Modified $f(R)$ gravity consistent with realistic cosmology: From a matter dominated epoch to a dark energy universe

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We develop the general scheme for modified $f(R)$ gravity reconstruction from any realistic Friedmann-Robertson-Walker (FRW) cosmology. We formulate several versions of modified gravity compatible with solar system tests where the following sequence of cosmological epochs occurs: (a) matter dominated phase (with or without usual matter), transition from deceleration to acceleration, accelerating epoch consistent with recent WMAP data, (b) ΛCDM cosmology without cosmological constant. As a rule, such modified gravities are expressed implicitly (in terms of special functions) with late-time asymptotics of known type (for instance, the model with negative and positive powers of curvature). In the alternative approach, it is demonstrated that even simple versions of modified gravity may lead to the unification of matter dominated and accelerated phases at the price of the introduction of compensating dark energy.

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I. INTRODUCTION

Modified gravity is an extremely promising approach to dark energy. Within a gravitational alternative for dark energy (for review, see [1]), the cosmic speedup is explained by the universe expansion where some subdominant terms (like $1/R$ [2,3] or $\ln R$ [4] which may be caused by string/M theory [5]) may become essential at small curvature. It also explains naturally the unification of earlier and later cosmological epochs as the manifestation of a different role of gravitational terms relevant at the small and large curvature as it happens in the model with negative and positive powers of curvature [6]. Moreover, modified gravity may serve as dark matter.

Special attention is paid to $f(R)$ modified gravity which may be constrained from cosmological/astrophysical observational data [7] and solar system tests [6,8–10]. Recently, a very interesting attempt to constrain such a model (with positive and negative powers of curvature) from fifth force/big bang nucleosynthesis (BBN) considerations has been made in [11] where it was shown that it is not easy to fulfill the known constraints and to describe the sequence of known cosmological epochs within the simple theory with positive and negative powers of curvature [6]. The cosmological dynamics of $1/R$ and other related $f(R)$ theories leading to late-time acceleration has been studied in Refs. [2–4,6,12] while Schwarzschild-de Sitter (SdS) black holes solutions were discussed in [13].

In the situation when general relativity cannot naturally describe the dark energy epoch of the universe the search of alternative, modified gravity which is consistent with solar system tests/observational data is of primary interest. Of course, it is too strong (at least, at the first step) to request that such a theory should correctly reproduce all known sequence of cosmological epochs (including an inflationary universe where quantum effects may be essential). Nevertheless, it is reasonable to constrain such a theory (if this is really an alternative theory for general relativity) by the condition that it reproduces a well-established sequence of classical cosmological phases (matter dominated phase, transition from deceleration to acceleration, and current universe speedup) being consistent with solar system tests. In the present paper we construct several examples of such modified gravity where $f(R)$ is presented implicitly, in terms of special functions.

The important remark is in order. It is a well-known fact that arbitrary $f(R)$ gravity may be presented in a mathematically equivalent form as a minimal scalar-tensor theory (Einstein gravity with scalar self-interacting field). Even more, it was shown [14,15] that $f(R)$ gravity may be formulated in a mathematically equivalent form as Einstein gravity with the ideal field having inhomogeneous equation of state (EOS). Then, it may look like it is not necessary to study modified gravity and it is enough to limit the consideration only by Einstein gravity with scalars and/or ideal fluid. However, the situation is more complicated. For instance, modified gravity of specific

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form which describes an acceptable accelerating universe (with realistic effective equation of state) is not physically equivalent to scalar-tensor theory [14,15]. Hence, the equivalent scalar-tensor gravity may not lead to accelerating FRW universe (or, it may lead but with a significantly different effective equation of state). The corresponding examples were given in [14,15]. Moreover, a specific form of modified gravity leads to a specific form of scalar potential. As a result, such specific modified gravity may comply with solar system tests (acceptable Newton law, etc.) while the corresponding scalar-tensor theory may not comply with it and vice versa. Hence, one should consider all three classes of theories: modified gravity, scalar-tensor gravity, and Einstein gravity with ideal fluid for description of dark energy and the early universe. One should fit all these three classes of theories with astrophysical and cosmological constraints (the corresponding cosmological parameters are defined from different bounds in [16]) in order to find finally what is the realistic gravitational theory compatible with observational data. In principle, in this work we are not so ambitious to comply with all bounds. We intend to present the first realistic example of modified gravity which is compatible with solar system tests, which is cosmologically viable (the sequence of matter dominated phase, transition from deceleration to acceleration, and acceleration phases) and which may lead even to ΛCDM cosmology.

The paper is organized as follows. In the next section we present general formulation to reconstruct the modified $f(R)$ gravity for any FRW given cosmology (using the auxiliary scalar field). This formulation is applied to work out several models. The explicit example of the model where the matter dominated phase may be realized by pure $f(R)$ gravity (no matter) with subsequent transition to acceleration phase is presented. There is constructed the model where the function $f(R)$ is expressed in terms of Gauss hypergeometric functions and where the standard ΛCDM cosmology is reproduced. For the specific version of the modified $f(R)$ gravity with matter it is shown that not only the matter dominated phase with subsequent transition to acceleration occurs, but the acceleration epoch complies with three years of Wilkinson Microwave Anisotropy Probe (WMAP) data. In order to ensure that transition from deceleration to acceleration indeed occurs, the (in)stability analysis of the cosmological solutions is fulfilled. We also demonstrate that corrections to Newton law for suggested versions of modified gravity are negligible. The third section is devoted to the consideration of the modified gravity model [6] with compensating dark energy (ideal fluid). It is shown that the role of such compensating dark energy may be to ensure the transition from the matter dominated to acceleration phase while during the current speedup such compensating dark energy quickly disappears. Some outlook is given in the discussion section. In the appendix it is shown that our formulation is just equivalent to standard metric formulation of $f(R)$ gravity (without an extra scalar).

II. RECONSTRUCTION OF MODIFIED GRAVITY WHICH DESCRIBES MATTER DOMINATED AND ACCELERATED PHASES

A. General formulation

In the present section we develop the general formulation of the reconstruction scheme for modified gravity with the $f(R)$ action. It is shown how any cosmology may define the implicit form of the function f. The starting action of modified gravity is:

$$S = \int d^4x\sqrt{-g}f(R). \tag{1}$$

First we consider the proper Hubble rate H, which describes the evolution of the universe, with radiation dominance, matter dominance, and accelerating expansion. It turns out that one can find $f(R)$-theory realizing such a cosmology (with or without matter). The construction is not explicit and it is necessary to solve the second order differential equation and algebraic equation. It shows, however, that, at least in principle, we could obtain any cosmology by properly reconstructing a function $f(R)$ on the theoretical level.

