{\infty} X_n$ converges almost surely. Then the support of the sum of this series is the closure of the set of all $\underline{f} \in H^m(D)$ which may be written as a convergent series $\underline{f} := \sum_{n=1}^{\infty} \underline{f}_n$, $\underline{f}_n \in S_{X_n}$.*

We quote some results on Hilbert spaces from [14, Chapter 6]. The subset $L \subset X$ is called a linear manifold if for all $x, y \in L$ and for all $\alpha, \beta \in \mathbb{C}$ the linear combination $\alpha x + \beta y \in L$. Let L be a linear manifold of X . The set of elements $x \in X$ such that $x \perp L$ is called the orthogonal complement of L and is denoted by L^\perp .

Lemma 2.2.6 ([14, Theorem 6.1.8]). *Let L be a linear manifold of X . Then L is dense in X if and only if $L^\perp = \{0\}$.*

Let X be a Hilbert space with an inner product $\langle x, y \rangle$ and a norm $\|x\| := \sqrt{\langle x, x \rangle}$.

Lemma 2.2.7 ([14, Theorem 6.1.11]). *Let f be a continuous linear functional on a Hilbert space X . Then there exists a unique element $y \in X$ such that $f(x) = \langle x, y \rangle$ for all $x \in X$.*

Lemma 2.2.8 ([14, Theorem 6.1.16]). *Let $\{x_n\}$ be a sequence in a Hilbert space X satisfying the following conditions :*

- (a) $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$;
- (b) $\sum_{n=1}^{\infty} |\langle x_n, x \rangle| = \infty$ for all $0 \neq x \in X$.

Then the set of all convergent series $\sum_{n=1}^{\infty} a_n x_n$, $|a_n| = 1$, $n \in \mathbb{N}$, is dense in X .

Finally, we quote some results on Hardy spaces. Let D_0 be an arbitrary simply connected domain with at least two boundary points. A set of functions \underline{f} analytic in D_0 is said to belong to the Hardy space $H_2^m(D_0)$, if the subharmonic function $\sum_{l=1}^m |f_l(s)|^2$ has a harmonic majorant in D_0 . We remark that $H_2^m(D_0)$ is a Hilbert space. A proof of the following lemmas in the case of $m = 1$ is given in [14, Theorem 6.3.6 and 6.3.7]. The proof of the general case is obtained in a similar way.

Lemma 2.2.9 ([14, Theorem 6.3.6]). Let $\{\underline{f}_n\}$ be a sequence in $H_2^m(D_0)$ such that

$$\lim_{n \rightarrow \infty} \underline{f}_n(s) = \underline{f}(s)$$

in the topology of $H_2^m(D_0)$. Then this relation is true uniformly on every compact subset of D_0 .

Lemma 2.2.10 ([14, Theorem 6.3.7]). Let $\underline{g} \in H_2^m(D_0)$. There exist complex Borel measures μ_{g_l} ($1 \leq l \leq m$) with their support contained in the boundary ∂D_0 of D_0 such that if $\underline{f} \in H_2^m(D_0)$ has a continuous extension to $\overline{D_0}$, then the inner product can be expressed by the formula

$$\langle \underline{f}, \underline{g} \rangle = \sum_{l=1}^m \int_{\partial D_0} f_l d\mu_{g_l}.$$

We define the norm of $\underline{f} \in H_2^m(D_0)$ by

$$\|\underline{f}\| := \sqrt{\langle \underline{f}, \underline{f} \rangle}.$$

Lemma 2.2.11 ([14, Theorem 6.3.9]). Let the boundary of D_0 be an analytic simple closed curve. The set of polynomials is dense in the space $H_2^m(D_0)$.

2.2.3 Joint universality I

In this section, we prove Theorems 2.2.1. We define the Hilbert space X^m by $X^m = X \times \cdots \times X$. For convenience, we define the next symbols :

$$\underline{a} \cdot \underline{x} := \{a_1 x_1, \dots, a_m x_m\}, \quad \underline{a} \in \mathbb{C}^m, \quad \underline{x} \in X^m,$$

$$\Pi^m := \{\underline{c} = (c_1, \dots, c_m) \in \mathbb{C}^m ; |c_l| = 1, 1 \leq l \leq m\}.$$

The following theorem is a generalization of Lemma 2.2.8.

Theorem 2.2.12. Let $\{\underline{x}_n\} := \{(x_{1,n}, \dots, x_{m,n}) ; n \in \mathbb{N}\}$ be a sequence in the Hilbert space X^m satisfying the following conditions :

(a) $\sum_{n=1}^{\infty} \|\underline{x}_n\|^2 < \infty;$

(b) There exist $\underline{c}_n \in \Pi^m$, $n \in \mathbb{N}$, such that $\sum_{n=1}^{\infty} |\langle \underline{c}_n \cdot \underline{x}_n, \underline{x} \rangle| = \infty$ for all $\underline{0} \neq \underline{x} \in X^m$.

Then the set of all convergent series $\sum_{n=1}^{\infty} \underline{a}_n \cdot \underline{x}_n$, $\underline{a}_n \in \Pi^m$, $n \in \mathbb{N}$, is dense in X^m .