The equivalent form of the above action is

$$S = \int d^4x\sqrt{-g}\{P(\phi)R + Q(\phi) + \mathcal{L}_{\rm matter}\}. \tag{2}$$

Here P and Q are proper functions of the scalar field ϕ and $\mathcal{L}_{\rm matter}$ is the matter Lagrangian density. Since the scalar field does not have a kinetic term, it may be regarded as an auxiliary field (compare with the ideal fluid representation of $f(R)$ gravity [14]). In fact, by the variation of ϕ, it follows

$$0 = P'(\phi)R + Q'(\phi), \tag{3}$$

which may be solved with respect to ϕ:

$$\phi = \phi(R). \tag{4}$$

By substituting (4) into (2), one obtains $f(R)$ gravity:

$$S = \int d^4x\sqrt{-g}\{f(R) + \mathcal{L}_{\rm matter}\}, \qquad f(R) \equiv P(\phi(R))R + Q(\phi(R)). \tag{5}$$

By the variation of the action (2) with respect to the metric $g_{\mu\nu}$, we obtain

$$0 = -\tfrac{1}{2}g_{\mu\nu}\{P(\phi)R + Q(\phi)\} - R_{\mu\nu}P(\phi) + \nabla_\mu\nabla_\nu P(\phi) - g_{\mu\nu}\nabla^2P(\phi) + \tfrac{1}{2}T_{\mu\nu}. \tag{6}$$

The equations corresponding to the standard spatially flat FRW universe are

$$0 = -6H^2 P(\phi) - Q(\phi) - 6H\frac{dP(\phi(t))}{dt} + \rho, \quad (7)$$

$$0 = (4\dot{H} + 6H^2)P(\phi) + Q(\phi) + 2\frac{d^2P(\phi(t))}{dt} + 4H\frac{dP(\phi(t))}{dt} + p. \quad (8)$$

By combining (6) and (7) and deleting $Q(\phi)$, we find the following equation

$$0 = 2\frac{d^2P(\phi(t))}{dt^2} - 2H\frac{dP(\phi(t))}{d\phi} + 4\dot{H}P(\phi) + p + \rho. \quad (9)$$

As one can redefine the scalar field ϕ properly, we may choose

$$\phi = t. \quad (10)$$

It is assumed that ρ and p are the sum from the contribution of the matters with a constant equation of state parameters w_i. Especially, when it is assumed a combination of the radiation and dust, one gets the standard expression

$$\rho = \rho_{r0} a^{-4} + \rho_{d0} a^{-3}, \qquad p = \frac{\rho_{r0}}{3} a^{-4}, \quad (11)$$

with constants ρ_{r0} and ρ_{d0}. If the scale factor a is given by a proper function $g(t)$ as

$$a = a_0 \mathrm{e}^{g(t)}, \quad (12)$$

with a constant a_0, Eq. (8) reduces to the second rank differential equation (see also [15]):

$$0 = 2\frac{d^2P(\phi)}{d\phi^2} - 2g'(\phi)\frac{dP(\phi))}{d\phi} + 4g''(\phi)P(\phi) + \sum_i (1 + w_i)\rho_{i0} a_0^{-3(1+w_i)} \mathrm{e}^{-3(1+w_i)g(\phi)}. \quad (13)$$

In principle, by solving (13) we find the form of $P(\phi)$. Using (7) (or equivalently (8)), we also find the form of $Q(\phi)$ as

$$Q(\phi) = -6(g'(\phi))^2 P(\phi) - 6g'(\phi)\frac{dP(\phi)}{d\phi} + \sum_i \rho_{i0} a_0^{-3(1+w_i)} \mathrm{e}^{-3(1+w_i)g(\phi)}. \quad (14)$$

Hence, in principle, any cosmology expressed as (12) can be realized by some specific $f(R)$ gravity.

B. Exactly solvable example I: unification of matter dominated and accelerated phases

As an example, we consider the case

$$g'(\phi) = g_0 + \frac{g_1}{\phi}, \quad (15)$$

without matter $\rho = p = 0$ for simplicity. Equation (13) reduces as

$$0 = \frac{d^2P}{d\phi^2} - \left(g_0 + \frac{g_1}{\phi}\right)\frac{dP}{d\phi} - \frac{2g_1}{\phi^2}P, \quad (16)$$

whose solutions are given by the Kummer functions (hypergeometric function of confluent type) as [15]

$$P = z^\alpha F_K(\alpha, \gamma; z), \qquad z^{1-\gamma} F_K(\alpha - \gamma + 1, 2 - \gamma; z). \quad (17)$$

Here

$$z \equiv g_0 \phi, \qquad \alpha \equiv \frac{1 + g_1 \pm \sqrt{g_1^2 + 2g_1 + 9}}{4}, \qquad \gamma \equiv 1 \pm \frac{\sqrt{g_1^2 + 2g_1 + 9}}{2}, \quad (18)$$

and the Kummer function is defined by

$$F_K(\alpha, \gamma; z) = \sum_{n=0}^{\infty} \frac{\alpha(\alpha + 1)\cdots(\alpha + n - 1)}{\gamma(\gamma + 1)\cdots(\gamma + n - 1)} \frac{z^n}{n!}. \quad (19)$$

Equation (15) tells that the Hubble rate H is given by

$$H = g_0 + \frac{g_1}{t}. \quad (20)$$

When t is small, as $H \sim g_1/t$, the universe behaves as the one filled with a perfect fluid with the EOS parameter $w = -1 + 2/3g_1$. On the other hand when t is large, H approaches to constant $H \to g_0$ and the universe looks like de Sitter space. This shows the possibility of the transition from the matter dominated phase to the accelerating phase (compare with [15]). We should note that in this case, there is no matter and $f(R)$-terms contribution plays the role of the matter instead of the real matter. We will investigate later (in the next subsection) the example that there is a real matter. Similarly, one can construct modified gravity action describing other epochs bearing in mind that the form of the modified gravity action is different at different epochs (for instance, in the inflationary epoch it is different from the form at late-time universe).

We now investigate the asymptotic forms of $f(R)$ in (5) corresponding to (15). When ϕ and therefore t are small, we find

$$P \sim P_0 \phi^\alpha, \qquad Q \sim -6P_0 g_1(g_1 + \alpha)\phi^{\alpha-2}. \quad (21)$$

Here P_0 is a constant. Using (3), it follows

$$\phi^2 \sim \frac{6g_1(g_1 + \alpha)(\alpha - 2)}{\alpha R}, \quad (22)$$

which gives

$$f(R) \sim -\frac{2P_0}{\alpha - 2}\left\{\frac{6g_1(g_1 + \alpha)(\alpha - 2)}{\alpha}\right\}^{\alpha/2} R^{1-(\alpha/2)}. \quad (23)$$

On the other hand, when ϕ and therefore t are positive and large, one gets

$$P \sim \tilde{P}_0 \phi^{2\alpha-\gamma} e^{g_0\phi}\left(1 + \frac{(1-\alpha)(\gamma-\alpha)}{g_0\phi}\right),$$
$$Q \sim -12 g_0^2 \tilde{P}_0 \phi^{2\alpha-\gamma} e^{g_0\phi} \times\left(1 + \frac{-9+12\alpha-5\gamma-2\alpha\gamma+2\alpha^2}{2g_0\phi}\right), \tag{24}$$
$$\phi \sim \frac{-\frac{9}{2}+9\alpha-\frac{7}{2}\gamma}{g_0(\frac{R}{12g_0^2}-1)}.$$

Here $\tilde{P}_0$ is a constant. Then we find

$$f(R) \sim 12 g_0^2 \tilde{P}_0 \left\{\frac{1}{g_0}\left(-\frac{9}{2}+9\alpha-\frac{7}{2}\gamma\right)\right\}^{2\alpha-\gamma} \times\left(\frac{R}{12g_0^2}-1\right)^{-2\alpha+\gamma+1} \exp\left(\frac{-\frac{9}{2}+9\alpha-\frac{7}{2}\gamma}{\frac{R}{12g_0^2}-1}\right). \tag{25}$$

This shows the principal possibility of unification of the matter dominated phase (even without matter), transition to acceleration and late-time speedup of the universe for a specific, implicitly given model of $f(R)$ gravity.

C. Exactly solvable example II: model reproducing ΛCDM-type cosmology

Let us investigate if ΛCDM-type cosmology could be reproduced by $f(R)$ gravity in the present formulation.

In the Einstein gravity, when there is a matter with the EOS parameter w and cosmological constant, the FRW equation has the following form:

$$\frac{3}{\kappa^2}H^2 = \rho_0 a^{-3(1+w)} + \frac{3}{\kappa^2 l^2}. \tag{26}$$

Here l is the length parameter coming from the cosmological constant. The solution of (26) is given by

$$a = a_0 e^{g(t)},$$
$$g(t) = \frac{2}{3(1+w)} \ln\left(\alpha \sinh\left(\frac{3(1+w)}{2l}(t-t_0)\right)\right). \tag{27}$$

Here t_0 is a constant of the integration and

$$\alpha^2 \equiv \tfrac{1}{3}\kappa^2 l^2 \rho_0 a_0^{-3(1+w)}. \tag{28}$$

It is possible to reconstruct $f(R)$ gravity reproducing (27). When the matter contribution is neglected, Eq. (13) has the following form:

$$0 = 2\frac{d^2P(\phi)}{d\phi^2} - \frac{2}{l}\coth\left(\frac{3(1+w)}{2l}(t-t_0)\right)\frac{dP(\phi)}{d\phi} - \frac{6(1+w)}{l^2}\sinh^{-2}\left(\frac{3(1+w)}{2l}(t-t_0)\right)P(\phi). \tag{29}$$

By changing the variable from ϕ to z as follows,

$$z \equiv -\sinh^{-2}\left(\frac{3(1+w)}{2l}(t-t_0)\right), \tag{30}$$

Eq. (29) can be rewritten in the form of Gauss's hypergeometric differential equation:

$$0 = z(1-z)\frac{d^2P}{dz^2} + [\tilde{\gamma} - (\tilde{\alpha}+\tilde{\beta}+1)z]\frac{dP}{dz} - \tilde{\alpha}\tilde{\beta}P,$$
$$\tilde{\gamma} \equiv 4 + \frac{1}{3(1+w)}, \qquad \tilde{\alpha}+\tilde{\beta}+1 \equiv 6 + \frac{1}{3(1+w)},$$
$$\tilde{\alpha}\tilde{\beta} \equiv -\frac{1}{3(1+w)}, \tag{31}$$

whose solution is given by Gauss's hypergeometric function:

$$P = P_0 F(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}; z) \equiv P_0 \frac{\Gamma(\tilde{\gamma})}{\Gamma(\tilde{\alpha})\Gamma(\tilde{\beta})} \sum_{n=0}^{\infty} \frac{\Gamma(\tilde{\alpha}+n)\Gamma(\beta+n)}{\Gamma(\tilde{\gamma}+n)} \frac{z^n}{n!}. \tag{32}$$

Here Γ is the Γ function. There is one more linearly independent solution like $(1-z)^{\tilde{\gamma}-\tilde{\alpha}-\tilde{\beta}}F(\tilde{\gamma}-\tilde{\alpha}, \tilde{\gamma}-\tilde{\beta}, \tilde{\gamma}; z)$ but we drop it, for simplicity. Using (14), one finds the form of $Q(\phi)$:

$$Q = -\frac{6(1-z)P_0}{l^2}F(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}; z) - \frac{3(1+w)z(1-z)P_0}{l^2(13+12w)}F(\tilde{\alpha}+1, \tilde{\beta}+1, \tilde{\gamma}+1; z). \tag{33}$$

From (30), it follows $z \to 0$ when $t = \phi \to +\infty$. Then in the limit, one arrives at

$$P(\phi)R + Q(\phi) \to P_0 R - \frac{6P_0}{l^2}. \tag{34}$$

Identifying

$$P_0 = \frac{1}{2\kappa^2}, \qquad \Lambda = \frac{6}{l^2}, \tag{35}$$

the Einstein theory with cosmological constant Λ can be reproduced. The action is not singular even in the limit of $t \to \infty$. Note that a slightly different approach to construct ΛCDM cosmology from $f(R)$ gravity is developed in Ref. [17].

Therefore even without the cosmological constant nor cold dark matter, the cosmology of the ΛCDM model could be reproduced by $f(R)$ gravity. It was shown in Ref. [18] that some versions of modified gravity contain big rip singularities [19] (for their classification, see [20]). Hence, the above model without future singularity and with typical ΛCDM behavior looks quite realistic.

D. Models of $f(R)$ gravity with transition of the matter dominated phase to the acceleration phase

Let us consider more realistic examples where the total action contains also usual matter. The starting form of $g(\phi)$ is

$$g(\phi) = h(\phi) \ln\left(\frac{\phi}{\phi_0}\right), \tag{36}$$

with a constant ϕ_0. It is assumed that $h(\phi)$ is a slowly changing function of ϕ. We use adiabatic approximation and neglect the derivatives of $h(\phi)$ $(h'(\phi) \sim h''(\phi) \sim 0)$. Equation (13) has the following form:

$$0 = \frac{d^2P(\phi)}{d\phi^2} - \frac{h(\phi)}{\phi}\frac{dP(\phi))}{dt} - \frac{2h(\phi)}{\phi^2}P(\phi) + \sum_i \rho_{i0} a_0^{-3(1+w_i)} \mathrm{e}^{-3(1+w_i)g(\phi)}. \tag{37}$$

The solution for $P(\phi)$ is found to be

$$P(\phi) = p_+\phi^{n_+(\phi)} + p_-\phi^{n_-(\phi)} + \sum_i p_i(\phi)\phi^{-3(1+w_i)h(\phi)+2}. \tag{38}$$

Here $p_\pm$ are arbitrary constants and

$$\begin{aligned} n_\pm(\phi) &\equiv \frac{h(\phi) - 1 \pm \sqrt{h(\phi)^2 + 6h(\phi) + 1}}{2}, \\ p_i(\phi) &\equiv -\{(1+w)\rho_{i0} a_0^{-3(1+w_i)} \phi_0^{3(1+w)h(\phi)}\} \\ &\quad \times \{6(1+w)(4+3w)h(\phi)^2 \\ &\quad - 2(13+9w)h(\phi) + 4\}^{-1}. \end{aligned} \tag{39}$$

Especially for the radiation and dust, one has

$$\begin{aligned} p_r(\phi) &\equiv -\frac{4\rho_{r0}\phi_0^{4h(\phi)}}{3a_0^4(40h(\phi)^2 - 32h(\phi) + 4)}, \\ p_d(\phi) &\equiv -\frac{\rho_{d0}\phi_0^{3h(\phi)}}{a_0^3(24h(\phi)^2 - 26h(\phi) + 4)}. \end{aligned} \tag{40}$$

We also find the form of $Q(\phi)$ as

$$\begin{aligned} Q(\phi) = &-6h(\phi)p_+(h(\phi) + n_+(\phi))\phi^{n_+(\phi)-2} \\ &- 6h(\phi)p_-(h(\phi) + n_-(\phi))\phi^{n_-(\phi)-2} \\ &+ \sum_i \{-6h(\phi)(-(2+3w)h(\phi) + 2)p_i(\phi) \\ &+ p_{i0} a_0^{-3(1+w)} \phi_0^{3(1+w)h(\phi)}\}\phi^{-3(1+w)h(\phi)}. \end{aligned} \tag{41}$$

Equation (36) tells that

$$H \sim \frac{h(t)}{t} \tag{42}$$

and

$$R \sim \frac{6(-h(t) + 2h(t)^2)}{t^2}. \tag{43}$$

Let us assume $\lim_{\phi\to 0} h(\phi) = h_i$ and $\lim_{\phi\to\infty} h(\phi) = h_f$. Then if $0 < h_i < 1$, the early universe is in deceleration phase and if $h_f > 1$, the late universe is in acceleration phase. We may consider the case $h(\phi) \sim h_m$ is almost constant when $\phi \sim t_m$ $(0 \ll t_m \ll +\infty)$. If h_1, $h_f > 1$, and $0 < h_m < 1$, the early universe is also accelerating, which could be inflation. After that the universe becomes decelerating, which corresponds to the matter dominated phase with $h(\phi) \sim 2/3$ there. Furthermore, after that, the universe could be in the acceleration phase.