Proof. Put $\underline{y}_n := \underline{c}_n \cdot \underline{x}_n$. By using Lemma 2.2.8 as $X = X^m$, the set of all convergent series

$$\sum_{n=1}^{\infty} b_n \underline{y}_n, \quad |b_n| = 1, \quad n \in \mathbb{N},$$

is dense in X^m . Hence by taking $\underline{d}_n := b_n \underline{c}_n$, the set of all convergent series

$$\sum_{n=1}^{\infty} \underline{d}_n \cdot \underline{x}_n, \quad \underline{d}_n \in \Pi^m,$$

is dense in X^m . Since this set is contained in the set of all convergent series $\sum_{n=1}^{\infty} \underline{a}_n \cdot \underline{x}_n$, $\underline{a}_n \in \Pi^m$, we obtain this theorem. \square

The following theorem is a generalization of [14, Theorem 6.3.10] and [17, Lemma 3].

Theorem 2.2.13. Let D_1 be a simply connected domain in \mathbb{C} . Let $\{\underline{f}_n\}$ be a sequence in $H^m(D_1)$ which satisfies :

(a) If $\mu_l, 1 \leq l \leq m$ are complex measures on $(\mathbb{C}, \mathfrak{B}(\mathbb{C}))$ with compact supports contained in D_1

such that there exist $\underline{c}_n \in \Pi^m, n \in \mathbb{N}$, which satisfy $\sum_{n=1}^{\infty} \left| \sum_{l=1}^m \int_{\mathbb{C}} c_{l,n} f_{l,n} d\mu_l \right| < \infty$,

then $\int_{\mathbb{C}} s^r d\mu_l = 0$ for all $r \in \mathbb{N} \cup \{0\}, 1 \leq l \leq m$;

(b) There exist $\underline{d}_n \in \Pi^m, n \in \mathbb{N}$, for which the series $\sum_{n=1}^{\infty} \underline{d}_n \cdot \underline{f}_n$ converges in $H^m(D_1)$;

(c) For any compact set $K \subseteq D_1$, we have $\sum_{n=1}^{\infty} \sup_{1 \leq l \leq m} \sup_{s \in K} |f_{l,n}(s)|^2 < \infty$.

Then the set of all convergent series $\sum_{n=1}^{\infty} \underline{a}_n \cdot \underline{x}_n, \underline{a}_n \in \Pi^m, n \in \mathbb{N}$, is dense in $H^m(D_1)$.

Proof. We modify the proof of [14, Theorem 6.3.10]. Let K be a compact subset of D_1 . We choose a simply connected domain G such that $K \subseteq G, \bar{G}$ is a compact subset of D_1 and the boundary of G is an analytic simple closed curve. We will consider the space $H_2^m(D_1)$. In view of Lemma 2.2.10 (see [14, proof of Theorem 6.3.10]), we have

$$\|\underline{f}_n\|^2 = \int_{\partial G} \sum_{l=1}^m f_{l,n} d\mu_{f_{l,n}} \leq \sup_{1 \leq l \leq m} \sup_{s \in \bar{G}} |f_{l,n}(s)| \int_{\partial G} |d\mu_{f_{l,n}}| \leq c \sup_{1 \leq l \leq m} \sup_{s \in \bar{G}} |f_{l,n}(s)|^2.$$

Hence by assumption (c), we have

$$\sum_{n=1}^{\infty} \|\underline{f}_n\|^2 < \infty. \quad (2.2.2)$$

Suppose $\underline{c}_n \in \Pi^m, n \in \mathbb{N}$. Let $\underline{g} \in H^m(G)$ be such that

$$\sum_{n=1}^{\infty} |\langle \underline{c}_n \cdot \underline{f}_n, \underline{g} \rangle| < \infty. \quad (2.2.3)$$

By Lemma 2.2.10 again, we have

$$\langle \underline{c}_n \cdot \underline{f}_n, \underline{g} \rangle = \sum_{l=1}^m \int_{\mathbb{C}} c_{l,n} f_{l,n} d\mu_{g_l}, \quad (2.2.4)$$

where $\mu_{g_l}, 1 \leq l \leq m$ are complex Borel measure with support contained in the boundary of G . Thus in view of (2.2.3), we have

$$\sum_{n=1}^{\infty} \left| \sum_{l=1}^m \int_{\mathbb{C}} c_{l,n} f_{l,n} d\mu_{g_l} \right| < \infty.$$

This and assumption (a) give that

$$\int_{\mathbb{C}} s^r d\mu_{g_l} = 0, \quad \text{for all } r \in \mathbb{N} \cup \{0\}, 1 \leq l \leq m.$$

Hence in view of (2.2.4), we deduce that \underline{g} is orthogonal to all polynomials. Therefore it follows from Lemmas 2.2.6 and 2.2.11 that \underline{g} is the zero element of $H_2^m(G)$. Consequently,

$$\sum_{n=1}^{\infty} |\langle \underline{c}_n \cdot \underline{f}_n, \underline{g} \rangle| = \infty, \quad \text{for all } 0 \neq \underline{g} \in H_2^m(G).$$

Whence and from (2.2.2), using Theorem 2.2.12, we obtain that the set of all convergent series

$$\sum_{n=1}^{\infty} \underline{a}_n \cdot \underline{f}_n, \quad \underline{a}_n \in \Pi^m,$$

is dense in $H_2^m(G)$.

Let $f \in H^m(D_1)$ and $\varepsilon > 0$. Then by Theorem 2.2.12 and Lemma 2.2.9, there exists a series

$$\sum_{n=1}^{\infty} \underline{\alpha}_n \cdot \underline{f}_n, \quad \underline{\alpha}_n \in \Pi^m,$$

convergent uniformly on K and

$$\sup_{1 \leq l \leq m} \sup_{s \in K} \left| \sum_{n=1}^{\infty} \alpha_{l,n} f_{l,n}(s) - f_l(s) \right| < \frac{\varepsilon}{4}.$$

Thus we can choose a positive integer M such that

$$\sup_{1 \leq l \leq m} \sup_{s \in K} \left| \sum_{n=1}^M \alpha_{l,n} f_{l,n}(s) - f_l(s) \right| < \frac{\varepsilon}{2}, \quad (2.2.5)$$

and in view of (b)

$$\sup_{1 \leq l \leq m} \sup_{s \in K} \left| \sum_{n=M+1}^{\infty} d_{l,n} f_{l,n}(s) \right| < \frac{\varepsilon}{2}. \quad (2.2.6)$$

Now let

$$\underline{a}_n := \begin{cases} \underline{\alpha}_n & \text{if } 1 \leq m \leq M, \\ \underline{d}_n & \text{if } m > M. \end{cases}$$

Then we have the convergent series $\sum_{n=1}^{\infty} \underline{a}_n \cdot \underline{x}_n$ in $H^m(D)$ and the inequalities (2.2.5) and (2.2.6) yield the inequality

$$\sup_{1 \leq l \leq m} \sup_{s \in K} \left| \sum_{n=1}^{\infty} a_{l,n} f_{l,n}(s) - f_l(s) \right| < \varepsilon$$

which completes the proof of this theorem. \square

Proof of Theorem 2.2.1. We modify the proof of [18, Theorem 10]. It follows from the definition of Ω^m that $\{\underline{\omega}(n)\}$ is a sequence of independent random variables with respect to the measure m_{H_m} . Hence $\{\underline{f}_n(s, \underline{\omega}(n)), n \in \mathbb{N} \cup \{0\}\}$ is a sequence of independent $H^m(D)$ -random elements, where

$$\underline{f}_n(s, \underline{\omega}(n)) := \left(\frac{e^{2\pi i \lambda_1 n} \omega_1(n)}{(n + a_1)^s}, \dots, \frac{e^{2\pi i \lambda_m n} \omega_m(n)}{(n + a_m)^s} \right).$$

The support of each $\omega_l(n)$ ($n \in \mathbb{N} \cup \{0\}$, $1 \leq l \leq m$) is the unit circle γ . Therefore the set $\{\underline{f}_n(s, \underline{\alpha}); \underline{\alpha} \in \Pi^m\}$, is the support of the random elements $\underline{f}_n(s, \underline{\omega}(n))$. Consequently, by Lemma 2.2.5 the closure of the set of all convergent series

$$\sum_{n=0}^{\infty} \left(\frac{e^{2\pi i \lambda_1 n} \alpha_{1,n}}{(n + a_1)^s}, \dots, \frac{e^{2\pi i \lambda_m n} \alpha_{m,n}}{(n + a_m)^s} \right), \quad \alpha_{l,n} \in \gamma, \quad n \in \mathbb{N} \cup \{0\}, \quad 1 \leq l \leq m,$$

is the support of the random element $\underline{L}(s, \underline{\omega}) := (L(\lambda_1, a_1, s, \omega_1), \dots, L(\lambda_m, a_m, s, \omega_m))$. It remains to check that the latter set is dense in $H^m(D)$. First, we check assumption (c) of Theorem 2.2.13. By the definition of D , for every compact subset K of D ,

$$\sup_{1 \leq l \leq m} \sup_{s \in K} \sum_{n=0}^{\infty} \left| \frac{e^{2\pi i \lambda_l n}}{(n + a_l)^{2\sigma}} \right| < \infty.$$

Next, we verify assumptions (a) and (b) of Theorem 2.2.13. We put $\eta_l := \eta + l/m$, $\eta \in \mathbb{R} \setminus \mathbb{Q}$, $1 \leq l \leq m$. We define $\underline{c}_n \in \Pi^m$ by

$$e^{2\pi i \lambda_l n} c_{l,n} := e^{2\pi i \eta_l n}, \quad n \in \mathbb{N} \cup \{0\}, \quad 1 \leq l \leq m.$$

By the definition of \underline{c}_n and Abel's partial summation, we can check assumption (b). Therefore it remains to confirm only assumption (a) of Theorem 2.2.13. Let μ_l , $1 \leq l \leq m$ be complex measures on $(\mathbb{C}, \mathfrak{B}(\mathbb{C}))$ with compact supports contained in D such that

$$\sum_{n=0}^{\infty} \left| \int_{\mathbb{C}} \sum_{l=1}^m \frac{e^{2\pi i \eta_l n}}{(n+a_l)^s} d\mu_l \right| < \infty. \quad (2.2.7)$$

By the same argument as in the proof of [34, Theorem 4.1] (using the ‘‘positive density method’’), we deduce that

$$\int_{\mathbb{C}} s^r d\mu_l = 0 \quad \text{for all } r \in \mathbb{N} \cup \{0\}, \quad 1 \leq l \leq m.$$

Therefore we obtain that all assumptions of Theorem 2.2.13 are satisfied. Hence we obtain Theorem 2.2.1 by the same argument as in the proof of [17, Theorem 1]. \square

Remark 2.2.14. *In the proof of [18, Theorem 10], the authors showed $S_{P_{\underline{L}_0}} = H^m(D)$ where*

$$\underline{L}_0(s, \omega) := (L(\lambda_1, a_1, s, \omega), \dots, L(\lambda_m, a_m, s, \omega)), \quad \omega \in \Omega.$$

Clearly $S_{P_{\underline{L}_0}} \subseteq S_{P_{\underline{L}}}$. The fact $S_{P_{\underline{L}}} = H^m(D)$ is therefore an immediate consequence of $S_{P_{\underline{L}_0}} = H^m(D)$ in the situation of [18]. However, if at least two of λ_l 's are equal, which is a special case of the present weaker assumptions, we will show $S_{P_{\underline{L}_0}} \neq H^m(D)$ in Proposition 2.2.20, hence we have shown $S_{P_{\underline{L}}} = H^m(D)$ directly in the proof of Theorem 2.2.1.