The simplest example is

$$h(\phi) = \frac{h_i + h_f q\phi^2}{1 + q\phi^2}, \tag{44}$$

with constants h_i, h_f, and q, when $\phi \to 0$, $h(\phi) \to h_i$ and when $\phi \to \infty$, $h(\phi) \to h_f$. If q is small enough, $h(\phi)$ can be a slowly varying function of ϕ. By using the expression of (43), we find

$$\phi^2 = \Phi_0(R), \qquad \Phi_\pm(R), \qquad \Phi_0 \equiv \alpha_+^{1/3} + \alpha_-^{1/3}, \qquad \Phi_\pm \equiv \alpha_\pm^{1/3}\mathrm{e}^{2\pi i/3} + \alpha_\mp^{1/3}\mathrm{e}^{-2\pi i/3}, \qquad \alpha_\pm \equiv \frac{-\beta_0 \pm \sqrt{\beta_0^2 - \frac{4\beta_1^3}{27}}}{2},$$
$$\beta_0 \equiv \frac{2(2R + 6h_f q - 12h_f^2 q)^3}{27q^3R^3} - \frac{(2R + 6h_f q - 12h_f^2 q)(R + 6h_i q + 6h_f q - 4h_i h_f q)}{3qR} + 6h_i - 12h_i^2,$$
$$\beta_1 \equiv -\frac{(2R + 6h_f q - 12h_f^2 q)^2}{3q^2R^2} - \frac{R + 6h_i q + 6h_f q - 4h_i h_f q}{q^2 R}. \tag{45}$$

There are three branches Φ_0 and $\Phi_\pm$ in (45). Equations (43) and (44) show that when the curvature is small ($\phi = t$ is large), we find $R \sim 6(-h_f + 2h_f)/\phi^2$ and when the curvature is large ($\phi = t$ is small), $R \sim 6(-h_i + 2h_i)/\phi^2$. This asymptotic behavior indicates that we should choose Φ_0 in (45). Then an explicit form of $f(R)$ could be given by using the expressions of $P(\phi)$ (38) and $Q(\phi)$ (41) as

$$f(R) = P(\sqrt{\Phi_0(R)})R + Q(\sqrt{\Phi_0(R)}). \quad (46)$$

One may check the asymptotic behavior of $f(R)$ in (46). For simplicity, it is considered the case that the matter is only dust ($w = 0$) and that $p_- = 0$. Then we find

$$P(\phi) = p_+\phi^{n_+(\phi)} + p_d(\phi)\phi^{-3h(\phi)+2}. \quad (47)$$

One may always get

$$n_+ - (-3h + 2) > 0 \quad (48)$$

in (47). Here $n_+ = (h(\phi) - 1 \pm \sqrt{h(\phi)^2 + 6h(\phi) + 1})/2$ is defined in (39). Then when ϕ is large, the first term in (47) dominates and when ϕ is small, the last term dominates. When ϕ is large, the curvature is small and $\phi^2 \sim 6(-h_f + 2h_f)/R$ and $h(\phi) \to h(\infty) = h_f$. Hence, Eq. (47) shows that

$$P(\phi) \sim p_+\left(\frac{6(-h_f + 2h_f)}{R}\right)^{(h_f-1+\sqrt{h_f^2+6h_f+1})/4}, \quad (49)$$

and therefore

$$f(R) \sim R^{-(h(\phi)-5+\sqrt{h_f^2+6h_f+1})/4}. \quad (50)$$

Especially when $h \gg 1$, we find

$$f(R) \sim R^{-h_f/2}. \quad (51)$$

Therefore there appears the negative power of R. As $H \sim h_f/t$, if $h_f > 1$, the universe is in acceleration phase.

On the other hand, when the curvature is large, we find $\phi^2 \sim 6(-h_i + 2h_i)/R$ and $h(\phi) \to h(0) = h_i$. Then (38) shows

$$P(\phi) \sim p_d(0)\phi^{-3h_i+2}. \quad (52)$$

If the universe era corresponds to the matter dominated phase ($h_i = 2/3$), $P(\phi)$ becomes a constant and therefore

$$f(R) \sim R, \quad (53)$$

which reproduces the Einstein gravity.

Thus, in the above model, the matter dominated phase evolves into the acceleration phase and $f(R)$ behaves as $f(R) \sim R$ initially while $f(R) \sim R^{-(h(\phi)-5+\sqrt{h_f^2+6h_f+1})/4}$ at late time.

Three years of WMAP data were recently analyzed in Ref. [21], which shows that the combined analysis of WMAP with supernova Legacy survey (SNLS) constrains the dark energy equation of state $w_{\rm DE}$ pushing it towards the cosmological constant. The marginalized best fit values of the equation of state parameter at 68% confidence level are given by $-1.14 \le w_{\rm DE} \le -0.93$. In the case of a prior that the universe is flat, the combined data gives $-1.06 \le w_{\rm DE} \le -0.90$.

In our model, we can identify

$$w_{\rm DE} = -1 + \frac{2}{3h_f}, \quad (54)$$

or

$$h_f = \frac{2}{3(1 + w_{\rm DE})}, \quad (55)$$

which tells that h_f should be large if h_f is positive. For example, if $w_{\rm DE} = -0.93$, $h_f \sim 9.51\cdots$ and if $w_{\rm DE} = -0.90$, $h_f \sim 6.67\cdots$. Thus, we presented the example of $f(R)$ gravity which describes the matter dominated stage, transition from deceleration to acceleration, and the acceleration epoch which is consistent with three years of WMAP.

E. (In)stability of the cosmological solutions

Let us investigate the stability of the obtained solutions. We assume

$$a = a_0 e^{g(\phi)}, \quad (56)$$

which corresponds to (12) and $P(\phi)$ should be given by a solution of (13) (and $Q(\phi)$ should be given by (14)). Under the above assumptions, we consider the perturbations in (7) and (8). By deleting $Q(\phi)$ in (7) and (8), one obtains

$$\begin{aligned} 0 &= 2\frac{d^2P(\phi)}{dt^2} - 2g'(\phi)\frac{dP(\phi))}{dt} + 4g''(\phi)P(\phi) \\ &\quad + \sum_i(1 + w_i)\rho_{i0}a_0^{-3(1+w_i)}e^{-3(1+w_i)g(\phi)} \\ &= 2\frac{d^2P(\phi)}{d\phi^2}\left(\frac{d\phi}{dt}\right)^2 + 2\frac{dP(\phi)}{d\phi}\frac{d^2\phi}{dt^2} \\ &\quad - 2g'(\phi)\frac{dP(\phi))}{d\phi}\left(\frac{d\phi}{dt}\right)^2 \\ &\quad + 4\left\{g''(\phi)\left(\frac{d\phi}{dt}\right)^2 + g'(\phi)\frac{d^2\phi}{dt^2}\right\}P(\phi) \\ &\quad + \sum_i(1 + w_i)\rho_{i0}a_0^{-3(1+w_i)}e^{-3(1+w_i)g(\phi)}. \end{aligned} \quad (57)$$