By Theorem 2.2.1, we obtain the following theorem. The proof of this theorem is completely the same as in the proof of [17, Theorem 2].

Theorem 2.2.15. *Let λ_l and a_l , ($1 \leq l \leq m$) be as in Theorem 2.2.1. Suppose F_k , $0 \leq k \leq n$ be continuous functions on \mathbb{C}^{Nm} . Suppose*

$$\sum_{k=0}^n s^k F_k(L(\lambda_1, a_1, s), \dots, L(\lambda_m, a_m, s), L'(\lambda_1, a_1, s), \dots, L'(\lambda_m, a_m, s), \dots, L^{(N-1)}(\lambda_1, a_1, s), \dots, L^{(N-1)}(\lambda_m, a_m, s)) = 0$$

identically for all $s \in \mathbb{C}$. Then $F_k \equiv 0$, $0 \leq k \leq n$.

2.2.4 Joint universality II

In this section, we will prove Theorem 2.2.3. Firstly we show the limit theorem for $\mathfrak{L}(\lambda, a, b, c; s)$. Define on $(H^m(D), \mathfrak{B}(H^m(D)))$ the probability measure

$$P_{\underline{\mathfrak{L}}}^T(A) := \nu_T^\tau((\mathfrak{L}(\lambda_1, a_1, b_1, c; s + i\tau), \dots, \mathfrak{L}(\lambda_m, a_m, b_m, c; s + i\tau)) \in A), \quad A \in \mathfrak{B}(H^m(D)).$$

We define the $H^m(D)$ -valued random elements $\underline{\mathfrak{L}}(s, \underline{\omega})$ by

$$\underline{\mathfrak{L}}(s, \underline{\omega}) := (\mathfrak{L}(\lambda_1, a_1, b_1, c; s, \omega_1), \dots, \mathfrak{L}(\lambda_m, a_m, b_m, c; s, \omega_m)),$$

where

$$\mathfrak{L}(\lambda_l, a_l, b_l, c; s, \omega_l) := \sum_{n=0}^{\infty} \frac{e^{2\pi i \lambda_l n \omega_l(n)}}{(n+a_l)^{s-c} (n+b_l)^c}, \quad s \in D, \quad \omega_l \in \Omega, \quad 1 \leq l \leq m.$$

Let $P_{\underline{\mathfrak{L}}}$ be the distribution of the random element $\underline{\mathfrak{L}}(s, \underline{\omega})$.

Proposition 2.2.16. *Suppose $0 < a_l < 1$ be algebraically independent numbers and $0 < \lambda_l \leq 1$ for $1 \leq l \leq m$. The probability measure $P_{\underline{\mathfrak{L}}}^T$ converges weakly to $P_{\underline{\mathfrak{L}}}$ as $T \rightarrow \infty$.*

To prove this proposition, we prepare notations. For $\sigma_{1l} > 1/2$, $1 \leq l \leq m$ we define functions

$$\mathfrak{L}_r(\lambda_l, a_l, b_l, c; s) := \sum_{n=0}^{\infty} \frac{e^{2\pi i \lambda_l n} v_l(n, r)}{(n + a_l)^{s-c} (n + b_l)^c},$$

$$L_r(\lambda_l, a_l, s) := \sum_{n=0}^{\infty} \frac{e^{2\pi i \lambda_l n} v_l(n, r)}{(n + a_l)^s}, \quad v_l(n, r) := \exp \left\{ - \left(\frac{n + a_l}{r + a_l} \right)^{\sigma_{1l}} \right\}.$$

Let $\{C_k\}$ be a sequence of compact subsets of D such that $\bigcup_{k=1}^{\infty} C_k$, $C_k \subset C_{k+1}$, and if C is a compact subset of D , then $C \subseteq C_k$ for some k .

Lemma 2.2.17. *We have*

$$\lim_{r \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \sup_{s \in C_k} |\mathfrak{L}(\lambda_l, a_l, b_l, c; s + i\tau) - \mathfrak{L}_r(\lambda_l, a_l, b_l, c; s + i\tau)| d\tau = 0. \quad (2.2.8)$$

Proof. By the triangle inequality, we have

$$\begin{aligned} & |\mathfrak{L}(\lambda_l, a_l, b_l, c; s + i\tau) - \mathfrak{L}_r(\lambda_l, a_l, b_l, c; s + i\tau)| \\ & \leq |\mathfrak{L}(\lambda_l, a_l, b_l, c; s + i\tau) - L(\lambda_l, a_l, s + i\tau) + L_r(\lambda_l, a_l, s + i\tau) - \mathfrak{L}_r(\lambda_l, a_l, b_l, c; s + i\tau)| \\ & \quad + |L(\lambda_l, a_l, s + i\tau) - L_r(\lambda_l, a_l, s + i\tau)| \\ & := |f_r(\lambda_l, a_l, b_l, c; s + i\tau)| + |g_r(\lambda_l, a_l, b_l, c; s + i\tau)|, \end{aligned}$$

say. We have

$$\lim_{r \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \sup_{s \in C_k} |g_r(\lambda_l, a_l, b_l, c; s + i\tau)| d\tau = 0, \quad (2.2.9)$$

by [16, Lemma 5.2.11]. Hence we have to show

$$\lim_{r \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \sup_{s \in C_k} |f_r(\lambda_l, a_l, b_l, c; s + i\tau)| d\tau = 0.$$

By the binomial theorem, we have

$$1 - \left(\frac{n + b_l}{n + a_l} \right)^c = O(n^{-1}).$$

Hence, for some positive constant K , we have

$$\begin{aligned} & \sup_{s \in C_k} |f_r(\lambda_l, a_l, b_l, c; s + i\tau)| \\ & \leq \sum_{n=0}^{\infty} \frac{1}{(n + a_l)^{\Re(s-c)} (n + b_l)^{\Re(c)}} \left| 1 - \left(\frac{n + b_l}{n + a_l} \right)^c \right| \left| 1 - \exp \left\{ - \left(\frac{n + a_l}{r + a_l} \right)^{\sigma_{1l}} \right\} \right| \\ & \leq K \sum_{n=0}^{\infty} \frac{1}{(n + a_l)^{3/2}} \left| 1 - \exp \left\{ - \left(\frac{n + a_l}{r + a_l} \right)^{\sigma_{1l}} \right\} \right| := M_r(\sigma_{1l}), \end{aligned}$$

say. Therefore we obtain

$$\begin{aligned} & \lim_{r \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \sup_{s \in C_k} |f_r(\lambda_l, a_l, b_l, c; s + i\tau)| d\tau \\ & \leq \lim_{r \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T M_r(\sigma_{1l}) d\tau = \lim_{r \rightarrow \infty} M_r(\sigma_{1l}) = 0. \end{aligned} \quad (2.2.10)$$

□

Proof of Proposition 2.2.16. We modify the proof of [18, Theorem 1]. The only different point from the proof of [18, Theorem 1] is to use (2.2.8) instead of (2.2.9). Therefore by Lemma 2.2.17 we can prove Proposition 2.2.16. \square

Proof of Theorem 2.2.3. Similarly to the argument of the first part of the proof of Theorem 2.2.1, we have to check that the set of all convergent series

$$\sum_{n=0}^{\infty} \left(\frac{e^{2\pi i \lambda_1 n} \alpha_{1,n}}{(n+a_1)^{s-c}(n+b_1)^c}, \dots, \frac{e^{2\pi i \lambda_m n} \alpha_{m,n}}{(n+a_l)^{s-c}(n+b_m)^c} \right), \quad \underline{\alpha}_n \in \Pi^m, \quad n \in \mathbb{N} \cup \{0\},$$

is dense in $H^m(D)$. Hence we will confirm the assumptions of Theorem 2.2.13. Let μ_l , $1 \leq l \leq m$ be complex measures on $(\mathbb{C}, \mathfrak{B}(\mathbb{C}))$ with compact supports contained in D such that

$$\sum_{n=0}^{\infty} \left| \int_{\mathbb{C}} \sum_{l=1}^m \frac{e^{2\pi i \lambda_l n}}{(n+a_l)^{s-c}(n+b_l)^c} d\mu_l \right| < \infty.$$

By the same argument as in [17, (12)], we have

$$(n+a_l)^{-s} = n^{-s} + Bn^{-1-\sigma} |s| e^{B|s|}$$

where B a positive constant. Hence the above formula is equivalent to

$$\sum_{n=1}^{\infty} \left| \int_{\mathbb{C}} \sum_{l=1}^m \frac{e^{2\pi i \lambda_l n}}{n^s} d\mu_l \right| < \infty.$$

Therefore we can easily confirm all assumptions by modifying the proof of Theorem 2.2.1. Hence we obtain Theorem 2.2.3 by the same argument as in the proof of [17, Theorem 1]. \square

2.2.5 Non-denseness lemma

In this section, we reconsider the proof of Theorems 2.2.1 and 2.2.3, especially Remark 2.2.14. The next theorem is a kind of counter-proposition for Lemma 2.2.8.

Theorem 2.2.18 (Non-denseness lemma). *Let $\{\underline{x}_n\}$ be a sequence in a Hilbert space X^m satisfying the following condition.*

$$(a) \quad \text{There exists a non-zero } \underline{x} \in X^m \text{ such that } \sum_{n=1}^{\infty} |\langle \underline{x}_n, \underline{x} \rangle| < \infty.$$

Then the set of all convergent series $\sum_{n=1}^{\infty} a_n \underline{x}_n$, $|a_n| = 1$, $n \in \mathbb{N}$ is not dense in X^m .