Then combining (13) with (56) it follows

$$\begin{aligned} 0 &= 2\left\{\frac{d^2P(\phi)}{d\phi^2} - g'(\phi)\frac{dP(\phi))}{d\phi} + 2g''(\phi)P(\phi)\right\} \\ &\quad \times \left\{\left(\frac{d\phi}{dt}\right)^2 - 1\right\} + 2\left\{\frac{dP(\phi)}{d\phi} + 2g'(\phi)P(\phi)\right\}\frac{d^2\phi}{dt^2}. \end{aligned} \quad (58)$$

By defining δ as

$$\delta \equiv \frac{d\phi}{dt} - 1, \quad (59)$$

we consider the perturbation from the solution (10). Using (58), one gets

$$\frac{d\delta}{dt} = -\omega(t)\delta,$$
$$\omega(t) \equiv 2\frac{\frac{d^2P(\phi)}{d\phi^2} - g'(\phi)\frac{dP(\phi))}{d\phi} + 2g''(\phi)P(\phi)}{\frac{dP(\phi)}{d\phi} + 2g'(\phi)P(\phi)}\Bigg|_{\phi=t}. \tag{60}$$

Then when $\omega > 0$ ($\omega < 0$), the perturbation becomes small (large) and the system is stable (unstable).

As an example, the case (36) may be considered. It gives (38) with (39). Especially when $p_\pm = p_{i0} = 0$ except p_{di}, we find

$$\omega = \frac{2(12h^2 - 13h + 2)}{(2-h)t}. \tag{61}$$

Here the derivatives of $h(\phi)$ like $h'(\phi)$ are neglected again. Then ω goes to infinity when $h = 2$ and $h \to \pm\infty$ and ω vanishes when

$$h = h_\pm = \frac{13 \pm \sqrt{63}}{24} = 0.872\,38\cdots, 0.210\,947\,8\cdots. \tag{62}$$

Therefore $\omega > 0$ and the system is stable when $h < h_-$ or $h_+ < h < 2$. Hence, for the case that the universe starts from the deceleration phase with $h = h_0 < 1$, if $h_0 > h_+$, there is a stable solution where the universe develops to the acceleration phase $h \to 2 > 1$. Even if we started with $h(\phi) = 2/3$, which corresponds to the matter dominated phase $a \sim t^{2/3}$, the solution is unstable since $h_- < 2/3 < h_+$, the perturbed solution could develop into the stable solution with $h > h_+$ and therefore there could be a transition into the acceleration phase. If h goes to 2 from the region with $h < 2$, since $\omega \to +\infty$, the solution becomes extremely stable. Hence, h may pass through the point $h = 2$ and h could become larger than 2, where the effective EOS parameter $w = 1 - 2/3h = 7/9$.

Note that when $p_\pm = p_{i0} = 0$ ($i \neq d$: dust), one gets

$$R \sim \phi^{-2}, \qquad f(R) \sim R^{3h/2}. \tag{63}$$

Therefore in the matter dominated phase $h \sim 2/3$, the action behaves as the Hilbert-Einstein action.

In the more general case where there is only one kind of matter with w and $p_\pm = 0$, we find

$$R = \frac{3h(\phi)\{12(1+w)h(\phi)^2 - 2(7+9w)h(\phi) + 4\}}{\{3(1+w)h(\phi) + 2\}\phi^2},$$
$$f(R) \sim R^{3/2(1+w)h(\phi(R))}. \tag{64}$$

Note that ω in (60) is given by

$$\omega = \frac{3(1+w)(4+3w)h(\phi)^2 - (13+9w)h(\phi) + 2}{\{-(1+3w)h(\phi) + 2\}t}. \tag{65}$$

Next we consider the case that $p_{d0} \neq 0$ and $p_{r0} \neq 0$ but $p_\pm = 0$ and $p_{i0} = 0$ except $i \neq d, r$. Then it follows

$$\omega = -\frac{1}{\phi}\{4p_r(10h^2 - 8h + 1) + 2p_d(12h^2 - 13h + 2)\phi^h\} \times \{2p_r(h-1) + p_d(h-2)\phi^h\}^{-1}, \tag{66}$$

or by using (39),

$$\omega = -\frac{2}{\phi}\left(\frac{4\rho_{r0}\phi_0^{4h}}{3a_0^4} + \frac{\rho_{d0}\phi_0^{3h}\phi^h}{a_0^3}\right)(10h^2 - 8h + 1) \times (12h^2 - 13h + 2)\left\{\frac{4\rho_{r0}\phi_0^{4h}}{3a_0^4}(12h^2 - 13h + 2) \times (h-1) + \frac{\rho_{d0}\phi_0^{3h}}{a_0^3}\phi^h(10h^2 - 8h + 1)(h-2)\right\}^{-1}. \tag{67}$$

As is clear from (11), (36), and (56), $\rho_{r0}3a_0^{-4}$ and $\rho_{d0}a_0^{-3}$ correspond to the energy density for radiation and dust (usual matter plus cold dark matter), respectively, when $t = \phi = \phi_0$. Let us choose $t = \phi_0$ corresponding to the present universe. It may be assumed

$$\epsilon \equiv \left(\frac{4\rho_{r0}\phi_0^{4h}\phi^h}{3a_0^4}\right)\left(\frac{\rho_{d0}\phi_0^{3h}}{a_0^3}\right)^{-1} \ll 1. \tag{68}$$

In the expression (67), ω vanishes when $h = h_\pm$ (62) and $h = \tilde{h}_\pm$, defined by

$$\tilde{h}_\pm \equiv \frac{4 \pm \sqrt{6}}{10} = 0.6449\cdots, 0.155\,05\cdots. \tag{69}$$

Under the assumption (68), ω diverges at

$$h = 2 - \frac{24}{25}\epsilon, \qquad h = \tilde{h}_\pm + \epsilon\delta_\pm,$$
$$\delta_\pm \equiv \mp\frac{(28 \pm 17\sqrt{6})(-6 \pm \sqrt{6})(16 \pm \sqrt{6})}{25\,000}. \tag{70}$$

Here $\delta_\pm > 0$. Near the singularities, if ω is negative, there could be very large instability, which should be avoided. In the case $p_\pm \neq 0$, the singularity could be avoided. The points where ω vanishes could remain even if $p_\pm \neq 0$ but the instability becomes finite and there can be a solution which describes the transition from the matter dominated phase to the acceleration phase.

We now consider the case that the contribution from the matter could be neglected in (38) and therefore we could assume that $p_{i0} = 0$. Furthermore when one of $p_\pm$ vanishes, one finds

$$R \sim \phi^{-2}, \qquad f(R) \sim R^{N_\pm}, \qquad N_\pm \equiv 1 - \frac{n_\pm}{2}. \tag{71}$$

The behavior of $N_\pm$ is as the following:

$$\text{when } h \to 0, \qquad N_+ \to 1 \quad \text{and} \quad N_- \to \frac{3}{3},$$
$$h \to +\infty, \qquad N_+ \to -\frac{h}{2} \quad \text{and} \quad N_- \to 2,$$
$$h = 1, \qquad N_\pm \to 1 \mp \frac{1}{\sqrt{2}},$$
$$h = \frac{2}{3}, \qquad N_+ \to \frac{1}{3} \quad \text{and} \quad N_- \to \frac{3}{2}. \tag{72}$$

Thus, in the case $p_+ = 0$ but $p_- \neq 0$, when $h \to +\infty$, the higher derivative inflationary model $f(R) \sim R^2$ appears. We should also note ω in (60) has the following form:

$$\omega = \omega_\pm \equiv -\frac{2(h(\phi) - 1) \pm 4\sqrt{h(\phi)^2 + 6h(\phi) + 1}}{5h(\phi) - 1 \pm \sqrt{h(\phi)^2 + 6h(\phi) + 1}}. \tag{73}$$

Now $\hat{h}_\pm$ may be defined as

$$\hat{h}_\pm = \frac{13 \pm 4\sqrt{10}}{3} = 8.5497\cdots, 0.116\,90\cdots. \tag{74}$$

Then $\omega_+ < 0$ when $h > \hat{h}_+$ or $h < 0$, and $\omega_+ > 0$ when $0 < h < \hat{h}_+$. We also find that when $\omega_- > 0$, $h > 2/3$, or $h < h_-$, and $\omega_- < 0$ when $\hat{h}_- < h < 2/3$. Hence, the model where $p_+ = 0$ but $p_- \neq 0$ is stable when $h > \frac{2}{3}$. Thus, one can make a stable model where the matter dominated phase $h \sim 3/2$ evolves to the acceleration phase $h > 1$.