Proof. Firstly, we consider the case of $\sum_{n=1}^{\infty} |\langle \underline{x}_n, \underline{x} \rangle| = 0$. We take an \underline{x} which satisfies this condition. By the assumption, we have

$$\left\| \underline{x} - \sum_{n=1}^{\infty} a_n \underline{x}_n \right\|^2 = \|\underline{x}\|^2 + \left\| \sum_{n=1}^{\infty} a_n \underline{x}_n \right\|^2 \geq \|\underline{x}\|^2.$$

Hence in this case, the set of all convergent series $\sum_{n=1}^{\infty} a_n \underline{x}_n$ is not dense in X^m . Next we consider the case of $0 \neq \sum_{n=1}^{\infty} |\langle \underline{x}_n, \underline{x} \rangle| < \infty$. We take an \underline{x} which satisfies this condition and choose $b_n \in \mathbb{C}$ so that $|b_n| = 1$ and

$$\begin{cases} b_n \langle \underline{x}_n, \underline{x} \rangle = -|\langle \underline{x}_n, \underline{x} \rangle| & \text{if } \langle \underline{x}_n, \underline{x} \rangle \neq 0, \\ b_n = 1 & \text{if } \langle \underline{x}_n, \underline{x} \rangle = 0. \end{cases}$$

We can assume that $|\langle \underline{x}_1, \underline{x} \rangle| \neq 0$ without loss of generality. Let M be a sufficiently large integer which satisfies

$$\left| 2\Re \left(\sum_{n=M+1}^{\infty} a_n \langle \underline{x}_n, \underline{x} \rangle \right) - \left\| \sum_{n=M+1}^{\infty} a_n \underline{x}_n \right\|^2 \right| < \frac{|\langle \underline{x}_1, \underline{x} \rangle|}{2}.$$

By the trigonometric inequality, we have

$$\begin{aligned} & \left\| 2 \sum_{n=1}^M b_n \underline{x}_n - \sum_{n=1}^{\infty} a_n \underline{x}_n \right\| = \left\| \sum_{n=1}^M 2b_n \underline{x}_n - \underline{x} + \underline{x} - \sum_{n=1}^{\infty} a_n \underline{x}_n \right\| \\ & \geq \left\| \sum_{n=1}^M (2b_n - a_n) \underline{x}_n - \underline{x} \right\| - \left\| \underline{x} - \sum_{n=M+1}^{\infty} a_n \underline{x}_n \right\| := |A - B|, \end{aligned}$$

say. Then we obtain

$$\begin{aligned} A^2 - B^2 &= \\ & \left\| \sum_{n=1}^M (2b_n - a_n) \underline{x}_n \right\|^2 - 2\Re \left(\sum_{n=1}^M (2b_n - a_n) \langle \underline{x}_n, \underline{x} \rangle \right) + 2\Re \left(\sum_{n=M+1}^{\infty} a_n \langle \underline{x}_n, \underline{x} \rangle \right) - \left\| \sum_{n=M+1}^{\infty} a_n \underline{x}_n \right\|^2. \end{aligned}$$

By the definition of \underline{x} and b_n , we have

$$-\Re \left(\sum_{n=1}^M (2b_n - a_n) \langle \underline{x}_n, \underline{x} \rangle \right) \geq \sum_{n=1}^M |\langle \underline{x}_n, \underline{x} \rangle| \geq |\langle \underline{x}_1, \underline{x} \rangle|.$$

Therefore we have the inequality

$$\left\| 2 \sum_{n=1}^M b_n \underline{x}_n - \sum_{n=1}^{\infty} a_n \underline{x}_n \right\| \geq A - B > \frac{|\langle \underline{x}_1, \underline{x} \rangle|}{2(A+B)}.$$

Hence the set of all convergent series $\sum_{n=1}^{\infty} a_n \underline{x}_n$ is not dense in X^m . \square

Lemma 2.2.19. *If $\{\underline{x}_n\}$ is not dense in $H_2^m(D)$, then $\{\underline{x}_n\}$ is not dense in $H^m(D)$.*

Proof. We show the contraposition, that is, if $\{\underline{x}_n\}$ is dense in $H^m(D)$, then $\{\underline{x}_n\}$ is dense also in the Hardy space $H_2^m(D)$. By Lemma 2.2.10, we have

$$\|\underline{f} - \underline{g}\|^2 = \sum_{l=1}^m \int_{\partial D} (f_l - g_l) d\mu_{(f_l - g_l)} \leq c_2 \sup_{1 \leq l \leq m} \sup_{s \in \bar{D}} |f_l - g_l|^2.$$

This implies the contraposition. \square

Proposition 2.2.20. *Suppose $0 < a_l < 1$ be algebraically independent numbers and $0 < \lambda_l \leq 1$ for $1 \leq l \leq m$. If at least two of λ_l 's are equal, it holds that $S_{P_{\underline{L}_0}} \neq H^m(D)$ (see Remark 2.2.14).*

Proof. Similarly to the argument of the first part of the proof of Theorem 2.2.1, we have to check that

$$\sum_{n=0}^{\infty} \left(\frac{e^{2\pi i \lambda_1 n} \omega_n}{(n + a_1)^s}, \dots, \frac{e^{2\pi i \lambda_m n} \omega_n}{(n + a_m)^s} \right), \quad \omega_n \in \gamma, \quad n \in \mathbb{N} \cup \{0\},$$

is not dense in $H^m(D)$. First, we consider the case of $m = 2$, $\lambda := \lambda_1 = \lambda_2$. Let μ_1 and μ_2 be complex measures on $(\mathbb{C}, \mathfrak{B}(\mathbb{C}))$ with compact supports contained in D such that

$$\sum_{n=0}^{\infty} \left| \int_{\mathbb{C}} \sum_{l=1}^2 \frac{e^{2\pi i \lambda n}}{(n + a_l)^s} d\mu_l \right| < \infty.$$

By the same argument as in [17, (12)], the above formula is equivalent to

$$\sum_{n=1}^{\infty} \left| \int_{\mathbb{C}} \sum_{l=1}^2 \frac{e^{2\pi i \lambda n}}{n^s} d\mu_l \right| < \infty. \quad (2.2.11)$$