We should note that $p_- = 0$ and $p_{i0} = 0$ but $p_+ \neq 0$, if we put

$$h = \tfrac{10}{3}, \tag{75}$$

we obtain $N_+ = -1$ or $f(R) \sim 1/R$. Let us put

$$h(\phi) = \tfrac{10}{3} + \delta h, \qquad |\delta h| \ll 1. \tag{76}$$

It follows that

$$N_+ = -1 - \tfrac{18}{17}\delta h. \tag{77}$$

When $h \sim 10/3$, from (3), the curvature is given as

$$R \sim \frac{6h(h + n_+)(n_+ - 2)}{n_+ \phi^2} \sim \frac{220}{3\phi^2}. \tag{78}$$

Hence, with the choice

$$\delta h \sim -\frac{17}{18 \ln\frac{220}{3\mu^2\phi^2}}\left(\frac{220^2}{18\kappa^2\mu^6\phi^4} + \frac{220^3\beta}{27\mu^6\phi^6}\right) \sim -\frac{17}{18 \ln\frac{R}{\mu^2}}\left(\frac{R^2}{2\kappa^2\mu^6} + \frac{\beta R^3}{\mu^6}\right), \tag{79}$$

with a constant μ, which has a dimension of mass, one arrives at

$$f(R) \sim \frac{\mu^6}{R} + \frac{R}{2\kappa^2} + \beta R^2, \tag{80}$$

which reproduces the action proposed in [6]. (Note that such a class of actions does not describe the sequence of the matter dominated/acceleration phase [22]). Thus, using stability analysis we demonstrated that indeed the matter dominated phase may transit to the acceleration phase for some implicit model of $f(R)$ gravity found in the previous subsection. Moreover, it is shown that the model of Ref. [6] with positive and negative powers of the curvature is just an asymptotic form of such a consistent $f(R)$ theory (at some specific values of parameters) at the acceleration epoch. The complete, implicit version of $f(R)$ theory found in previous subsection describes the sequence of the matter dominated phase, transition from deceleration to acceleration, and then the acceleration epoch of the universe.

F. Corrections to Newton law

In the present subsection we will discuss the contributions to Newton law in the modified gravity under consideration. Note that there is a big number of papers devoted to the study of the Newtonian regime in modified $f(R)$ gravity [6,8]. However, these papers are mainly devoted to the study of the Newtonian regime for $1/R$ models. We now check when the corrections to the Newton law are not essential in $f(R)$ gravity under consideration. For this purpose, we put a point source at $\boldsymbol{r} = 0$:

$$\rho_m = \frac{m}{a(t)^3}\delta(\boldsymbol{r}). \tag{81}$$

Transforming

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}, \qquad \phi \to \phi + \delta\phi, \tag{82}$$

one finds the (t, t) component of (6) has the following form:

$$\begin{aligned} 0 &= G_0 + G_1, \\ G_0 &\equiv -\frac{1}{2}\delta g_{tt}\{P(\phi)(6\dot{H} + 12H^2) + Q(\phi)\} - \frac{1}{2}P(\phi)\delta R - P(\phi)\delta R_{tt}, \\ G_1 &= \frac{1}{2}\{P'(\phi)(6\dot{H} + 12H^2) + Q'(\phi)\}\delta\phi + 3(\dot{H} + H^2)P'(\phi)\delta\phi + \frac{1}{2}P'(\phi)\nabla_t\delta g_{tt} \\ &\quad + 3HP'(\phi)\delta g_{tt} + P''(\phi)\delta g_{tt} - HP'(\phi)g^{ij}\delta g_{ij} + \frac{1}{2}P'(\phi)\nabla_t(g^{\mu\nu}\delta g_{\mu\nu}) + \frac{m}{2a(t)^3}\delta(\boldsymbol{r}). \end{aligned} \tag{83}$$

Here the gauge condition is chosen

$$\nabla^\mu \delta g_{\mu\nu} = 0. \tag{84}$$

On the other hand, Eq. (3) gives

$$0 = (P''(\phi) + Q''(\phi)R)\delta\phi + Q'(\phi)\delta R. \tag{85}$$

We now consider the region for $r = |\boldsymbol{r}|$ as

$$\frac{1}{m} \ll r \ll \frac{1}{H}, \tag{86}$$

or

$$m \gg \frac{\partial}{\partial r} \gg H. \tag{87}$$

We should also note that

$$\frac{\partial}{\partial t} \sim \frac{1}{\phi} \sim H \sim \sqrt{R}. \tag{88}$$

Then if

$$P'(\phi) \ll HP(\phi), \qquad P''(\phi) \ll H^2 P(\phi), \tag{89}$$

G_1 (83) could be neglected if compared with G_0. Then since ϕ can be regarded as a constant and therefore $\delta\phi = 0$ as long as (86) is satisfied, Eq. (83) reduces to that in the Einstein gravity by identifying $P(\phi)$ with $1/\kappa^2$ and $Q(\phi) = \Lambda^2/\kappa^2$ since $P(\phi)$ and $Q(\phi)$ are slowly varying (almost constant) functions. From (85), $\delta\phi = 0$ implies $\delta R = 0$. Then Eq. (83) gives the (t, t) component of the usual perturbation in the Einstein equation:

$$\begin{aligned} 0 &= G_0 \\ &= -\frac{1}{2\kappa^2}\delta g_{tt}\{R_0 + \Lambda^2\} - \frac{1}{\kappa^2}\left(\frac{1}{2}\delta R - \delta R_{tt}\right) \\ &\quad + \frac{m}{2a_0^3}\delta(\boldsymbol{r}), \end{aligned} \tag{90}$$
$$R_0 \equiv 6\dot{H} + 12H^2.$$

Here a_0 is the scale factor in the present universe $a_0 = a(t)$, which may be chosen to be unity $a_0 = 1$. For the almost flat universe as our current one, we can neglect the first term in (90): $R_0 + \Lambda^2 \sim 0$ and we obtain

$$0 = -\frac{1}{\kappa^2}\left(\frac{1}{2}\delta R - \delta R_{tt}\right) + m\delta(\boldsymbol{r}), \tag{91}$$

which further reduces, since the universe is almost flat and the source and the universe are static in a region where we are investigating the Newton law, to

$$0 = \frac{1}{2\kappa^2}(\triangle\delta g_{tt} - \triangle(g^{\mu\nu}\delta g_{\mu\nu})) + m\delta(\boldsymbol{r}). \tag{92}$$

Here $\triangle$ is the usual Laplacian. Therefore if the conditions in (88) are satisfied, the correction to the Newton law could be neglected. In the case of (42), if we choose p_+ and p_- to satisfy (89) in the present universe, the correction to the Newton law could be small. (It is interesting to note that standard Newton law is valid also in an arbitrary $F(G)$ gravity [23] where G is Gauss-Bonnett invariant). One may consider an even simpler situation: admitting that Newton law is satisfied only in the current universe. Then, the form of Newton law should be fixed only at the present universe (in other words, the initial value of $f(R)$ should be restricted). We should also note that if $P(\phi)$ changes its sign, as we identified $P(\phi)$ with $1/\kappa^2$, there could appear the antigravitation effect. For instance, the form of modified gravity may be changed in the future, at the end of the acceleration epoch, driving the Newton law to its opposite sign form.

III. MODIFIED GRAVITY AND COMPENSATING DARK ENERGY

In the present section we will present another approach to modified gravity. Specifically, we discuss the modified gravity which successfully describes the acceleration epoch but may be not viable in the matter dominated stage. In this case, it is demonstrated that one can introduce the compensating dark energy (some ideal fluid) which helps to realize the matter dominated and deceleration-acceleration transition phases. The role of such compensating dark energy is negligible in the acceleration epoch.

We now start with general $f(R)$ gravity action:

$$S = \int d^4x\{f(R) + \mathcal{L}_{\rm matter}\}. \tag{93}$$

In the FRW metric with flat spatial dimensions one gets

$$\begin{aligned} \rho &= f(R) - 6\left(\dot{H} + H^2 - H\frac{d}{dt}\right)f'(R), \\ p &= -f(R) - 2\left(-\dot{H} - 3H^2 + \frac{d^2}{dt^2} + 2H\frac{d}{dt}\right)f'(R), \\ R &= 6\dot{H} + 12H^2. \end{aligned} \tag{94}$$

If the Hubble rate is given (say, by observational data) as a function of t: $H = H(t)$, by substituting such an expression into (94), we find the t dependence of ρ and p as $\rho = \rho(t)$ and $p = p(t)$. If one can solve the first equation with respect to t as $t = t(\rho)$, by substituting it into the second equation, an equation of state follows:

$$p = p(t(\rho)). \tag{95}$$

Of course, ρ and p could be a sum with the contribution of several kinds of fluids with simple EOS.

We now concentrate on the case that $f(R)$ is given by [6]

$$f(R) = -\frac{\alpha}{R^n} + \frac{R}{2\kappa^2} + \beta R^2. \tag{96}$$

Furthermore, we write $H(t)$ as Eq. (42) and assume $h(t)$ is a slowly varying function of t and neglect the derivatives of $h(t)$ with respect to t. Then one gets (43).

First we consider the case that the last term in (80) dominates $f(R) \sim \beta R^2$, which may correspond to the early (inflationary) epoch of the universe. It is not difficult to find

$$\begin{aligned} \rho &\sim -\frac{36\beta(-1 + 2h(t))h(t)^2}{t^{4n}}, \\ p &\sim -\frac{36\beta(-1 + 2h(t))h(3h(t) + 1)}{t^{4n}}. \end{aligned} \tag{97}$$

If h goes to infinity, which corresponds to the de Sitter universe, we find $\rho \sim p \sim h^3$ although, from (43), $R \sim h^4$. Therefore ρ, $p \ll \beta R^2$ and contribution from the matter could be neglected. Then the inflation could be generated only by the contribution from the higher curvature term.

Second, we consider the case that the second term in (80) dominates $f(R) \sim \frac{R}{2\kappa^2}$, which may correspond to the matter dominated epoch after the inflation. In this case ρ and p behave as

$$\rho \sim \frac{12h(t) + 6h(t)^2}{\kappa^2 t^2}, \qquad p \sim -\frac{4h(t) - 6h(t)^2}{\kappa^2 t^2}. \quad (98)$$

In the matter dominated epoch, we expect $h \sim 2/3$ ($a \sim t^{2/3}$). Hence, one gets

$$\rho \sim \frac{32}{3\kappa^2 t^2}, \qquad p \sim 0. \quad (99)$$

Therefore in the matter sector, dust with $w = 0$ ($p = 0$) should dominate, as usually expected.

Finally we consider the case that the first term in (93) dominates $f(R) \sim -\alpha/R^n$, which might describe the acceleration of the present universe. The behavior of ρ and p is given by

$$\begin{aligned}\rho &\sim \alpha\{6(n+1)(2n+1)h(t) + 6(n-2)h(t)^2\} \\ &\quad \times \{-6h(t) + 12h(t)^2\}^{-n-1} t^{2n}, \\ p &\sim \alpha\{-4n(n+1)(2n+1) - 2(8n^2+5n+3)h(t) \\ &\quad - 6(n-2)h(t)^2\}\{-6h(t) + 12h(t)^2\}^{-n-1} t^{2n}. \end{aligned} \quad (100)$$

Thus, the effective EOS parameter w_l is given by

$$\begin{aligned} w_l \equiv \frac{p}{\rho} &\sim \{-4n(n+1)(2n+1) - 2(8n^2+5n+3)h(t) \\ &\quad - 6(n-2)h(t)^2\}\{6(n+1)(2n+1)h(t) \\ &\quad + 6(n-2)h(t)^2\}^{-1}. \end{aligned} \quad (101)$$

In order that the acceleration of the universe could occur, we find $h > 1$. Let us now assume that $h(t) \to h_f$ when $t \to \infty$. Then one obtains

$$\begin{aligned} w_l \to w_f \equiv &\{-4n(n+1)(2n+1) \\ &- 2(8n^2+5n+3)h_f - 6(n-2)h_f^2\} \\ &\times \{6(n+1)(2n+1)h_f + 6(n-2)h_f^2\}^{-1}, \end{aligned} \quad (102)$$

and $H(t) \to h_f/t$. Since the matter energy density ρ_{w_f} with the EoS parameter w_f behaves as

$$\rho_{w_f} \propto a^{-3(1+w_f)} \propto \exp\left(-3(1+w_f)\int dt H(t)\right), \quad (103)$$

the energy density is

$$\rho_{w_f} \propto t^{-3(1+w_f)h_f}. \quad (104)$$

Comparing (104) with (100), we find

$$2n = -3(1+w_f)h_f, \quad (105)$$

which can be confirmed directly from (102).

From the above consideration, we find ρ and p contain mainly contributions from dust with $w = 0$, $\rho_d(t)$, $p(t) = 0$, and "dark energy" with $w = w_l$ in (101), $\rho_l(t)$, $p_l(t)$. In the expressions of $\rho(t)$ and $p(t)$ in (94), there might be a remaining part:

$$\rho_R(t) \equiv \rho(t) - \rho_d(t) - \rho_l(t), \qquad p_R(t) \equiv p(t) - p_l(t), \quad (106)$$

which may help the transition from the matter dominated epoch to the acceleration epoch. By deleting t in the expression of (106), we obtain the EOS for the remaining part:

$$p_R = p_R(\rho_R), \quad (107)$$

which may be called the compensating dark energy. More concretely, according to (99), one may have

$$\rho_d \sim \frac{32}{3\kappa^2 t_0^2} \mathrm{e}^{-3\int_{t_0}^t dt (h(t)/t)}, \quad (108)$$

and according to (100),

$$\begin{aligned}\rho_l \sim &\alpha\{6(n+1)(2n+1)h_f + 6(n-2)h_f^2\} \\ &\times \{-6h_f + 12h_f^2\}^{-n-1} t_1^{2n} \mathrm{e}^{-3(1+w_f)\int_{t_1}^t dt(h(t)/t)}. \end{aligned} \quad (109)$$

In (109), we choose t_1 to be large enough. When $t \sim t_0$, $\rho(t) \sim \rho_d$ and when $t \to \infty$, $\rho(t) \sim \rho_l$. Thus, ρ_R only dominates after $t = t_1$ but it becomes smaller in late times. Hence, the role of ρ_R (which perhaps may be identified partially with dark matter) is only to connect the matter dominated epoch to the acceleration epoch.