If $0 \neq \mu_1 = -\mu_2$, then we see that the left-hand side of (2.2.11) is equal to zero, hence the measures satisfy condition (2.2.11). Applying Lemma 2.2.7 with $x = x_n = e^{2\pi i \lambda n} n^{-s}$ and

$$f : x_n \mapsto \int_{\mathbb{C}} x_n d\mu_l,$$

we can rewrite (2.2.11) in terms of inner products. Because of Lemma 2.2.18, the set of all convergent series $\sum_{n=0}^{\infty} \underline{f}_n(s, \omega_n)$ with $\omega_n \in \gamma$ is not dense in the Hardy space $H_2^2(D)$. Therefore the set of all convergent series is not dense in $H^2(D)$ by Lemma 2.2.19. If $m \geq 3$, we can put $\lambda := \lambda_1 = \lambda_2$ without loss of generality. In this case, we take $0 \neq \mu_1 = -\mu_2, 0 = \mu_3 = \dots = \mu_m$. \square

2.2.6 Examples of non-existence of universality

In this section, we will show three examples which imply the non-existence of joint universality for Lerch zeta functions and generalized Lerch zeta functions. We remark that the parameter a_1, \dots, a_m of the next two examples are not algebraically independent.

Proposition 2.2.21. *If we put $a_1 = 1$ and $a_2 = 1/2$, then there exists an $\varepsilon > 0$ and analytic functions $f_l(s)$ on K_l , for which there does not exist τ satisfying*

$$\sup_{1 \leq l \leq m} \sup_{s \in K_l} |\zeta(a_l, s + i\tau) - f_l(s)| \leq \varepsilon.$$

Proof. Let $K := \{s; |s - 3/4| \leq R\}$, $0 < R < 1/4$. We put $\varepsilon = 1/3$, $f_1(s) \equiv 1$ and $f_2(s) \equiv 8$. Suppose

$$\sup_{s \in K} |\zeta(s + i\tau) - 1| < 1/3. \quad (2.2.12)$$

For every τ satisfying (2.2.12), by the well known formula

$$\zeta(1/2, s) = (2^s - 1)\zeta(s), \quad (2.2.13)$$

we have

$$\begin{aligned} \sup_{s \in K} |\zeta(1/2, s + i\tau) - 8| &= \sup_{s \in K} |(2^{s+i\tau} - 1)(\zeta(s + i\tau) - 1) + 2^{s+i\tau} - 9| \\ &\geq \sup_{s \in K} \left| |(2^{s+i\tau} - 1)(\zeta(s + i\tau) - 1)| - |2^{s+i\tau} - 9| \right| \\ &\geq \sup_{s \in K} |1 - 7| = 6. \end{aligned}$$

\square

This proposition implies that the set of Hurwitz zeta functions does not necessarily have the joint t -universality. Proposition 2.2.21 is a rather obvious example, but we can observe that the key of the proof is the functional relation (2.2.13). By using another functional relation, we can show the following result.

Proposition 2.2.22. *Let a be a positive number and λ be a real number. If we put $\lambda_n = \lambda + n/m$, $a_n = ma$ for $0 \leq n \leq m - 1$, and $\lambda_m = m\lambda$, $a_m = a + j/m$, ($0 \leq j \leq m - 1$), then there exists an $\varepsilon > 0$ and analytic functions $f_l(s)$ on K_l , for which there does not exist τ satisfying*

$$\sup_{0 \leq l \leq m} \sup_{s \in K_l} |L(\lambda_l, a_l, s + i\tau) - f_l(s)| \leq \varepsilon.$$

Proof. We define ω_m^j by $\omega_m^j := \exp(2\pi i j/m)$, $0 \leq j \leq m-1$, $j, m \in \mathbb{N}$. By using the inversion formula (2.1.2) and modifying the proof of Proposition 2.2.21, we can show this proposition. \square

Remark 2.2.23. *These propositions show that the existence of functional relations implies the non-existence of joint t -universality. Therefore we can see that joint t -universality is essentially more difficult than single t -universality (for example [14, p.111, Theorem 1.1]) because of its connection with functional relations. These facts should be compared with Theorem 2.2.15 concerning functional independence, deduced by joint t -universality Theorem 2.2.1.*

In the case of $a_1 = \dots = a_m$, we have the following non-existence of joint t -universality for $\mathfrak{L}(\lambda_l, a, b, c; s)$.

Proposition 2.2.24. *If at least two of λ_l s are equal, then there exists an $\varepsilon > 0$ and analytic functions $f_l(s)$ on K_l , for which there does not exist τ satisfying*

$$\sup_{1 \leq l \leq m} \sup_{s \in K_l} |\mathfrak{L}(\lambda_l, a, b_l, c; s + i\tau) - f_l(s)| \leq \varepsilon. \quad (2.2.14)$$

Proof. We assume $m = 2$ and $\lambda := \lambda_1 = \lambda_2$ with out loss of generality. For some positive constant C_1 and C_2 , we have

$$\begin{aligned} & |\mathfrak{L}(\lambda, a, b_1, c; s + i\tau) - \mathfrak{L}(\lambda, a, b_2, c; s + i\tau)| \\ &= \left| \sum_{n=0}^{\infty} \frac{e^{2\pi i \lambda n}}{(n+a)^{s+i\tau-c}(n+b_1)^c} \left(1 - \frac{(n+b_1)^c}{(n+b_2)^c} \right) \right| \leq C_1 \sum_{n=0}^{\infty} \frac{1}{n(n+a)^{\Re(s)}} \leq C_2. \end{aligned} \quad (2.2.15)$$