IV. DISCUSSION

In summary, we developed the general formulation of modified $f(R)$ gravity which may be reconstructed for any given FRW metric. The resulting action is given in the implicit form (usually, in terms of some special functions). Nevertheless, its early and late times asymptotics may be defined, it turns out to have quite a simple form, for instance, as model [6] with negative and positive powers of curvature. Several examples predicted by realistic cosmology are constructed. These specific $f(R)$ theories describe the sequence of cosmological epochs: the matter dominated stage (if necessary even without matter), transition from deceleration to acceleration, and current cosmic speedup consistent with three years of WMAP data. Moreover, the study of their Newtonian regime indicates that such models are consistent with solar system tests. It is not difficult to extend such a formulation to include consistently also the radiation dominated phase (perhaps, even inflation). Hence, modified $f(R)$ gravity indeed represents the realistic alternative to general relativity, being more consistent in the dark epoch. It is also shown that some implicit version of $f(R)$ gravity may describe ΛCDM cosmology without the need to introduce the cosmological constant and without singularity near $R = 0$.

In the alternative approach we also demonstrate that even simple models like the ones of Refs. [6,15] become cosmologically viable if compensating dark energy is introduced. It remains to study if such compensating dark energy or the version of $f(R)$ gravity which mimics the matter dominated phase without matter may serve as dark matter of the universe.

Definitely, more careful study of modified gravity and fitting the above models against the observational data/various constraints [16] should be done. First of all, one should study the linear perturbations in the matter dominated epoch. Such a study made for the scalar-Gauss-Bonnet gravity model of Ref. [24] in Ref. [25] shows that there is no really strong change if compared with usual general relativity. Hence, one may expect that the same will occur in the present model while a careful study should be made, of course. It is also known [7] that Supernovae Ia constraints are easy to fit in the class of models under consideration (subject to the assumption that they are treated as usual candles). Second, the cosmic microwave background radiation (CMBR) peak locations are weakly model dependent. Nevertheless, it is important to check constraints appearing from the CMBR shift parameter and baryon oscillations as well as nucleosynthesis bounds which restrict the amount of dark energy in the current universe. Third, solar system constraints (time variation of the effective gravitational constant, study of parametrized post-Newtonian (PPN) parameters) should be considered in detail. The preliminary expectation is that this may be achieved due to the fact of freedom of the form of modified gravity (as well as first time derivative of the gravitational Lagrangian) at some specific (initial) time. This will be investigated in detail elsewhere. Having in mind that new, more precise observational data will be available soon, one may expect that the question: is modified gravity suitable as dark energy will be answered in near future.

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APPENDIX

Let us show for the explicit example of $f(R)$ that our formulation with the auxiliary scalar fixed as time is equivalent to the usual metric formulation. The starting action of the modified gravity coupled with matter is:

$$S = \int d^4x\sqrt{-g}\{f_0R^\alpha + \mathcal{L}_{\rm matter}\}. \tag{A1}$$

The FRW equation is given by

$$0 = f_0\{-\tfrac{1}{2}(6\dot{H} + 12H^2)^\alpha + 3\alpha(\dot{H} + H^2)(6\dot{H} + 12H^2)^{\alpha-1} - 3\alpha H\partial_t\{(6\dot{H} + 12H^2)^{\alpha-1}\}\} + \tfrac{1}{2}\rho_0 a^{-3(1+w)}. \tag{A2}$$

An exact solution of (A2) is given by

$$a = a_0 t^{h_0}, \qquad h_0 \equiv \frac{2\alpha}{3(1+w)},$$
$$a_0 \equiv \left[-\frac{6f_0h_0}{\rho_0}(-6h_0 + 12h_0^2)^{\alpha-1}\{(1-2\alpha)(1-\alpha) - (2-\alpha)h_0\}\right]^{-(1/3(1+w))}. \tag{A3}$$

Instead of action (A1), one now starts with the action (2). Equation (A3) shows

$$H = \frac{h_0}{t} \quad \text{or} \quad g(t) = h_0 \ln t. \tag{A4}$$

Hence

$$\begin{aligned}0 &= 2\frac{d^2P(\phi)}{d\phi^2} - 2g'(\phi)\frac{dP(\phi))}{d\phi} + 4g''(\phi)P(\phi) + (1+w)\rho_0 a_0^{-3(1+w)} e^{-3(1+w)g(t)} \\ &= 2\frac{d^2P(\phi)}{d\phi^2} - \frac{2h_0}{\phi}\frac{dP(\phi))}{d\phi} - \frac{4h_0}{\phi^2}P(\phi) + (1+w)\rho_0 a_0^{-3(1+w)}\phi^{-2\alpha}.\end{aligned} \tag{A5}$$

Here we have used a relation $h_0 = (2/3)(\alpha/(1+w))$ in (A3). A solution of (A5) is given by

$$P = P_0\phi^{-2\alpha+2}, \qquad P_0 \equiv \frac{(1+w)\rho_0 a_0^{-3(1+w)}}{4\{(1-\alpha)(1-2\alpha) - h_0(2-\alpha)\}}. \tag{A6}$$

Then we find

$$Q(\phi) = -6(g'(\phi))^2 P(\phi) - 6g'(\phi)\frac{dP(\phi)}{d\phi} + \rho_0 a_0^{-3(1+w)} e^{-3(1+w)g(t)} = Q_0 \phi^{-2\alpha}, \quad Q_0 \equiv -6h_0(h_0 - 2\alpha + 2)P_0 + \rho_0 a_0^{-3(1+w)} = \frac{6(1-\alpha)h_0(2h_0 - 1)P_0}{\alpha}. \tag{A7}$$

In the last line, we used the definition of P_0 in (A6). Therefore it follows

$$0 = P'(\phi)R + Q'(\phi) = 2(1-\alpha)\phi^{-2\alpha+1}R - 2\alpha Q_0 \phi^{-2\alpha}, \tag{A8}$$

which gives

$$\phi^2 = \frac{\alpha Q_0}{(1-\alpha)P_0 R} = \frac{12h_0^2 - 6h_0}{R}. \tag{A9}$$

Then the action (2) has the following form

$$S = \int d^4x \sqrt{-g}\{f_0' R^\alpha + \mathcal{L}_{\rm matter}\}, \quad f_0' \equiv \left(\frac{\alpha}{1-\alpha}\right)^{-\alpha+1} \frac{Q_0^{-\alpha+1} P_0^\alpha}{\alpha} = \frac{(12h_0^2 - 6h_0)^{-\alpha+1}\rho_0 a_0^{-3(1+w)}}{6\{(1-\alpha)(1-2\alpha) - h_0(2-\alpha)\}}. \tag{A10}$$

Comparing f_0' (A10) with (A3), one gets

$$f_0 = f_0', \tag{A11}$$

which shows the action (A1) is surely reproduced. On the other hand, Eq. (A3) shows

$$R = \frac{12h_0^2 - 6h_0}{t^2}. \tag{A12}$$

Comparing (A12) with (A9), we find

$$\phi = t. \tag{A13}$$

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