Let $K_1 = K_2 = K := \{s : |s - 3/4| \leq 1/5\}$. We put $\varepsilon = 1/3$, $f_1(s) \equiv 1$ and $f_2(s) \equiv C_2 + 1$. Suppose

$$\sup_{s \in K} |\mathfrak{L}(\lambda, a, b_1, c; s + i\tau) - 1| \leq 1/3$$

For every τ satisfying the above formula, we have

$$\sup_{s \in K} |\mathfrak{L}(\lambda, a_1, b_2, c; s + i\tau) - (C_2 + 1)| > 1/3.$$

Hence we have (2.2.14) in this case. \square

In the case when a is transcendental, we obtain another proof of Proposition 2.2.24 by using Theorem 2.2.18. Firstly we show the limit theorem for $\mathfrak{L}(\lambda, a, b, c; s)$. Denote on $(H^m(D), \mathfrak{B}(H^m(D)))$ the probability measure

$$P_{\underline{\mathfrak{L}}_0}^T(A) := \nu_T^\tau((\mathfrak{L}(\lambda_1, a, b_1, c; s + i\tau), \dots, \mathfrak{L}(\lambda_m, a, b_m, c; s + i\tau)) \in A), \quad A \in \mathfrak{B}(H^m(D)).$$

We define the $H^m(D)$ -valued random element $\underline{\mathfrak{L}}_0(s, \underline{\omega})$ by

$$\underline{\mathfrak{L}}_0(s, \underline{\omega}) := (\mathfrak{L}(\lambda_1, a, b_1, c; s, \omega_1), \dots, \mathfrak{L}(\lambda_m, a, b_m, c; s, \omega_m)),$$

where

$$\mathfrak{L}_0(\lambda_l, a, b_l, c; s, \omega_l) := \sum_{n=0}^{\infty} \frac{e^{2\pi i \lambda_l n \omega_l(n)}}{(n+a)^{s-c}(n+b_l)^c}, \quad s \in D, \quad \omega_l \in \Omega, \quad 1 \leq l \leq m.$$

Let $P_{\underline{\mathfrak{L}}_0}$ stands for the distribution of the random element $\underline{\mathfrak{L}}_0(s, \underline{\omega})$.

Proposition 2.2.25. *Suppose $0 < a < 1$ be a transcendental number and $0 < \lambda_l \leq 1$ for $1 \leq l \leq m$. The probability measure $P_{\underline{\mathfrak{L}}_0}^T$ converges weakly to $P_{\underline{\mathfrak{L}}_0}$ as $T \rightarrow \infty$.*

Proof. We can prove this theorem by modifying [15, Theorem 3] and using Lemma 2.2.17. \square

Similarly to the argument of Proposition 2.2.20, we can check

$$\sum_{n=0}^{\infty} \left(\frac{e^{2\pi i \lambda_1 n} \alpha_n}{(n+a)^{s-c} (n+b_1)^c}, \dots, \frac{e^{2\pi i \lambda_m n} \alpha_n}{(n+a)^{s-c} (n+b_m)^c} \right), \quad \alpha_n \in \gamma, \quad n \in \mathbb{N} \cup \{0\},$$

is not dense in $H^m(D)$, since

$$\sum_{n=0}^{\infty} \left| \int_{\mathbb{C}} \sum_{l=1}^2 \frac{e^{2\pi i \lambda_n}}{(n+a)^{s-c} (n+b_l)^c} d\mu_l \right| < \infty.$$

is equivalent to (2.2.11).

Suppose that functions $F_l(s)$ for $1 \leq l \leq m$ can be continued analytically to the whole of D . Denote by V_k the set of $g \in H^m(D)$ such that

$$\sup_{1 \leq l \leq m} \sup_{s \in K_l} |g_l(s) - F_l(s)| < (k+1)\varepsilon, \quad k = 1, 2.$$

We recall that the support S_P consists of all $\underline{f} \in H^m(D)$ such that for every neighborhood V of \underline{f} the inequality $P(V) > 0$ is satisfied. Since the support of the random element $\underline{\mathfrak{L}}_0(s, \omega)$ is not whole of $H^m(D)$, there exists a set of analytic functions $f_l(s)$ and its neighborhood V_2 satisfying $P_{\underline{\mathfrak{L}}_0}(V_2) = 0$. Since $\overline{V_1} \subset V_2$, we have $P_{\underline{\mathfrak{L}}_0}(\overline{V_1}) = 0$. Let P_n and P be probability measures defined on $(S, \mathfrak{B}(S))$. It is well known that P_n converges weakly to P as $n \rightarrow \infty$ if and only if

$$\limsup_{n \rightarrow \infty} P_n(C) \leq P(C)$$

for all closed set C . The set of $\overline{V_1}$ is closed, hence by Lemma 2.2.4, we obtain

$$\limsup_{T \rightarrow \infty} \nu_T^{\tau} \left\{ \sup_{1 \leq l \leq m} \sup_{s \in K_l} |\mathfrak{L}(\lambda_l, a, b_l, c; s + i\tau) - f_l(s)| \leq 2\varepsilon \right\} \leq P_{\underline{\mathfrak{L}}_0}(\overline{V_1}) = 0.$$

This formula yields the assertion of non-existence of joint t -universality.

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